NONLINEAR
PHYSICAL
SCIENCE

Vladimir V. Uchaikin

Fractional Derivatives for Physicists and Engineers

Volume I Background and Theory

物理及工程中的分数维微积分

第I卷 数学基础及其理论

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NONLINEAR PHYSICAL SCIENCE

非线性物理科学

NONLINEAR PHYSICAL SCIENCE

Nonlinear Physical Science focuses on recent advances of fundamental theories and principles, analytical and symbolic approaches, as well as computational techniques in nonlinear physical science and nonlinear mathematics with engineering applications.

Topics of interest in *Nonlinear Physical Science* include but are not limited to:

- New findings and discoveries in nonlinear physics and mathematics
- Nonlinearity, complexity and mathematical structures in nonlinear physics
- Nonlinear phenomena and observations in nature and engineering
- Computational methods and theories in complex systems
- Lie group analysis, new theories and principles in mathematical modeling
- Stability, bifurcation, chaos and fractals in physical science and engineering
- Nonlinear chemical and biological physics
- Discontinuity, synchronization and natural complexity in the physical sciences

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To my friends
who made my life longer

Preface

"God made the integers; all else is the work of man"[1]. For centuries, the ancients were satisfied with using natural numbers called simply "numbers". What we call *irrational numbers* was not included into this notion by the Greeks. Not even rational fractions were called numbers.

Hence, *numbers* were conceived as discontinuous, while *magnitudes* were continuous. The two notations appeared, therefore, entirely distinct. The transfer from numbers to magnitudes (to lengths, for example) was a difficult and important step. Perhaps, the most dramatic confrontation of the notions exhibited in *Zeno's paradoxes*. One of them says: "Achilles cannot overtake a tortoise. Why? Achilles must first reach the place from which the tortoise started. By that time, the tortoise will have moved on a little way. Achilles must then traverse that, and still the tortoise will be ahead. He is always nearer, yet never makes up to it". The paradox resolution became possible only after extending the concept "number" over the whole real axis.

The real numbers have formed a basis of classical analysis whose major concept is the *continuity*. In frame of this conception, the set of natural and even rational numbers is vanishing (in cardinality) as compared with the continuum of real numbers. In numerical calculations, we use rational numbers as approximations to real ones, but namely irrational numbers reflect the real world. Nobody can make a rod with rational length and nobody can check that the length is rational.

Of course, when we are dealing with a set of isolated objects we use the natural numbers for counting the objects. However, if the objects are numbers and we consider the sum of the numbers, the situation may change. Thus, the number n in the expressions

$$y_n = x_1 + x_2 + \cdots + x_n = \sum_{j=1}^{n} x_j$$

and

[1] I'm very grateful to Prof. V. Kiryakova for her remark that this phrase is due to Leopold Kronecker (1886).

$$z_n = x_1 \cdot x_2 \cdot \cdots \cdot x_n = \prod_{j=1}^{n} x_j$$

needs no comments, it is natural. But if the numbers x_i are identical (and positive), these operations can be easily continued on the whole real axis: $n = \{1,2,3,\dots\} \to \nu \in (-\infty,\infty)$:

$$y_\nu = \nu x, \quad z_\nu = x^\nu, \quad \nu \in \mathbb{R}.$$

The famous Euler invention called Euler's Gamma function

$$\Gamma(\nu) = \int_0^\infty e^{-x} x^{\nu-1} dx, \quad \nu > 0,$$

$$\Gamma(n+1) = n!, \quad n = 1 \cdot 2 \cdot 3 \cdot \cdots$$

has played a crucial role in extending the concept "number of operations" on the noninteger values. With this function, there was made a scientific breakthrough in the differential calculus, which enriched it with differentiation and integration of fractional orders called shortly *fractional calculus*. The foundation of fractional calculus is connected with the names of Riemann, Liouille, Weyl, Grünwald, Letnikov and others. Though the first works in this direction were made of about two centuries ago, these ideas had not found any practical applications for a long time. However, the situation has been changed dramatically during a couple of last decades, while about 3 thousand works were published on the subject.

For better understanding of what the extension of this concept can bring, let us consider popular differential equations of theoretical physics of the form

$$a\frac{\partial^m f(x,t)}{\partial t^m} + b\frac{\partial^n f(x,t)}{\partial x^n} = F, \tag{0.1}$$

where x, t are space-time variables, a, b, and F are given functions of x and t, and m, $n = 0,1,2,\dots$ are integer numbers. If one of the numbers, say n, is zero, the corresponding variable x becomes a plain parameter. Omitting it, we arrive at the ordinary differential equation:

$$a\frac{d^m f(t)}{dt^m} + bf(t) = F.$$

Putting $n = 1$ in Eq. (0.1) and interpreting $f(t)$ as a velocity of a material point performing one-dimensional motion under action of the force $F - bf$, we recognize here the simplest version of the Newton equation. If $a,b > 0$, then the term $-bf$ can be interpreted as a friction force, and we meet the relaxation problem. When a denotes mass and f means the coordinate of a particle, we again see the Newton equation describing one-dimensional motion along the x-axis. This time, the term $-bf$ means the elastic force and the equation describes the harmonic oscillator driven by the force F (assuming $b = \text{const}$).

Choosing $m = n = 1$, we obtain a one-dimensional continuity equation. This is the simplest equation of partial differential equations of mathematical physics. The other popular versions of equations of mathematical physics are represented in Table 0.1. They are well-known and do not need any comments. In the last line, you can see the equations containing time- and space-derivatives of fractional orders. These operations are significantly less familiar to the majority of physicists and engineers. One could pay no serious attention to such exotic mathematical construction, but only one glance at Fig. 0.1 may shake the scepticism: we see that the set of

Table 0.1

m,n	1D-equations	3D-equations	Phys. sense	Math. type
1, 0	$\|a\|\frac{dv}{dt}+bv=F$	$\|a\|\frac{d\mathbf{v}}{dt}+b\mathbf{v}=F$	Damped motion	–
2, 0	$\|a\|\frac{d^2x}{dt^2}+bx=F$	$\|a\|\frac{d^2\mathbf{r}}{dt^2}+b\mathbf{r}=F$	Oscillation	–
1, 1	$\|a\|\frac{\partial f}{\partial t}+\frac{\partial(bf)}{\partial x}=F$	$\|a\|\frac{\partial f}{\partial t}+\nabla(\mathbf{b}f)=F$	Continuity	–
1, 2	$\|a\|\frac{\partial f}{\partial t}-\|b\|\frac{\partial^2 f}{\partial x^2}=F$	$\|a\|\frac{\partial f}{\partial t}-\|b\|\nabla^2 f=F$	Diffusion	Parabolic
2, 2	$\|a\|\frac{\partial^2 f}{\partial t^2}-\|b\|\frac{\partial^2 f}{\partial x^2}=F$	$\|a\|\frac{\partial^2 f}{\partial t^2}-\|b\|\nabla^2 f=F$	Waves	Hyperbolic
0, 2	$af+b\frac{\partial^2 f}{\partial x^2}=F$	$af+b\nabla^2 f=F$	Static fields	Elliptic
μ,ν non-integers	$a\frac{\partial^\mu f}{\partial t^\mu}+b\left(\frac{\partial^2}{\partial x^2}\right)^{\nu/2}f=F$ $t>0,\ -\infty<x<\infty$	$a\frac{\partial^\mu f}{\partial t^\mu}+b\Delta^{\nu/2}f=F$	?	No classified yet

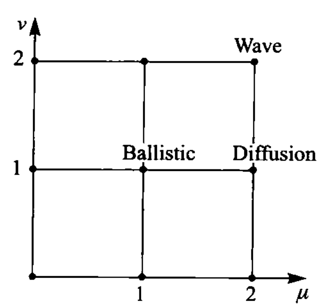

Fig. 0.1 Continuous manifold of fractional partial equations.

well-known and well-investigated differential equations of mathematical physics is represented by only a few points on (μ, ν) diagram, while the continuous set of all other points of the whole (μ, ν)-plane is a *terra incognita*, which can not but attract attention of graduate and postgraduate students, promising scientists and young engineers. This is the readership the book is addressed in the first place. Nevertheless, I hope that it can attract the attention of more experienced researchers, both physicists and mathematicians, as comparatively new tools for investigating and modeling of complex natural processes.

I was 55 years old when I went on a trip for years over this "fractional" land and found many treasures there: about three thousand articles using the tools for solving different problems in physics and engineering have been published during the last two decades. They include

- inverse mechanical problems
- stochastic kinetics and dynamical chaos
- motion in viscous fluid
- heat flow spreading
- electrochemistry of electrodes
- percolation through porous media
- rheology of viscoelastic materials
- electrical and radio engineering
- plasma physics
- quantum optics and nanophysics
- astrophysics and cosmology
- biophysics and medicine

Fortunately, our group managed to participate in developing fractional approach to description of anomalous (dispersive) transport in disordered semiconductors, non-Debye relaxation in solid dielectrics, penetration of light beam through a turbulent medium, transport of resonance radiation in plasma, blinking fluorescence of quantum dots, subrecoil laser cooling of atoms, penetration and acceleration of cosmic ray in the Galaxy, large-scale statistical cosmography and solving some other problems. These investigations allowed us to become aware of deep links between fractional calculus, non-Gaussian Lévy-stable statistics and stochastic fractals. The presence of a time-fractional derivative in the equation is interpreted as a special property of the process under consideration called the *memory*, the *after-effect*, or, when we handle with a stochastic process, the *non-Markovian property*. Fractional derivatives with respect to coordinates reflect a medium with inhomogeneities of some special kind called *selfsimilar inhomogeneities* or *fractals*. We meet such structures in turbulent flows, plasma, and interstellar media.

Like many my colleagues, I'm convinced that the fractional derivative, or, as it is often called, the *fractional differintegral* given by the expression

$$_af^{(\nu)}(x) = \frac{1}{\Gamma(n-\nu)} \frac{d^n}{dx^n} \int_0^x (x-\xi)^{n-\nu-1} f(\xi) d\xi$$

with $n-1 \leq \nu < n$ if $\nu \geq 0$ and $n = 0$ if $\nu < 0$, is much more than merely a sequence of differential and special integral operators[2]. Important properties of Nature underlie this mathematical concept. Starting to write this book, I decided to begin it with discussion of these properties, enveloping, in my opinion, *heredity*, *nonlocality*, *selfsimilarity*, and *stochasticity*. This is why the first three chapters of the book are united into the first part "Background" which contains description of various natural phenomena demonstrating such properties.

Chapter 1 contains a modern exposition of the Volterra heredity concept whose main tool is the integral operation

$$f(x) \mapsto \int_0^x K(x,\xi)f(\xi)d\xi$$

with the kernel $K(x,\xi)$ interpreted as a memory in case x and ξ are time, or as nonlocality in case the variables are spatial coordinates. The variety of physical processes including mechanical, molecular, hydrodynamical, thermodynamical phenomena demonstrating hereditary properties are described in this chapter.

In Chapter 2 we review physical processes characterized by power-type memory functions and basic mechanisms generated this property. This list may shake the opinion that the exponential function is the queen-function of theoretical physics and show that without power functions like without maids of honor the queen court would be more tiresome.

Chapter 3 opens a wide panorama of stochastic processes which shows that probabilistic long tails of power type results from selfsimilarity of the processes and the latter is connected with existence of limit distributions, namely with the Lévy-stable laws. The reader will go into the fractional Brownian motion continuous-time random walk processes, fractional Poisson process and walking on fractals.

The second part of the book, "Theory", contains the elements of fractional calculus theory with review of various fractional equations, and their analytical and numerical solutions.

Chapter 4 serves as a mathematical introduction to fractional calculus containing basic definitions of fractional operators, their properties and rules of applications. Readers can recognize many of them as corresponding generalization of well-known analogous from integer-order calculus such as the Leibnitz rule or the Taylor formula.

Chapter 5 shows how equations with fractional derivatives are solved. The reader will meet the description of some analytical methods of solution and many examples of their applications to ordinary and partial fractional equations. Of course, this review can not pretend on a strict and exhaustive exposition, but it will be useful for physicists and engineers as a first acquaintance.

Chapter 6 contains an introduction to numerical methods of solving fractional equation. Starting with the fractional difference operators and based on this con-

[2] Here are two remarkable quotations: "The fractional calculus is the calculus of the XXI century" (K. Nishimoto, 1989) and "We may say that Nature works with fractional time derivatives" (S. Westerlund, 1991).

cept the Grünwald-Letnikov definition of fractional derivatives, the reader is acquainted with the finite-difference methods of computing fractional integrals, fractional derivatives, and fractional equations of various kinds. The last section of this chapter is devoted to some aspects of Monte Carlo techniques.

The third part, opening the second volume of the book, exposes a wide field of applications of fractional calculus in modern physics including mechanics, hydrodynamics, viscoelasticity, thermodynamics, electrodynamics, plasma physics, quantum physics, and cosmic ray physics.

Finally, the fourth, last part of the book contains various auxiliary materials (special functions, notation of fractional derivatives, main formulas of fractional calculus, tables and graphs of some functions, which are playing a special role in the solution of fractional equations).

Ulyanovsk (Russia), March 2012 *Vladimir V. Uchaikin*

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