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Albert C. J. Luo

Continuous Dynamical Systems

连续动力系统

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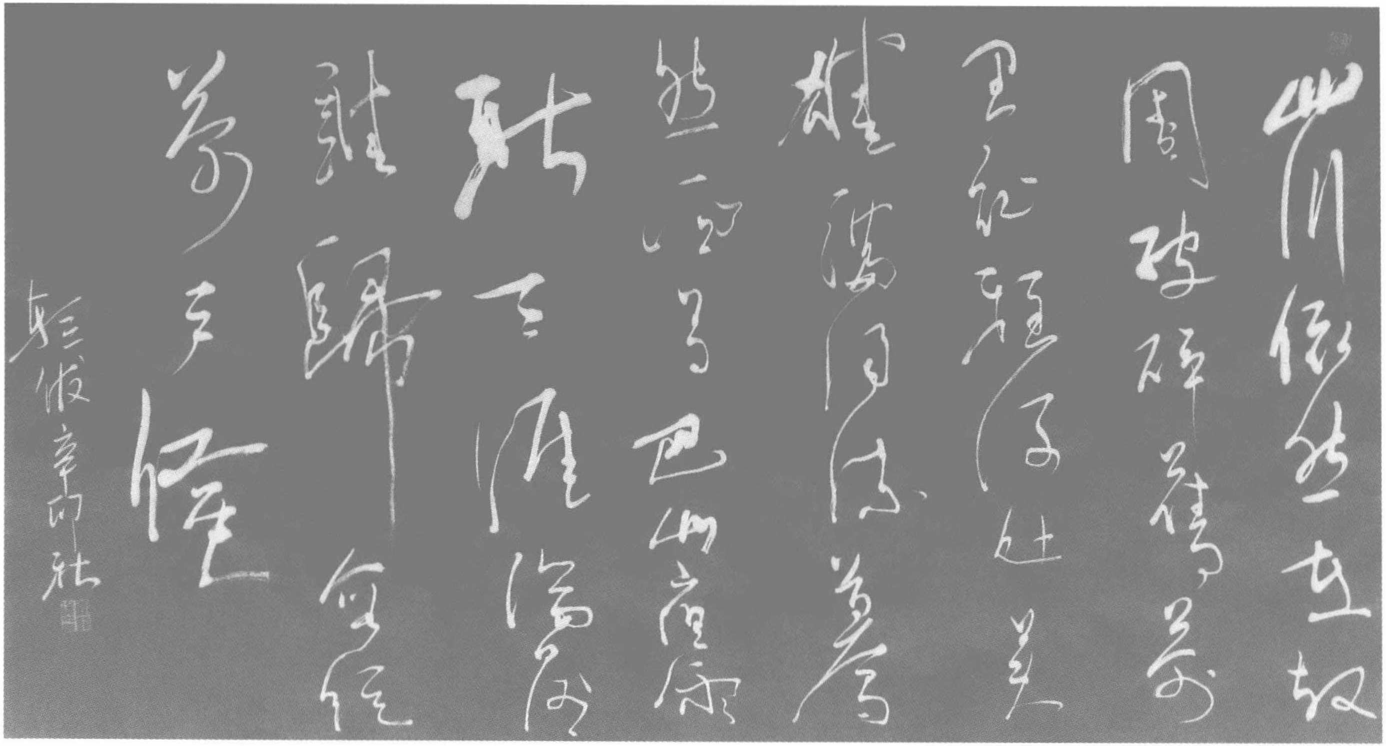

Preface

This book collects the recent development on the theory of continuous dynamical systems from a different point of view. From the author's 20 years researches and teaching experience, the author places materials in five chapters to provide a better understanding of stability, stability switching, bifurcation, complexity and chaos in nonlinear continuous dynamical systems. For completeness, the theory for discrete and switching systems with transports is presented in a different volume.

The stability theory of linear continuous dynamical systems is comprehensively discussed in Chapter 1 from the author's teaching experience. The materials presented in this book are a foundation to understand stability and bifurcation theory in nonlinear dynamical systems. In Chapter 2, the author used a different point of view to present the stability, stability switching and bifurcation of equilibriums for nonlinear continuous systems. Such presentation makes the stability and bifurcation theory much simpler, readable and doable. In Chapter 3, an analytical method is presented to obtain the analytical solutions of periodic flows and chaos in nonlinear dynamical systems. The analytical solution of chaos ends the history of chaos numerically simulated only. In Chapter 4, the global transversality of a flow to the generic separatrix in nonlinear dynamical systems is presented, and the theory of global transversality is the base to understand the complexity of flows in nonlinear dynamical systems. How to determine the separatrix and all possible first integral manifolds in n-dimensional dynamical systems is still unsolved. In Chapter 5, chaos in stochastic and resonant layers in nonlinear Hamiltonian systems is discussed. The physical mechanism of chaos in stochastic layers is the resonance interaction to form the hyperbolic characteristics.

Finally, I would like to appreciate my student, Jianzhe Huang for completing numerical computations in Chapter 3. Herein, I thank my wife (Sherry X. Huang) and my children (Yanyi Luo, Robin Ruo-Bing Luo, and Robert Zong-Yuan Luo) for tolerance, patience, understanding and support. I hope this book will be a good gift for them.

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Contents

Chapter 1
Linear Systems and Stability

In this Chapter, the theory of linear systems will be presented. Separated linear systems and diagonalization of square matrix will be discussed first. The linear operator exponentials will be presented. The fundamental solutions of autonomous linear systems will be given with the matrix possessing real eigenvalues, complex eigenvalues and repeated eigenvalues. The stability theory for autonomous linear systems will be discussed. The solutions of non-autonomous linear systems will be discussed and steady state solutions will be presented. A generalized "resonance" concept will be introduced, and the resonant solutions will be presented. Solutions and stability for lower dimensional linear systems will be discussed in details.

1.1 Linear systems with distinct eigenvalues

Definition 1.1 Consider a linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{Q}(t) \text{ for } t \in \mathscr{R} \text{ and } \mathbf{x} = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \in \mathscr{R}^n \quad (1.1)$$

where $\dot{\mathbf{x}} = d\mathbf{x}/dt$ is differentiation with respect to time t. $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{Q}(t)$ is a continuous vector function. If $\mathbf{Q}(t) = \mathbf{0}$, the linear system in Eq.(1.1) is autonomous. Equation (1.1) becomes

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \text{ for } t \in \mathscr{R} \text{ and } \mathbf{x} \in \mathscr{R}^n \quad (1.2)$$

which is called an autonomous linear system or a homogenous linear system. With an initial condition of $\mathbf{x}(t_0) = \mathbf{x}_0$, the solution of Eq.(1.2) is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0. \quad (1.3)$$

If $\mathbf{Q}(t) \neq \mathbf{0}$, the linear system in Eq.(1.1) is non-autonomous, and such a non-

Albert C. J. Luo, *Continuous Dynamical Systems*

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autonomous system is also called a nonhomogenous linear system. With an initial condition of $\mathbf{x}(t_0)=\mathbf{x}_0$, the solution of Eq.(1.1) is given by

$$\mathbf{x}(t)=\mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{x}_0+\int_{t_0}^{t}\mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(\tau)\mathbf{Q}(\tau)d\tau, \tag{1.4}$$

where $\mathbf{\Phi}(t)$ is a fundamental matrix of the homogenous linear system in Eq.(1.2) with

$$\dot{\mathbf{\Phi}}(t)=\mathbf{A}\mathbf{\Phi}(t) \text{ for all } t\in I\subseteq\mathscr{R}. \tag{1.5}$$

Definition 1.2 For a linear dynamical system in Eq.(1.2), if the linear matrix $\mathbf{A}=diag(\lambda_1,\lambda_2,\cdots,\lambda_n)$ is a diagonal matrix, then the linear system in Eq.(1.2) is called an uncoupled linear homogenous system. With an initial condition of $\mathbf{x}(t_0)=\mathbf{x}_0$, the solution of the uncoupled linear homogenous solution is

$$\mathbf{x}(t)=diag[e^{\lambda_1 t},e^{\lambda_2 t},\cdots,e^{\lambda_n t}]\mathbf{x}_0. \tag{1.6}$$

Theorem 1.1 *Consider a linear system* $\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}$ *in Eq.*(1.2) *with the initial condition of* $\mathbf{x}(t_0)=\mathbf{x}_0$. *If the real and distinct eigenvalues of the* $n\times n$ *matrix* $\mathbf{A}$ *are* $\lambda_1,\lambda_2,\cdots,\lambda_n$, *then a set of corresponding eigenvectors* $\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n\}$ *is determined by*

$$(\mathbf{A}-\lambda_i\mathbf{I})\mathbf{v}_i=\mathbf{0} \tag{1.7}$$

which forms a basis in $\Omega\subseteq\mathscr{R}^n$. *The eigenvector matrix of* $\mathbf{P}=[\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n]$ *is invertible and*

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P}=diag[\lambda_1,\lambda_2,\cdots,\lambda_n]. \tag{1.8}$$

Thus, with an initial condition of $\mathbf{x}(t_0)=\mathbf{x}_0$, *the solution of linear system in Eq.*(1.2) *is*

$$\begin{aligned}\mathbf{x}(t)&=\mathbf{P}diag[e^{\lambda_1(t-t_0)},e^{\lambda_2(t-t_0)},\cdots,e^{\lambda_n(t-t_0)}]\mathbf{P}^{-1}\mathbf{x}_0\\&=\mathbf{P}\mathbf{E}(t-t_0)\mathbf{P}^{-1}\mathbf{x}_0,\end{aligned} \tag{1.9}$$

where the diagonal matrix $\mathbf{E}(t)$ *is given by*

$$\mathbf{E}(t-t_0)=diag[e^{\lambda_1(t-t_0)},e^{\lambda_2(t-t_0)},\cdots,e^{\lambda_n(t-t_0)}]. \tag{1.10}$$

Proof: The proof can be referred to Luo (2011). ■

Computing the eigenvector is a key to obtain the general solution of linear systems. Consider an alternative method herein and the eigenvector of $\mathbf{v}_i$ is assumed as

$$\mathbf{v}_i = \begin{bmatrix} 1 \\ \mathbf{r}_i \end{bmatrix} v_i. \tag{1.11}$$

From Eq.(1.7), we have

$$\begin{bmatrix} a_{11} - \lambda_i & \mathbf{b}_{1\times(n-1)} \\ \mathbf{c}_{(n-1)\times 1} & \mathbf{A}_{11} - \lambda_i \mathbf{I}_{(n-1)\times(n-1)} \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{r}^{(i)} \end{bmatrix} v_i = \mathbf{0}, \tag{1.12}$$

where the minor of matrix $\mathbf{A}$ is $\mathbf{A}_{11}$, and other vectors are defined by

$$\begin{aligned} \mathbf{c}_{(n-1)\times 1} &= (a_{i1})_{(n-1)\times 1} \ (i = 2,3,\cdots,n), \\ \mathbf{b}_{1\times(n-1)} &= (a_{1j})_{1\times(n-1)} \ (j = 2,3,\cdots,n), \\ \mathbf{A}_{11} &= (a_{ij})_{(n-1)\times(n-1)} \ (i,j = 2,3,\cdots,n). \end{aligned} \tag{1.13}$$

Thus,

$$\mathbf{r}_i = (\mathbf{A}_{11} - \lambda_i \mathbf{I}_{(n-1)\times(n-1)})^{-1} \mathbf{c}_{(n-1)\times 1}. \tag{1.14}$$

The solution of linear system in Eq.(1.2) is

$$\begin{aligned} \mathbf{x}(t) &= \sum_{i=1}^{n} C_i \mathbf{v}_i e^{\lambda_i (t-t_0)} \\ &= [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n] diag[e^{\lambda_1 (t-t_0)}, e^{\lambda_2 (t-t_0)}, \cdots, e^{\lambda_n (t-t_0)}]\mathbf{C} \\ &= \mathbf{P} diag[e^{\lambda_1 (t-t_0)}, e^{\lambda_2 (t-t_0)}, \cdots, e^{\lambda_n (t-t_0)}]\mathbf{C}, \end{aligned} \tag{1.15}$$

where

$$\mathbf{C} = (C_1, C_2, \cdots, C_n)^{\mathrm{T}}. \tag{1.16}$$

For $t = t_0$, the initial conditions is $\mathbf{x}(t) = \mathbf{x}_0$. Thus,

$$\mathbf{C} = \mathbf{P}^{-1}\mathbf{x}_0. \tag{1.17}$$

Therefore, the solution is expressed by

$$\mathbf{x}(t) = \mathbf{P} diag[e^{\lambda_1 (t-t_0)}, e^{\lambda_2 (t-t_0)}, \cdots, e^{\lambda_n (t-t_0)}]\mathbf{P}^{-1}\mathbf{x}_0 = \mathbf{P}\mathbf{E}(t)\mathbf{P}^{-1}\mathbf{x}_0. \tag{1.18}$$

The two methods give the same expression.

Theorem 1.2 *Consider a linear system* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ *in Eq.*(1.2) *with the initial condition of* $\mathbf{x}(t_0) = \mathbf{x}_0$. *If distinct complex eigenvalues of the* $2n \times 2n$ *matrix* $\mathbf{A}$ *are* $\lambda_j = \alpha_j + \mathrm{i}\beta_j$ *and* $\bar{\lambda}_j = \alpha_j - \mathrm{i}\beta_j$ *with corresponding eigenvectors* $\mathbf{w}_j = \mathbf{u}_j + \mathrm{i}\mathbf{v}_j$ *and* $\bar{\mathbf{w}}_j = \mathbf{u}_j - \mathrm{i}\mathbf{v}_j$ ($j = 1,2,\cdots,n$ *and* $\mathrm{i} = \sqrt{-1}$), *then the corresponding eigenvectors* $\mathbf{u}_j$ and $\mathbf{v}_j$ ($j = 1,2,\cdots,n$) *are determined by*

$$\begin{aligned}&\big(\mathbf{A}-(\alpha_j+\mathrm{i}\beta_j)\mathbf{I}\big)(\mathbf{u}_j+\mathrm{i}\mathbf{v}_j)=\mathbf{0},\ \text{or}\\&\big(\mathbf{A}-(\alpha_j-\mathrm{i}\beta_j)\mathbf{I}\big)(\mathbf{u}_j-\mathrm{i}\mathbf{v}_j)=\mathbf{0},\end{aligned} \tag{1.19}$$

which forms a basis in $\Omega \subseteq \mathscr{R}^{2n}$. *The corresponding eigenvector matrix of* $\mathbf{P}=[\mathbf{u}_1,\mathbf{v}_1,\mathbf{u}_2,\mathbf{v}_2,\cdots,\mathbf{u}_n,\mathbf{v}_n]$ *is invertible and*

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P}=diag[\mathbf{B}_1,\mathbf{B}_2,\cdots,\mathbf{B}_n], \tag{1.20}$$

where

$$\mathbf{B}_j=\begin{bmatrix}\alpha_j & \beta_j\\ -\beta_j & \alpha_j\end{bmatrix}\ (j=1,2,\cdots,n). \tag{1.21}$$

Thus, with an initial condition of $\mathbf{x}(t_0)=\mathbf{x}_0$, *the solution of the linear system in Eq.*(1.2) *is*

$$\mathbf{x}(t)=\mathbf{P}\mathbf{E}(t-t_0)\mathbf{P}^{-1}\mathbf{x}_0, \tag{1.22}$$

where the diagonal matrix $\mathbf{E}(t-t_0)$ *is given by*

$$\begin{aligned}&\mathbf{E}(t-t_0)=diag\big[\mathbf{E}_1(t-t_0),\mathbf{E}_2(t-t_0),\cdots,\mathbf{E}_n(t-t_0)\big],\\&\mathbf{E}_j(t-t_0)=e^{\alpha_j(t-t_0)}\begin{bmatrix}\cos\beta_j(t-t_0) & \sin\beta_j(t-t_0)\\ -\sin\beta_j(t-t_0) & \cos\beta_j(t-t_0)\end{bmatrix}.\end{aligned} \tag{1.23}$$

Proof: The proof can be referred to Luo (2011). ■

Consider an alternative approach, and the conjugate complex eigenvectors are

$$\mathbf{u}_i+\mathrm{i}\mathbf{v}_i=C_i\begin{bmatrix}1\\ \mathbf{r}_i\end{bmatrix}\ \text{and}\ \mathbf{u}_i-\mathrm{i}\mathbf{v}_i=\bar{C}_i\begin{bmatrix}1\\ \bar{\mathbf{r}}_i\end{bmatrix}, \tag{1.24}$$

where the conjugate complex constants are assumed as

$$\begin{aligned}&C_i=\frac{1}{2}(M_i-\mathrm{i}N_i)\ \text{and}\ \bar{C}_i=\frac{1}{2}(M_i+\mathrm{i}N_i).\\&\mathbf{r}_i=\mathbf{U}_i+\mathrm{i}\mathbf{V}_i\ \text{and}\ \bar{\mathbf{r}}_i=\mathbf{U}_i-\mathrm{i}\mathbf{V}_i.\end{aligned} \tag{1.25}$$

From Eq.(1.19), we have

$$\begin{bmatrix}a_{11}-\alpha_i-\mathrm{i}\beta_i & \mathbf{b}_{1\times(n-1)}\\ \mathbf{c}_{(n-1)\times 1} & \mathbf{A}_{11}-(\alpha_i+\mathrm{i}\beta_i)\mathbf{I}_{(n-1)\times(n-1)}\end{bmatrix}\begin{bmatrix}1\\ \mathbf{r}_i\end{bmatrix}C_i=\mathbf{0}, \tag{1.26}$$

Thus, the foregoing equation gives

$$\mathbf{c}+[(\mathbf{A}_{11}-\alpha_i\mathbf{I})-\mathrm{i}\beta_i\mathbf{I}]\mathbf{r}_i=\mathbf{0}, \tag{1.27}$$

$$\mathbf{r}_i = \left[(\mathbf{A}_{11} - \alpha_i \mathbf{I})^2 + \beta_i^2 \mathbf{I}\right]^{-1} \left[(\mathbf{A}_{11} - \alpha_i \mathbf{I}) + \mathrm{i}\beta_i \mathbf{I}\right]\mathbf{c} = \mathbf{U}_i + \mathrm{i}\mathbf{V}_i. \tag{1.28}$$

where

$$\begin{aligned} \mathbf{U}_i &= \left[(\mathbf{A}_{11} - \alpha_i \mathbf{I})^2 + \beta_i^2 \mathbf{I}\right]^{-1} (\mathbf{A}_{11} - \alpha_i \mathbf{I})\mathbf{c}, \\ \mathbf{V}_i &= \left[(\mathbf{A}_{11} - \alpha_i \mathbf{I})^2 + \beta_i^2 \mathbf{I}\right]^{-1} \beta_i \mathbf{c}. \end{aligned} \tag{1.29}$$

The solution of the linear system in Eq.(1.2) is

$$\begin{aligned} \mathbf{x}(t) &= \sum\nolimits_{i=1}^{n} \frac{1}{2}(M_i - \mathrm{i}N_i)\begin{bmatrix} 1 \\ \mathbf{U}_i + \mathrm{i}\mathbf{V}_i \end{bmatrix} e^{(\alpha_i + \mathrm{i}\beta_i)(t-t_0)} \\ &\quad + \frac{1}{2}(M_i + \mathrm{i}N_i)\begin{bmatrix} 1 \\ \mathbf{U}_i - i\mathbf{V}_i \end{bmatrix} e^{(\alpha_i - \mathrm{i}\beta_i)(t-t_0)} \\ &= \sum\nolimits_{i=1}^{n} e^{\alpha_i(t-t_0)}[\begin{bmatrix} M_i \\ M_i\mathbf{U}_i + N_i\mathbf{V}_i \end{bmatrix}\cos\beta_i(t-t_0) \\ &\quad + \begin{bmatrix} N_i \\ N_i\mathbf{U}_i - M_i\mathbf{V}_i \end{bmatrix}\sin\beta_i(t-t_0)] \\ &= \sum\nolimits_{i=1}^{n} e^{\alpha_i(t-t_0)}[(M_i\begin{bmatrix} 1 \\ \mathbf{U}_i \end{bmatrix} + N_i\begin{bmatrix} 0 \\ \mathbf{V}_i \end{bmatrix})\cos\beta_i(t-t_0) \\ &\quad + (N_i\begin{bmatrix} 1 \\ \mathbf{U}_i \end{bmatrix} - M_i\begin{bmatrix} 0 \\ \mathbf{V}_i \end{bmatrix})\sin\beta_i(t-t_0)] \\ &= \sum\nolimits_{i=1}^{n} e^{\alpha_i(t-t_0)}(\mathbf{u}_i, \mathbf{v}_i)\begin{bmatrix} \cos\beta_i(t-t_0) & \sin\beta_i(t-t_0) \\ -\sin\beta_i(t-t_0) & \cos\beta_i(t-t_0) \end{bmatrix}\begin{bmatrix} M_i \\ N_i \end{bmatrix} \\ &= \mathbf{PE}(t-t_0)\mathbf{C}, \end{aligned} \tag{1.30}$$

where

$$\begin{aligned} &\mathbf{P} = [\mathbf{u}_1, \mathbf{v}_1, \cdots, \mathbf{u}_n, \mathbf{v}_n], \\ &\mathbf{E}(t-t_0) = diag\left[\mathbf{E}_1(t-t_0), \mathbf{E}_2(t-t_0), \cdots, \mathbf{E}_n(t-t_0)\right], \\ &\mathbf{C} = (M_1, N_1, \cdots, M_n, N_n)^{\mathrm{T}}, \\ &\mathbf{E}_i(t-t_0) = e^{\alpha_i(t-t_0)}\begin{bmatrix} \cos\beta_i(t-t_0) & \sin\beta_i(t-t_0) \\ -\sin\beta_i(t-t_0) & \cos\beta_i(t-t_0) \end{bmatrix}, \\ &\mathbf{u}_i = \begin{bmatrix} 1 \\ \mathbf{U}_i \end{bmatrix} \text{ and } \mathbf{v}_i = \begin{bmatrix} 0 \\ \mathbf{V}_i \end{bmatrix}. \end{aligned} \tag{1.31}$$

For $t = t_0$, the initial conditions is $\mathbf{x}(t) = \mathbf{x}_0$. Thus,

$$\mathbf{C} = \mathbf{P}^{-1}\mathbf{x}_0. \tag{1.32}$$

Therefore, the solution is expressed by

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{P}diag[\mathbf{E}_1(t-t_0), \mathbf{E}_2(t-t_0), \cdots, \mathbf{E}_n(t-t_0)]\mathbf{P}^{-1}\mathbf{x}_0 \\ &= \mathbf{PE}(t-t_0)\mathbf{P}^{-1}\mathbf{x}_0.\end{aligned} \tag{1.33}$$

The two methods give the same expression.

Theorem 1.3 *Consider a linear system* $\dot{\mathbf{x}} = \mathbf{Ax}$ *in Eq.*(1.2) *with the initial condition of* $\mathbf{x}(t_0) = \mathbf{x}_0$. *If the eigenvalues of the* $n \times n$ *matrix* $\mathbf{A}$ *possesses* p*-pairs of distinct complex eigenvalues with* $\lambda_j = \alpha_j + \mathrm{i}\beta_j$ *and* $\bar{\lambda}_j = \alpha_j - \mathrm{i}\beta_j$ *with corresponding eigenvectors* $\mathbf{w}_j = \mathbf{u}_j + \mathrm{i}\mathbf{v}_j$ *and* $\bar{\mathbf{w}}_j = \mathbf{u}_j - \mathrm{i}\mathbf{v}_j$ $(j = 1, 2, \cdots, p$ *and* $\mathrm{i} = \sqrt{-1})$, *and* $(n-2p)$ *distinct real eigenvalues of* $\lambda_{2p+1}, \lambda_{2p+2}, \cdots, \lambda_n$, *then the corresponding eigenvectors* $\mathbf{u}_j$ *and* $\mathbf{v}_j$ *for complex eigenvalues* $(\lambda_j, \bar{\lambda}_j)$ $(j = 1, 2, \cdots, p)$ *are determined by*

$$\begin{aligned}&\left(\mathbf{A} - (\alpha_j + \mathrm{i}\beta_j)\mathbf{I}\right)(\mathbf{u}_j + \mathrm{i}\mathbf{v}_j) = \mathbf{0}, \text{ or} \\ &\left(\mathbf{A} - (\alpha_j - \mathrm{i}\beta_j)\mathbf{I}\right)(\mathbf{u}_j - \mathrm{i}\mathbf{v}_j) = \mathbf{0}\end{aligned} \tag{1.34}$$

and the eigenvectors $\{\mathbf{v}_{2p+1}, \mathbf{v}_{2p+2}, \cdots, \mathbf{v}_n\}$ *for real eigenvalues are determined by*

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{v}_i = \mathbf{0} \tag{1.35}$$

which forms a basis in $\Omega \subseteq \mathscr{R}^n$. *The eigenvector matrix of*

$$\mathbf{P} = [\mathbf{u}_1, \mathbf{v}_1, \mathbf{u}_2, \mathbf{v}_2, \cdots, \mathbf{u}_p, \mathbf{v}_p, \mathbf{v}_{2p+1}, \mathbf{v}_{2p+2}, \cdots, \mathbf{v}_n] \tag{1.36}$$

is invertible and

$$\mathbf{P}^{-1}\mathbf{AP} = diag[\mathbf{B}_1, \mathbf{B}_2, \cdots, \mathbf{B}_p, \lambda_{2p+1}, \lambda_{2p+2}, \cdots, \lambda_n], \tag{1.37}$$

where

$$\mathbf{B}_j = \begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix} \ (j = 1, 2, \cdots, p). \tag{1.38}$$

Thus, for $\mathbf{x}(t_0) = \mathbf{x}_0$ *the solution of linear system in Eq.*(1.2) *is*

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{P}diag[\mathbf{E}_1(t-t_0), \mathbf{E}_2(t-t_0), \cdots, \mathbf{E}_p(t-t_0), \\ &\qquad e^{\lambda_{2p+1}(t-t_0)}, e^{\lambda_{2p+2}(t-t_0)}, \cdots, e^{\lambda_n(t-t_0)}]\mathbf{P}^{-1}\mathbf{x}_0 \\ &= \mathbf{PE}(t-t_0)\mathbf{P}^{-1}\mathbf{x}_0,\end{aligned} \tag{1.39}$$

where the diagonal matrix $\mathbf{E}(t-t_0)$ *is given by*

$$\begin{aligned}\mathbf{E}(t-t_0) &= diag[\mathbf{E}_1(t-t_0), \mathbf{E}_2(t-t_0), \cdots, \mathbf{E}_p(t-t_0),\\ &\qquad e^{\lambda_{2p+1}(t-t_0)}, e^{\lambda_{2p+2}(t-t_0)}, \cdots, e^{\lambda_n(t-t_0)}];\\ \mathbf{E}_j(t-t_0) &= e^{\alpha_j(t-t_0)}\begin{bmatrix}\cos\beta_j(t-t_0) & \sin\beta_j(t-t_0)\\ -\sin\beta_j(t-t_0) & \cos\beta_j(t-t_0)\end{bmatrix}\ (j=1,2,\cdots,p).\end{aligned} \tag{1.40}$$

Proof: The proof can be referred to Luo (2011). ■

1.2 Operator exponentials

Definition 1.3 Consider a linear operator $\mathbf{A}: \mathscr{R}^n \to \mathscr{R}^n$ in linear operator space (i.e., $\mathbf{A} \in L(\mathscr{R}^n)$). The operator norm of $\mathbf{A}$ is defined by

$$\| \mathbf{A} \| = \max_{\|\mathbf{x}\| \le 1} \| \mathbf{A}(\mathbf{x}) \|, \tag{1.41}$$

where $\| \mathbf{x} \|$ is the Euclidean norm of $\mathbf{x} \in \mathscr{R}^n$. The operator norm has the following properties for $\mathbf{A}, \mathbf{B} \in L(\mathscr{R}^n)$:

(i) $\| \mathbf{A} \| \ge 0$ and $\| \mathbf{A} \| = 0$ if and only if $\mathbf{A} = \mathbf{0}$,
(ii) $\| k\mathbf{A} \| = k \| \mathbf{A} \|$ for $k \in \mathscr{R}$,
(iii) $\| \mathbf{A} + \mathbf{B} \| \le \| \mathbf{A} \| + \| \mathbf{B} \|$.

Definition 1.4 Consider a sequence of linear operator $\mathbf{A}_k \in L(\mathscr{R}^n)$ and the linear operator $\mathbf{A} \in L(\mathscr{R}^n)$. For any $\varepsilon > 0$, there exists an N such that for $k \ge N$,

$$\| \mathbf{A} - \mathbf{A}_k \| < \varepsilon. \tag{1.42}$$

Then, the sequence of linear operator $\mathbf{A}_k$ is called to be convergent to a linear operator $\mathbf{A}$ as $k \to \infty$, i.e.,

$$\lim_{k \to \infty} \mathbf{A}_k = \mathbf{A}. \tag{1.43}$$

Theorem 1.4 *For* $\mathbf{A}, \mathbf{B} \in L(\mathscr{R}^n)$ *and* $\mathbf{x} \in \mathscr{R}^n$,

(i) $\| \mathbf{A}\mathbf{x} \| \le \| \mathbf{A} \| \cdot \| \mathbf{x} \|$,
(ii) $\| \mathbf{A}\mathbf{B} \| \le \| \mathbf{A} \| \cdot \| \mathbf{B} \|$,
(iii) $\| \mathbf{A}^k \| \le \| \mathbf{A} \|^k$.

Proof: The proof can be referred to Luo (2011). ■

Theorem 1.5 *For* $\mathbf{A} \in L(\mathscr{R}^n)$ *and* $t \in \mathscr{R}$ *with* $t_0 > 0$, *a series* $\mathbf{I} + \sum_{k=1}^{\infty} \mathbf{A}^k t^k / k!$ *is absolutely and uniformly convergent for* $|t| \leq t_0$.

Proof: The proof can be referred to Luo (2011). ■

Definition 1.5 The exponential of a linear operator $\mathbf{A} \in L(\mathscr{R}^n)$ is defined by

$$e^{\mathbf{A}} = \mathbf{I} + \sum_{k=1}^{\infty} \frac{1}{k!} \mathbf{A}^k. \tag{1.44}$$

If $\mathbf{A}$ is an $n \times n$ matrix, for $t \in \mathscr{R}$,

$$e^{\mathbf{A}t} = \mathbf{I} + \sum_{k=1}^{\infty} \frac{1}{k!} \mathbf{A}^k t^k. \tag{1.45}$$

Theorem 1.6 *For* $\mathbf{A}, \mathbf{B}, \mathbf{P} \in L(\mathscr{R}^n)$,

(i) *if* $\mathbf{P}$ *is nonsingular, then* $e^{\mathbf{P}^{-1}\mathbf{A}\mathbf{P}} = \mathbf{P}^{-1} e^{\mathbf{A}} \mathbf{P}$,
(ii) *if* $\mathbf{AB} = \mathbf{BA}$, *then* $e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}} e^{\mathbf{B}}$,
(iii) $e^{-\mathbf{A}} = (e^{\mathbf{A}})^{-1}$,
(iv) *If* $\mathbf{A} = \lambda \mathbf{I}$, *then* $e^{\mathbf{A}} = e^{\lambda} \mathbf{I}$.

Proof: The proof can be referred to Luo (2011). ■

Lemma 1.1 *For an* $n \times n$ *matrix* $\mathbf{A}$,

$$\frac{d}{dt} e^{\mathbf{A}t} = \mathbf{A} e^{\mathbf{A}t}. \tag{1.46}$$

Theorem 1.7 *Consider a linear system* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ *in Eq.*(1.2) *with the initial condition of* $\mathbf{x}(t_0) = \mathbf{x}_0$. *The solution of the linear system is unique, i.e.,*

$$\mathbf{x} = e^{\mathbf{A}t} \mathbf{x}_0. \tag{1.47}$$

Proof: The proof can be referred to Luo (2011). ■

1.3 Linear systems with repeated eigenvalues

Definition 1.6 Consider a linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ in Eq.(1.2) with the initial condition of $\mathbf{x}(t_0) = \mathbf{x}_0$. If the $n \times n$ matrix $\mathbf{A}$ has an m-repeated real eigenvalue of λ

with ($m \le n$), then any nonzero eigenvector of

$$(\mathbf{A} - \lambda \mathbf{I})^m \mathbf{v} = \mathbf{0} \tag{1.48}$$

is called a generalized eigenvector of $\mathbf{A}$.

Definition 1.7 An $n \times n$ matrix $\mathbf{N}$ is called a nilpotent matrix of order k if $\mathbf{N}^{k-1} \ne \mathbf{0}$ and $\mathbf{N}^k = \mathbf{0}$.

Theorem 1.8 *Consider a linear system* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ *in Eq.*(1.2) *with the initial condition of* $\mathbf{x}(t_0) = \mathbf{x}_0$. *There is a repeated eigenvalue* λ_j *with m -times among the real eigenvalues* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *of the* $n \times n$ *matrix* $\mathbf{A}$. *If a set of generalized eigenvectors* $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ *forms a basis in* $\Omega \subseteq \mathscr{R}^n$. *The eigenvector matrix of* $\mathbf{P} = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n]$ *is invertible. For the repeated eigenvalue* λ_j *, the matrix* $\mathbf{A}$ *can be decomposed by*

$$\mathbf{A} = \mathbf{S} + \mathbf{N}, \tag{1.49}$$

where

$$\mathbf{P}^{-1}\mathbf{S}\mathbf{P} = diag[\lambda_j]_{n \times n}, \tag{1.50}$$

and the matrix $\mathbf{N} = \mathbf{A} - \mathbf{S}$ *is nilpotent of order* $m \le n$ *with* $\mathbf{SN} = \mathbf{NS}$,

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = diag[\lambda_1, \cdots, \lambda_{j-1}, \underbrace{\lambda_j, \cdots, \lambda_j}_{m}, \lambda_{j+m}, \cdots, \lambda_n]. \tag{1.51}$$

Thus, with an initial condition of $\mathbf{x}(t_0) = \mathbf{x}_0$ *, the solution of linear system in Eq.* (1.2) *is*

$$\mathbf{x}(t) = \mathbf{P}\mathbf{E}(t)\mathbf{P}^{-1}\left[\mathbf{I} + \sum_{k=1}^{m-1} \frac{\mathbf{N}^k t^k}{k!}\right]\mathbf{x}_0, \tag{1.52}$$

where

$$\mathbf{E}(t) = diag[e^{\lambda_1 t}, \cdots, e^{\lambda_{j-1} t}, \underbrace{e^{\lambda_j t}, \cdots, e^{\lambda_j t}}_{m}, e^{\lambda_{j+m} t}, \cdots, e^{\lambda_n t}]. \tag{1.53}$$

Proof: The proof can be referred to Luo (2011). ■

From a different point of view, consider a solution for repeated eigenvalues of a linear system as

$$\mathbf{x}^{(j)}(t) = \sum_{k=0}^{m-1} e^{\lambda_j t}(C_k^{(j)} \mathbf{v}_k^{(j)} t^k), \tag{1.54}$$

$$\dot{\mathbf{x}}^{(j)}(t) = \sum_{k=0}^{m-1} [\lambda_j e^{\lambda_j t}(C_k^{(j)} \mathbf{v}_k^{(j)} t^k) + e^{\lambda_j t}(k C_k^{(j)} \mathbf{v}_k^{(j)} t^{k-1})]. \tag{1.55}$$

Submission of Eqs.(1.54) and (1.55) into $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ in Eq.(1.2) gives

$$\sum_{k=0}^{m-1}[(\mathbf{A}-\lambda_j\mathbf{I})e^{\lambda_j t}(C_k^{(j)}\mathbf{v}_k^{(j)}t^k) - e^{\lambda_j t}(kC_k^{(j)}\mathbf{v}_k^{(j)}t^{k-1})] = \mathbf{0}, \tag{1.56}$$

and for $k = 0,1,2,\cdots,m-2$

$$(\mathbf{A}-\lambda_j\mathbf{I})C_k^{(j)}\mathbf{v}_k^{(j)} - (k+1)C_{k+1}^{(j)}\mathbf{v}_{k+1}^{(j)} = \mathbf{0}, \tag{1.57}$$

with $(\mathbf{A}-\lambda_j\mathbf{I})\mathbf{v}_{m-1}^{(j)} = 0$. Once eigenvectors are determined, the constants $C_k^{(j)}$ are obtained. On the other hand, let

$$C_k^{(j)} = (k+1)C_{k+1}^{(j)}. \tag{1.58}$$

Thus, one obtains

$$\begin{aligned} &(\mathbf{A}-\lambda_j\mathbf{I})\mathbf{v}_{m-1}^{(j)} = \mathbf{0}, \\ &(\mathbf{A}-\lambda_j\mathbf{I})\mathbf{v}_k^{(j)} = \mathbf{v}_{k+1}^{(j)} \quad (k=0,1,2,\cdots,m-2). \end{aligned} \tag{1.59}$$

Deformation of Eq.(1.59) gives

$$\begin{aligned} &\mathbf{A}(\underbrace{\mathbf{0},\cdots,\mathbf{0}}_{j-1},\underbrace{\mathbf{v}_0^{(j)},\mathbf{v}_1^{(j)},\cdots,\mathbf{v}_{m-1}^{(j)}}_{m},\underbrace{\mathbf{0},\cdots,\mathbf{0}}_{n-m-j+1}) \\ &= (\underbrace{\mathbf{0},\cdots,\mathbf{0}}_{j-1},\underbrace{\mathbf{v}_0^{(j)},\mathbf{v}_1^{(j)},\cdots,\mathbf{v}_{m-1}^{(j)}}_{m},\underbrace{\mathbf{0},\cdots,\mathbf{0}}_{n-m-j+1})\mathbf{B}, \end{aligned} \tag{1.60}$$

where the Jordan matrix is

$$\mathbf{B}^{(j)} = \begin{bmatrix} \lambda_j & 0 & 0 & \cdots & 0 & 0 \\ 1 & \lambda_j & 0 & \cdots & 0 & 0 \\ 0 & 1 & \lambda_j & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_j & 0 \\ 0 & 0 & 0 & \cdots & 1 & \lambda_j \end{bmatrix}_{m\times m}, \tag{1.61}$$

$$\mathbf{B} = diag[\mathbf{0}_{(j-1)\times(j-1)},\mathbf{B}^{(j)},\mathbf{0}_{(n-m-j+1)\times(n-m-j+1)}]. \tag{1.62}$$

Thus

$$\begin{aligned} \mathbf{A}\mathbf{P} &= \mathbf{P}diag[\lambda_1,\cdots,\lambda_{j-1},\mathbf{B}^{(j)}|_{m\times m},\lambda_{j+m},\cdots,\lambda_n], \\ \mathbf{P}^{-1}\mathbf{A}\mathbf{P} &= diag[\lambda_1,\cdots,\lambda_{j-1},\mathbf{B}^{(j)}|_{m\times m},\lambda_{j+m},\cdots,\lambda_n], \end{aligned} \tag{1.63}$$

where

$$\begin{aligned} \mathbf{P} &= (\mathbf{v}^{(1)},\cdots,\mathbf{v}^{(j-1)},\mathbf{v}_0^{(j)},\mathbf{v}_1^{(j)},\cdots,\mathbf{v}_{m-1}^{(j)},\mathbf{v}^{(j+m)},\cdots,\mathbf{v}^{(n)}) \\ &= (\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n). \end{aligned} \tag{1.64}$$