

Graduate Texts in Mathematics

Melvyn B. Nathanson

Additive Number Theory

The Classical Bases

加性数论 经典基

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continued after index

To Marjorie

Preface

[Hilbert's] style has not the terseness of many of our modern authors in mathematics, which is based on the assumption that printer's labor and paper are costly but the reader's effort and time are not.

H. Weyl [143]

The purpose of this book is to describe the classical problems in additive number theory and to introduce the circle method and the sieve method, which are the basic analytical and combinatorial tools used to attack these problems. This book is intended for students who want to learn additive number theory, not for experts who already know it. For this reason, proofs include many “unnecessary” and “obvious” steps; this is by design.

The archetypical theorem in additive number theory is due to Lagrange: Every nonnegative integer is the sum of four squares. In general, the set A of nonnegative integers is called an *additive basis of order h* if every nonnegative integer can be written as the sum of h not necessarily distinct elements of A . Lagrange's theorem is the statement that the squares are a basis of order four. The set A is called a *basis of finite order* if A is a basis of order h for some positive integer h . Additive number theory is in large part the study of bases of finite order. The classical bases are the squares, cubes, and higher powers; the polygonal numbers; and the prime numbers. The classical questions associated with these bases are Waring's problem and the Goldbach conjecture.

Waring's problem is to prove that, for every $k \geq 2$, the nonnegative k th powers form a basis of finite order. We prove several results connected with Waring's problem, including Hilbert's theorem that every nonnegative integer is the sum of

a bounded number of k th powers, and the Hardy–Littlewood asymptotic formula for the number of representations of an integer as the sum of s positive k th powers.

Goldbach conjectured that every even positive integer is the sum of at most two prime numbers. We prove three of the most important results on the Goldbach conjecture: Shnirel'man's theorem that the primes are a basis of finite order, Vinogradov's theorem that every sufficiently large odd number is the sum of three primes, and Chen's theorem that every sufficiently large even integer is the sum of a prime and a number that is a product of at most two primes.

Many unsolved problems remain. The Goldbach conjecture has not been proved. There is no proof of the conjecture that every sufficiently large integer is the sum of four nonnegative cubes, nor can we obtain a good upper bound for the least number s of nonnegative k th powers such that every sufficiently large integer is the sum of s k th powers. It is possible that neither the circle method nor the sieve method is powerful enough to solve these problems and that completely new mathematical ideas will be necessary, but certainly there will be no progress without an understanding of the classical methods.

The prerequisites for this book are undergraduate courses in number theory and real analysis. The appendix contains some theorems about arithmetic functions that are not necessarily part of a first course in elementary number theory. In a few places (for example, Linnik's theorem on sums of seven cubes, Vinogradov's theorem on sums of three primes, and Chen's theorem on sums of a prime and an almost prime), we use results about the distribution of prime numbers in arithmetic progressions. These results can be found in Davenport's *Multiplicative Number Theory* [19].

Additive number theory is a deep and beautiful part of mathematics, but for too long it has been obscure and mysterious, the domain of a small number of specialists, who have often been specialists only in their own small part of additive number theory. This is the first of several books on additive number theory. I hope that these books will demonstrate the richness and coherence of the subject and that they will encourage renewed interest in the field.

I have taught additive number theory at Southern Illinois University at Carbondale, Rutgers University—New Brunswick, and the City University of New York Graduate Center, and I am grateful to the students and colleagues who participated in my graduate courses and seminars. I also wish to thank Henryk Iwaniec, from whom I learned the linear sieve and the proof of Chen's theorem.

This work was supported in part by grants from the PSC-CUNY Research Award Program and the National Security Agency Mathematical Sciences Program.

I would very much like to receive comments or corrections from readers of this book. My e-mail addresses are nathansn@alpha.lehman.cuny.edu and nathanson@worldnet.att.net. A list of errata will be available on my homepage at <http://www.lehman.cuny.edu> or <http://math.lehman.cuny.edu/nathanson>.

Melvyn B. Nathanson
Maplewood, New Jersey
May 1, 1996

Notation and conventions

Theorems, lemmas, and corollaries are numbered consecutively in each chapter and in the Appendix. For example, Lemma 2.1 is the first lemma in Chapter 2 and Theorem A.2 is the second theorem in the Appendix.

The lowercase letter p denotes a prime number.

We adhere to the usual convention that the *empty sum* (the sum containing no terms) is equal to zero and the *empty product* is equal to one.

Let f be any real or complex-valued function, and let g be a positive function. The functions f and g can be functions of a real variable x or arithmetic functions defined only on the positive integers. We write

$$f = O(g)$$

or

$$f \ll g$$

or

$$g \gg f$$

if there exists a constant $c > 0$ such that

$$|f(x)| \leq cg(x)$$

for all x in the domain of f . The constant c is called the *implied constant*. We write

$$f \ll_{a,b,\dots} g$$

if there exists a constant $c > 0$ that depends on a, b, \dots such that

$$|f(x)| \leq cg(x)$$

for all x in the domain of f . We write

$$f = o(g)$$

if

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0.$$

The function f is asymptotic to g , denoted

$$f \sim g,$$

if

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1.$$

The real-valued function f is *increasing* on the interval I if $f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in I$ with $x_1 < x_2$. Similarly, the real-valued function f is *decreasing* on the interval I if $f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in I$ with $x_1 < x_2$. The function f is *monotonic* on the interval I if it is either increasing on I or decreasing on I .

We use the following notation for exponential functions:

$$\exp(x) = e^x$$

and

$$e(x) = \exp(2\pi i x) = e^{2\pi i x}.$$

The following notation is standard:

\mathbf{Z}	the integers $0, \pm 1, \pm 2, \dots$
\mathbf{R}	the real numbers
\mathbf{R}^n	n -dimensional Euclidean space
\mathbf{Z}^n	the integer lattice in \mathbf{R}^n
\mathbf{C}	the complex numbers
$ z $	the absolute value of the complex number z
$\Re z$	the real part of the complex number z
$\Im z$	the imaginary part of the complex number z
$[x]$	the integer part of the real number x , that is, the integer uniquely determined by the inequality $[x] \leq x < [x] + 1$.
$\{x\}$	the fractional part of the real number x , that is, $\{x\} = x - [x] \in [0, 1)$.
$\ x\ $	the distance from the real number x to the nearest integer, that is, $\ x\ = \min\{ x - n : n \in \mathbf{Z}\} = \min(\{x\}, 1 - \{x\}) \in [0, 1/2]$.
(a_1, \dots, a_n)	the greatest common divisor of the integers a_1, \dots, a_n
$[a_1, \dots, a_n]$	the least common multiple of the integers a_1, \dots, a_n
$ X $	the cardinality of the set X
hA	the h -fold sumset, consisting of all sums of h elements of A

Contents

Preface	vii
Notation and conventions	xiii
I Waring's problem	
1 Sums of polygons	3
1.1 Polygonal numbers	4
1.2 Lagrange's theorem	5
1.3 Quadratic forms	7
1.4 Ternary quadratic forms	12
1.5 Sums of three squares	17
1.6 Thin sets of squares	24
1.7 The polygonal number theorem	27
1.8 Notes	33
1.9 Exercises	34
2 Waring's problem for cubes	37
2.1 Sums of cubes	37
2.2 The Wieferich–Kempner theorem	38
2.3 Linnik's theorem	44
2.4 Sums of two cubes	49
2.5 Notes	71
2.6 Exercises	72
3 The Hilbert–Waring theorem	75
3.1 Polynomial identities and a conjecture of Hurwitz	75
3.2 Hermite polynomials and Hilbert's identity	77
3.3 A proof by induction	86
3.4 Notes	94

3.5	Exercises	94
4	Weyl's inequality	97
4.1	Tools	97
4.2	Difference operators	99
4.3	Easier Waring's problem	102
4.4	Fractional parts	103
4.5	Weyl's inequality and Hua's lemma	111
4.6	Notes	118
4.7	Exercises	118
5	The Hardy–Littlewood asymptotic formula	121
5.1	The circle method	121
5.2	Waring's problem for $k = 1$	124
5.3	The Hardy–Littlewood decomposition	125
5.4	The minor arcs	127
5.5	The major arcs	129
5.6	The singular integral	133
5.7	The singular series	137
5.8	Conclusion	146
5.9	Notes	147
5.10	Exercises	147

II The Goldbach conjecture

6	Elementary estimates for primes	151
6.1	Euclid's theorem	151
6.2	Chebyshev's theorem	153
6.3	Mertens's theorems	158
6.4	Brun's method and twin primes	167
6.5	Notes	173
6.6	Exercises	174
7	The Shnirel'man–Goldbach theorem	177
7.1	The Goldbach conjecture	177
7.2	The Selberg sieve	178
7.3	Applications of the sieve	186
7.4	Shnirel'man density	191
7.5	The Shnirel'man–Goldbach theorem	195
7.6	Romanov's theorem	199
7.7	Covering congruences	204
7.8	Notes	208
7.9	Exercises	208

8	Sums of three primes	211
8.1	Vinogradov's theorem	211
8.2	The singular series	212
8.3	Decomposition into major and minor arcs	213
8.4	The integral over the major arcs	215
8.5	An exponential sum over primes	220
8.6	Proof of the asymptotic formula	227
8.7	Notes	230
8.8	Exercise	230
9	The linear sieve	231
9.1	A general sieve	231
9.2	Construction of a combinatorial sieve	238
9.3	Approximations	244
9.4	The Jurkat–Richert theorem	251
9.5	Differential-difference equations	259
9.6	Notes	267
9.7	Exercises	267
10	Chen's theorem	271
10.1	Primes and almost primes	271
10.2	Weights	272
10.3	Prolegomena to sieving	275
10.4	A lower bound for $S(A, \mathcal{P}, z)$	279
10.5	An upper bound for $S(A_q, \mathcal{P}, z)$	281
10.6	An upper bound for $S(B, \mathcal{P}, y)$	286
10.7	A bilinear form inequality	292
10.8	Conclusion	297
10.9	Notes	298

III Appendix

Arithmetic functions	301
A.1 The ring of arithmetic functions	301
A.2 Sums and integrals	303
A.3 Multiplicative functions	308
A.4 The divisor function	310
A.5 The Euler φ -function	314
A.6 The Möbius function	317
A.7 Ramanujan sums	320
A.8 Infinite products	323
A.9 Notes	327
A.10 Exercises	327

Bibliography	331
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Index	341
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Part I

Waring's problem

