钱学森
力学手稿

5

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西安交通大学出版社
XI'AN JIAOTONG UNIVERSITY PRESS

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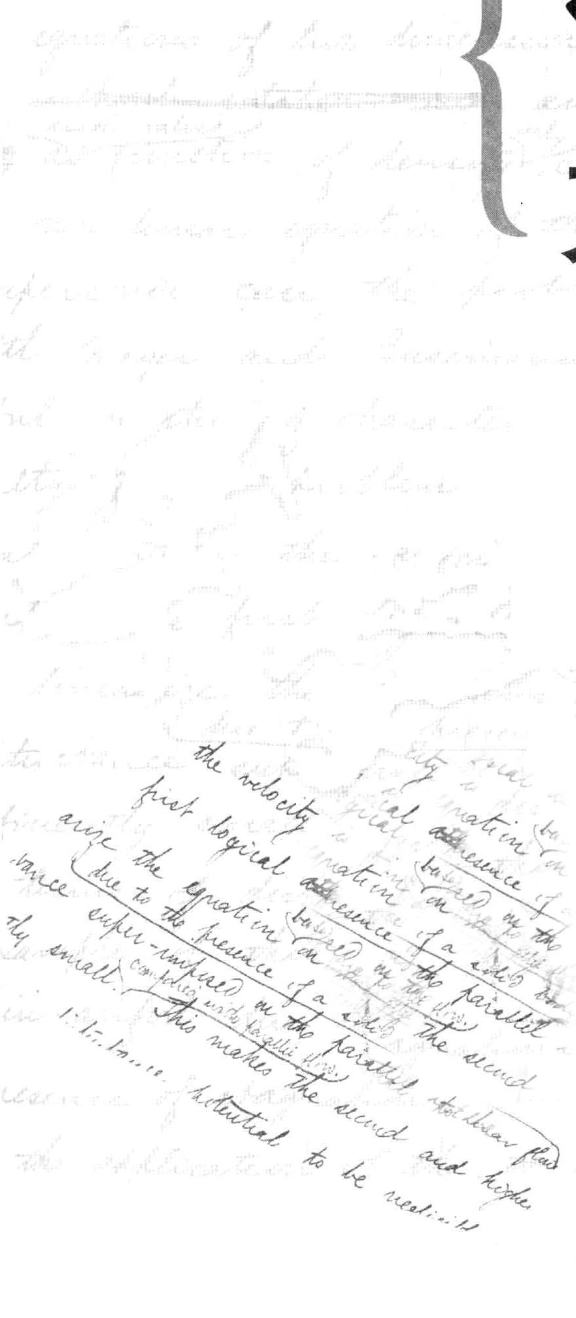

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出版前言

2011 年 12 月 11 日是西安交通大学杰出校友钱学森先生的百年诞辰。为缅怀钱学森学长，学习他的科学思想和卓越风范，展示其丰功伟绩和人格魅力，西安交通大学举办了“纪念钱学森诞辰 100 周年”系列活动：作为制片方之一，参与西部电影集团摄制传记故事片《钱学森》；与中央电视台合作，出品纪录片《实验班的故事——沿着钱学森走过的路》；扩建钱学森生平业绩展馆，向校内外开放；举办钱学森科学与教育思想研讨会；出版发行《钱学森力学手稿》、《钱学森年谱（初编）》、《钱学森第六次产业革命思想探微丛书》等。

钱学森先生在美国深造和工作期间留下大量珍贵手稿，这些手稿真实展示了钱学森先生博大精深的学识、开拓求实的精神和严谨奋进的作风，是钱老勇攀科学高峰和严谨治学的集中体现。这里，我们将部分原稿整理汇集成册，出版《钱学森力学手稿》，作为钱老百年诞辰的献礼。

《钱学森力学手稿》共 10 卷，包含两部分内容。第一部分是草稿，包括扁壳、球壳和圆柱壳屈曲分析的公式推导和数值演算。在研究圆柱壳轴压屈曲问题时，为了求得圆柱壳体的临界压力，在有关的五百多页草稿中，对多达二十多种可能的屈曲模

态逐一进行公式推演和数值计算，最终才找到满意的并在论文中采用的屈曲模态。仔细观察草稿中的数据列表，每个数字有效位数都长达八位，在手摇机械式计算机作为主要计算工具的年代，这串串数字凝聚着多少现今难以想象的艰辛劳动。

第二部分是手稿，以航空航天工程为核心，涵盖空气动力学、固体力学、火箭技术、工程控制论和物理力学等领域的部分学术论文手稿、打印稿和讲义。

《钱学森力学手稿》是在西安交通大学校领导的大力支持下，由西安交通大学航天航空学院沈亚鹏教授整理完成。图书出版过程中得到了西安交通大学党委宣传部、校友关系发展部、图书馆、航天航空学院等的积极协助，在此深表感谢。

Preliminary Calculation of

Circular Cylinder (Ⅳ)

PRINTED IN U. S. A.

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3-27-37

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ALLIS-CHALMERS MANUFACTURING CO.

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OF AXIAL BLOWER
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TUNNEL

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V VII

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Mr. C.C [illegible] Assigt.

We have the equation

$$\frac{\partial^2 v}{\partial x^2} + 2\frac{\partial^2 v}{\partial y^2} = -\left\{\frac{\partial w}{\partial y}\left(\frac{\partial^2 w}{\partial x^2}+2\frac{\partial^2 w}{\partial y^2}\right)+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y}\right\}$$

$$\frac{w}{R} = \frac{1}{2}\left(\frac{a}{R}\right)^2\left[1-\left(\frac{x}{a}\right)^2 - \frac{f}{2}\cos\frac{\pi x}{2a}\left(1+\cos\frac{\pi y}{b}\right)\right]$$

$$\frac{w_0}{R} = \frac{1}{2}\left(\frac{a}{R}\right)^2\left[1-\left(\frac{x}{a}\right)^2\right]$$

$$\frac{\partial w}{\partial x} = \left(\frac{a}{R}\right)\left[-\left(\frac{x}{a}\right) + \frac{f}{8}\pi \sin\frac{\pi x}{2a}\left(1+\cos\frac{\pi y}{b}\right)\right]$$

$$\frac{\partial w}{\partial y} = \left(\frac{a}{R}\right)\left[\quad + \frac{f}{4}\pi\lambda \cos\frac{\pi x}{2a}\sin\frac{\pi y}{b}\right]$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{R}\left[-1 + \frac{f}{16}\pi^2 \cos\frac{\pi x}{2a}\left(1+\cos\frac{\pi y}{b}\right)\right]$$

$$\frac{\partial^2 w}{\partial x\partial y} = \frac{1}{R}\left[\quad - \frac{f}{8}\pi^2\lambda \sin\frac{\pi x}{2a}\sin\frac{\pi y}{b}\right]$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{1}{R}\left[\quad + \frac{f}{4}\pi^2\lambda^2 \cos\frac{\pi x}{2a}\cos\frac{\pi y}{b}\right]$$

$$R\left[\frac{\partial^2 v}{\partial x^2}+2\frac{\partial^2 v}{\partial y^2}\right] = -\left(\frac{a}{R}\right)\left(\frac{f}{4}\pi\lambda\right)\Big[\cos\frac{\pi x}{2a}\sin\frac{\pi y}{b}\Big\{-1 + \frac{f\pi^2}{16}\cos\frac{\pi x}{2a}\left(1+\cos\frac{\pi y}{b}\right)$$
$$+ \frac{f\pi^2}{2}\lambda^2\cos\frac{\pi x}{2a}\cos\frac{\pi y}{b}\Big\}$$
$$- \frac{\pi}{2}\sin\frac{\pi x}{2a}\sin\frac{\pi y}{b}\Big\{-\left(\frac{x}{a}\right) + \frac{f\pi}{8}\sin\frac{\pi x}{2a}\left(1+\cos\frac{\pi y}{b}\right)\Big\}\Big]$$

$$= -\left(\frac{a}{R}\right)\frac{f}{4}\pi\lambda\Big[-\cos\frac{\pi x}{2a}\sin\frac{\pi y}{b} + \frac{f\pi^2}{64}\left(1+\cos\frac{\pi x}{a}\right)\left(2\sin\frac{\pi y}{b}+\sin\frac{2\pi y}{b}\right)$$
$$+\left(\frac{\pi x}{2a}\right)\sin\frac{\pi x}{2a}\sin\frac{\pi y}{b} + \frac{f\pi^2}{64}\left(-1+\cos\frac{\pi x}{a}\right)\left(2\sin\frac{\pi y}{b}+\sin\frac{2\pi y}{b}\right)$$
$$+\frac{f\pi^2}{8}\lambda^2\left(1+\cos\frac{\pi x}{a}\right)\sin\frac{2\pi y}{b}\Big]$$

$$R\left[\frac{\partial^2 v}{\partial x^2}+2\frac{\partial^2 v}{\partial y^2}\right]=-\left(\frac{a}{R}\right)\frac{f\pi\lambda}{4}\left[\left(\frac{\pi x}{2a}\right)\sin\frac{\pi x}{2a}\sin\frac{\pi y}{b}-\cos\frac{\pi x}{2a}\sin\frac{\pi y}{b}+\frac{f\pi^2}{16}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}\right.$$

$$\left.+\frac{f\pi^2}{32}\cos\frac{\pi x}{a}\sin\frac{2\pi y}{b}+\frac{f\pi^2}{8}\lambda^2\sin\frac{2\pi y}{b}+\frac{f\pi^2}{8}\lambda^2\cos\frac{\pi x}{a}\sin\frac{2\pi y}{b}\right]$$

$$=-\left(\frac{a}{R}\right)\frac{f\pi\lambda}{4}\left[\left(\frac{\pi x}{2a}\right)\sin\frac{\pi x}{2a}\sin\frac{\pi y}{b}-\cos\frac{\pi x}{2a}\sin\frac{\pi y}{b}+\frac{f\pi^2}{8}\lambda^2\sin\frac{2\pi y}{b}+\frac{f\pi^2}{16}(1+2\lambda^2)\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}\right.$$

$$\left.+\frac{f\pi^2}{32}\cos\frac{\pi x}{a}\sin\frac{2\pi y}{b}\right]$$

Investigate the particular solution of the following eqn.

$$\frac{\partial^2 v}{\partial x^2}+2\frac{\partial^2 v}{\partial y^2}=\left(\frac{\pi x}{2a}\right)\sin\frac{\pi x}{2a}\sin\frac{\pi y}{b}$$

Put $\quad v = X\sin\frac{\pi y}{b}$

$$\frac{d^2X}{dx^2}-2\left(\frac{\pi}{b}\right)^2X=\left(\frac{\pi x}{2a}\right)\sin\frac{\pi x}{2a}$$

Let $\quad X = A\cos\frac{\pi x}{2a}+B\left(\frac{\pi x}{2a}\right)\sin\frac{\pi x}{2a}$

$$\frac{d^2X}{dx^2}=\left(\frac{\pi}{a}\right)^2\left\{\frac{(2B-A)}{4}\cos\frac{\pi x}{2a}-\frac{B}{4}\left(\frac{\pi x}{2a}\right)\sin\frac{\pi x}{2a}\right\}$$

$$-2\left(\frac{\pi}{b}\right)^2X=\left(\frac{\pi}{a}\right)^2\left\{-2\lambda^2A\cos\frac{\pi x}{2a}-2\lambda^2B\left(\frac{\pi x}{2a}\right)\sin\frac{\pi x}{2a}\right\}$$

$$\therefore\ \left(\frac{\pi}{a}\right)^2\left\{\frac{2B-A}{4}-2\lambda^2A\right\}=0,\qquad \left(\frac{\pi}{a}\right)^2\left\{-\frac{B}{4}-2\lambda^2B\right\}=1$$

$$B=-\frac{1}{\left(\frac{\pi}{a}\right)^2}\frac{4}{(1+8\lambda^2)}$$

$$A=-\frac{1}{\left(\frac{\pi}{a}\right)^2}\frac{8}{(1+8\lambda^2)^2}$$

$$\frac{v}{R} = \left(\frac{a}{R}\right)^3 \frac{f}{4}\frac{1}{\pi}\lambda\Big[\frac{8}{(1+8\lambda^2)^2}\cos\frac{\pi x}{2a}\sin\frac{\pi y}{b} + \frac{4}{1+8\lambda^2}\left(\frac{\pi x}{2a}\right)\sin\frac{\pi x}{2a}\sin\frac{\pi y}{b} - \frac{4}{(1+8\lambda^2)}\cos\frac{\pi x}{2a}\sin\frac{\pi y}{b}$$

$$+ \frac{f\pi^2}{8}\lambda^2\frac{1}{8\lambda^2}\sin\frac{2\pi y}{b} + \frac{f\pi^2}{16}\frac{1+2\lambda^2}{1+2\lambda^2}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} + \frac{f\pi^2}{32}\frac{1}{1+8\lambda^2}\cos\frac{\pi x}{a}\sin\frac{2\pi y}{b}$$

$$+ a_0\left(\frac{\pi y}{b}\right) + a_2\cosh\sqrt{2}\frac{\pi\lambda x}{a}\sin\frac{\pi y}{b}\Big]$$

$$= \left(\frac{a}{R}\right)^3\frac{f}{4}\frac{1}{\pi}\lambda\Big[\frac{4(1-8\lambda^2)}{(1+8\lambda^2)^2}\cos\frac{\pi x}{2a}\sin\frac{\pi y}{b} + \frac{4}{1+8\lambda^2}\left(\frac{\pi x}{2a}\right)\sin\frac{\pi x}{2a}\sin\frac{\pi y}{b} + \frac{f\pi^2}{64}\sin\frac{2\pi y}{b}$$

$$+ \frac{f\pi^2}{16}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} + \frac{f\pi^2}{32}\frac{1}{(1+8\lambda^2)}\cos\frac{\pi x}{a}\sin\frac{2\pi y}{b} + a_0\left(\frac{\pi y}{b}\right) + a_2\cosh\frac{\sqrt{2}\pi\lambda x}{a}\sin\frac{\pi y}{b}\Big]$$

$$\frac{\partial v}{\partial x} = \left(\frac{a}{R}\right)^2\frac{f}{4}\lambda\Big[-\frac{2(1-8\lambda^2)}{(1+8\lambda^2)^2}\sin\frac{\pi x}{2a}\sin\frac{\pi y}{b} + \frac{2}{1+8\lambda^2}\sin\frac{\pi x}{2a}\sin\frac{\pi y}{b} + \frac{2}{(1+8\lambda^2)}\left(\frac{\pi x}{2a}\right)\cos\frac{\pi x}{2a}\sin\frac{\pi y}{b}$$

$$- \frac{f\pi^2}{16}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} - \frac{f\pi^2}{32}\frac{1}{(1+8\lambda^2)}\sin\frac{\pi x}{a}\sin\frac{2\pi y}{b} + a_2\sqrt{2}\lambda\sinh\frac{\sqrt{2}\pi\lambda x}{a}\sin\frac{\pi y}{b}\Big]$$

$$\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} = \left(\frac{a}{R}\right)^2\frac{f}{4}\lambda\Big[-2\left(\frac{\pi x}{2a}\right)\cos\frac{\pi x}{2a}\sin\frac{\pi y}{b} + \frac{f\pi^2}{32}\sin\frac{\pi x}{a}\left(2\sin\frac{\pi y}{b} + \sin\frac{2\pi y}{b}\right)\Big]$$

4ff

$$\left(\frac{a}{R}\right)^2 \frac{f}{4}\lambda^2 a_0 = -\frac{\sigma}{E} - \left(\frac{a}{R}\right)^2 \frac{f}{4}\lambda^2 \left[-\frac{16}{\pi}\frac{1}{(1+8\lambda^2)^2} + \frac{f\pi^2}{32} + \frac{16}{\pi}\frac{1}{(1+8\lambda^2)^2} \right]$$

$$= \underline{-\frac{\sigma}{E} - \left(\frac{a}{R}\right)^2 \frac{f}{4}\lambda^2 \left(\frac{f\pi^2}{32}\right)}$$

$$\boxed{\frac{\Delta p}{abtE} = -\left(\frac{\sigma}{E}\right)^2 - \left(\frac{\sigma}{E}\right)\left(\frac{a}{R}\right)^2 \left(\frac{f}{4}\right)^2 \left(\frac{\pi^2\lambda^2}{8}\right)}$$

$$\varepsilon_1 = \frac{1}{2}\left\{\left(\frac{\partial w}{\partial x}\right)^2 - \left(\frac{\partial w_0}{\partial x}\right)^2\right\} = \left(\frac{a}{R}\right)^2 \frac{f}{4}\left\{ -\left(\frac{\pi x}{2a}\right)\sin\frac{\pi x}{2a}\left(1+\cos\frac{\pi y}{b}\right) + \frac{f\pi^2}{128}\left(1-\cos\frac{\pi x}{a}\right)\left(3+4\cos\frac{\pi y}{b}+\cos\frac{2\pi y}{b}\right)\right\}$$

$$= \left(\frac{a}{R}\right)^2 \frac{f}{4}\left\{ -\left(\frac{\pi x}{2a}\right)\sin\frac{\pi x}{2a} + \frac{3f\pi^2}{128}\left(1-\cos\frac{\pi x}{a}\right)\right.$$

$$+ \left\{ -\left(\frac{\pi x}{2a}\right)\sin\frac{\pi x}{2a} + \frac{f\pi^2}{32}\left(1-\cos\frac{\pi x}{a}\right)\cos\frac{\pi y}{b}\right.$$

$$\left. + \frac{f\pi^2}{128}\left(1-\cos\frac{\pi x}{a}\right)\cos\frac{2\pi y}{b}\right\}$$

$$\frac{1}{2ab}\int_0^a\int_0^b \varepsilon_1^2\,dx\,dy = \left(\frac{a}{R}\right)^4\left(\frac{f}{4}\right)^2\left\{ \frac{3}{4}\left[\left(\frac{3}{128}\right)^2 + \frac{1}{2}\left(\frac{1}{32}\right)^2 + \frac{1}{2}\left(\frac{1}{128}\right)^2\right](f\pi^2)^2 + \frac{3}{4}\int_0^1\left(\frac{\pi x}{2a}\right)^2\sin^2\frac{\pi x}{2a}\,d\left(\frac{x}{a}\right)\right.$$

$$\left. - \frac{5}{128}(f\pi^2)\int_0^1\left(\frac{\pi x}{2a}\right)\sin\frac{\pi x}{2a}\left(1-\cos\frac{\pi x}{a}\right)d\left(\frac{x}{a}\right)\right\}$$

$$= \underline{\left(\frac{a}{R}\right)^4\left(\frac{f}{4}\right)^2\left[\frac{105}{8(128)^2}(f\pi^2)^2 - \frac{5}{128}\left(\frac{2}{\pi}+\frac{10}{9\pi}\right)(f\pi^2) + \left(\frac{\pi^2}{32}+\frac{3}{16}\right)\right]}$$

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$$\frac{\partial v}{\partial y} = \left(\frac{g}{R}\right)^2 \frac{f}{4}\lambda^2 \left[\frac{4(1-\delta\lambda^2)}{(1+\delta\lambda^2)^2} \cos\frac{\pi x}{2a} \cos\frac{\pi y}{b} + \frac{4}{(1+\delta\lambda^2)} \left(\frac{\pi x}{2a}\right) \sin\frac{\pi x}{2a} \cos\frac{\pi y}{b} + \frac{f\pi^2}{32} \cos\frac{2\pi y}{b} \right.$$

$$\left. + \frac{f\pi^2}{16} \cos\frac{\pi x}{a} \cos\frac{\pi y}{b} + \frac{f\pi^2}{16} \frac{1}{(1+\delta\lambda^2)} \cos\frac{\pi x}{a} \cos\frac{2\pi y}{b} - \frac{f\pi^2}{32} + a_2 \cosh\frac{\sqrt{2}\,\pi\lambda x}{a} \cos\frac{\pi y}{b} \right] - \frac{\sigma}{E}$$

$$\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{g}{R}\right)^2 \frac{f}{4}\lambda^2 \left[\frac{f\pi^2}{32} \left(1 + \cos\frac{\pi x}{a}\right)\left(1 - \cos\frac{2\pi y}{b}\right) \right] = \left(\frac{g}{R}\right)^2 \frac{f}{4}\lambda^2 \left[\frac{f\pi^2}{32} + \frac{f\pi^2}{32} \cos\frac{\pi x}{a} - \frac{f\pi^2}{32} \cos\frac{2\pi y}{b} \right.$$

$$\left. - \frac{f\pi^2}{32} \cos\frac{\pi x}{a} \cos\frac{2\pi y}{b} \right]$$

$$\varepsilon_2 = \left(\frac{g}{R}\right)^2 \frac{f}{4}\lambda^2 \left[\frac{4(1-\delta\lambda^2)}{(1+\delta\lambda^2)^2} \cos\frac{\pi x}{2a} \cos\frac{\pi y}{b} + \frac{4}{(1+\delta\lambda^2)} \left(\frac{\pi x}{2a}\right) \sin\frac{\pi x}{2a} \cos\frac{\pi y}{b} + \frac{f\pi^2}{16} \cos\frac{\pi x}{a} \cos\frac{\pi y}{b} + \frac{f\pi^2}{32} \cos\frac{\pi x}{a} \right.$$

$$\left. + \frac{f\pi^2}{32} \left(\frac{1-\delta\lambda^2}{1+\delta\lambda^2}\right) \cos\frac{\pi x}{a} \cos\frac{2\pi y}{b} + a_2 \cosh\frac{\sqrt{2}\,\pi\lambda x}{a} \cos\frac{\pi y}{b} \right] - \frac{\sigma}{E}$$

$$= \left(\frac{g}{R}\right)^2 \frac{f}{4}\lambda^2 \left[\frac{f\pi^2}{32} \cos\frac{\pi x}{a} \right.$$

$$+ \left\{ \frac{4(1-\delta\lambda^2)}{(1+\delta\lambda^2)^2} \cos\frac{\pi x}{2a} + \frac{f\pi^2}{16} \cos\frac{\pi x}{a} + \frac{4}{1+\delta\lambda^2} \left(\frac{\pi x}{2a}\right) \sin\frac{\pi x}{2a} + a_2 \cosh\frac{\sqrt{2}\,\pi\lambda x}{a} \right\} \cos\frac{\pi y}{b}$$

$$\left. + \left\{ \frac{f\pi^2}{32} \left(\frac{1-\delta\lambda^2}{1+\delta\lambda^2}\right) \cos\frac{\pi x}{a} \right\} \cos\frac{2\pi y}{b} \right] - \frac{\sigma}{E}$$

$$\frac{1}{2ab}\int_0^a\int_0^b \varepsilon_2^2\,dx\,dy = \left(\frac{a}{R}\right)^4\left(\frac{f}{4}\right)^2\lambda^4\Bigg[\frac{1}{4}\left(\frac{f\pi^2}{32}\right)^2 + \frac{1}{8}\left\{\frac{4(1-\delta\lambda^2)}{(1+\delta\lambda^2)^2}\right\}^2 + \frac{1}{8}\left(\frac{f\pi^2}{16}\right)^2$$

$$+\ \frac{2(1-\delta\lambda^2)}{(1+\delta\lambda^2)^2}\int_0^1 \cos\frac{\pi x}{2a}\left\{\frac{4}{1+\delta\lambda^2}\left(\frac{\pi x}{2a}\right)\sin\frac{\pi x}{2a} + a_2\cosh\frac{\sqrt{2}\pi\lambda x}{a}\right\}d\left(\frac{x}{a}\right)$$

$$+\ \frac{f\pi^2}{32}\int_0^1 \cos\frac{\pi x}{a}\left\{\frac{4}{1+\delta\lambda^2}\left(\frac{\pi x}{2a}\right)\sin\frac{\pi x}{2a} + a_2\cosh\frac{\sqrt{2}\pi\lambda x}{a}\right\}d\left(\frac{x}{a}\right)$$

$$+\ \frac{1}{4}\int_0^1\left\{\frac{4}{1+\delta\lambda^2}\left(\frac{\pi x}{2a}\right)\sin\frac{\pi x}{2a} + a_2\cosh\frac{\sqrt{2}\pi\lambda x}{a}\right\}^2 d\left(\frac{x}{a}\right)$$

$$+\ \frac{1}{8}\left\{\frac{f\pi^2}{32}\,\frac{1-\delta\lambda^2}{1+\delta\lambda^2}\right\}^2\Bigg] + \frac{1}{2}\left(\frac{\sigma}{E}\right)^2$$

$$\frac{1}{4ab}\int_0^a\int_0^b \gamma^2\,dx\,dy = \left(\frac{a}{R}\right)^2\left(\frac{f}{4}\right)^2\lambda^2\Bigg[\frac{1}{4}\left\{\frac{16\lambda^2}{(1+\delta\lambda^2)^2}\right\}^2 + \frac{16\lambda^2}{(1+\delta\lambda^2)^2}\int_0^1 \sin\frac{\pi x}{2a}\left\{-\frac{\delta\lambda^2}{1+\delta\lambda^2}\left(\frac{\pi x}{2a}\right)\cos\frac{\pi x}{2a} + \frac{a_2\lambda}{\sqrt{2}}\sinh\frac{\sqrt{2}\pi\lambda x}{a}\right\}d\left(\frac{x}{a}\right)$$

$$+\ \frac{1}{2}\int_0^1\left\{-\frac{\delta\lambda^2}{(1+\delta\lambda^2)}\left(\frac{\pi x}{2a}\right)\cos\frac{\pi x}{2a} + \frac{a_2\lambda}{\sqrt{2}}\sinh\frac{\sqrt{2}\pi\lambda x}{a}\right\}^2 d\left(\frac{x}{a}\right)$$

$$+\ \frac{1}{4}\left(\frac{f\pi^2}{8}\,\frac{\lambda^2}{1+\delta\lambda^2}\right)^2\Bigg]$$

$$\int_0^1 \left(\frac{\pi x}{2a}\right) \sin\frac{\pi x}{2a} \cos\frac{\pi x}{2a}\, d\left(\frac{x}{a}\right) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \theta \sin 2\theta\, d\theta = \frac{1}{4\pi}\int_0^{\pi} x \sin x\, dx = \frac{1}{4\pi}\Big[\sin x - x\cos x\Big]_0^{\pi} = \underline{\frac{1}{4}}$$

$$\int_0^1 \cos\frac{\pi x}{2a} \cosh\frac{\sqrt{2}\pi x}{a}\, d\left(\frac{x}{a}\right) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \Big[\cosh(2\sqrt{2}\lambda + i)\theta + \cosh(2\sqrt{2}\lambda - i)\theta\Big]\, d\theta$$

$$= \frac{1}{\pi}\left[\frac{\sinh(2\sqrt{2}+i)\frac{\pi}{2}}{2\sqrt{2}\lambda + i} + \frac{\sinh(2\sqrt{2}-i)\frac{\pi}{2}}{2\sqrt{2}\lambda - i}\right] = \underline{\frac{2}{\pi}\,\frac{\cosh\sqrt{2}\lambda\pi}{(1+8\lambda^2)}}$$

$$\int_0^1 \left(\frac{\pi x}{2a}\right) \sin\frac{\pi x}{2a} \cos\frac{\pi x}{a}\, d\left(\frac{x}{a}\right) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \theta\Big[\sin 3\theta - \sin\theta\Big]\, d\theta = \frac{1}{\pi}\left[\frac{1}{9}\Big\{\sin 3\theta - 3\theta\cos 3\theta\Big\} - \Big\{\sin\theta - \theta\cos\theta\Big\}\right]_0^{\frac{\pi}{2}}$$

$$= \underline{-\frac{10}{9\pi}}$$

$$\int_0^1 \cosh\frac{\sqrt{2}\pi\lambda x}{a} \cos\frac{\pi x}{a}\, d\left(\frac{x}{a}\right) = \frac{1}{2\pi}\int_0^{\pi} \Big[\cosh(\sqrt{2}\lambda + i)\theta + \cosh(\sqrt{2}\lambda - i)\theta\Big]\, d\theta$$

$$= \frac{1}{2\pi}\left[\frac{\sinh(\sqrt{2}\lambda + i)\pi}{\sqrt{2}\lambda + i} + \frac{\sinh(\sqrt{2}\lambda - i)\pi}{\sqrt{2}\lambda - i}\right] = \underline{-\frac{1}{\pi}\,\frac{\sqrt{2}\lambda\sinh\sqrt{2}\lambda\pi}{1+2\lambda^2}}$$

$$\int_0^1 \left(\frac{\pi x}{2a}\right)^2 \sin^2\frac{\pi x}{2a}\, d\left(\frac{x}{a}\right) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \theta^2\Big[1 - \cos 2\theta\Big]\, d\theta = \frac{1}{\pi}\left[\frac{1}{3}\left(\frac{\pi}{2}\right)^3 - \frac{1}{8}\Big\{-2x\cos x + (x^2-2)\sin x\Big\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{24}\pi^3 + \frac{1}{4}\pi\right] = \underline{\left[\frac{\pi^2}{24} + \frac{1}{4}\right]}$$

$$\int_0^1 \left(\frac{\pi x}{2a}\right) \sin\frac{\pi x}{2a} \cosh\frac{\sqrt{2}\pi\lambda x}{a}\, d\left(\frac{x}{a}\right) = \frac{1}{\pi i}\int_0^{\frac{\pi}{2}} \theta\left[\sinh(2\sqrt{2}\lambda+i)\theta - \sinh(2\sqrt{2}\lambda-i)\theta\right] d\theta$$

$$= \frac{1}{\pi i}\left[\frac{\frac{\pi}{2}}{2\sqrt{2}\lambda+i}\cosh(2\sqrt{2}\lambda+i)\frac{\pi}{2} - \frac{\frac{\pi}{2}}{2\sqrt{2}\lambda-i}\cosh(2\sqrt{2}\lambda-i)\frac{\pi}{2}\right.$$

$$\left. - \left\{\frac{1}{(2\sqrt{2}\lambda+i)^2}\sinh(2\sqrt{2}\lambda+i)\frac{\pi}{2} - \frac{1}{(2\sqrt{2}\lambda-i)^2}\sinh(2\sqrt{2}\lambda-i)\frac{\pi}{2}\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\sqrt{2}\pi\lambda \sinh\sqrt{2}\lambda\pi}{1+8\lambda^2} + \frac{2(1-8\lambda^2)\cosh\sqrt{2}\lambda\pi}{(1+8\lambda^2)^2}\right]$$

$$\int_0^1 \cosh^2\frac{\sqrt{2}\pi\lambda x}{a}\, d\left(\frac{x}{a}\right) = \frac{1}{2}\int_0^1\left[1+\cosh\frac{2\sqrt{2}\pi\lambda x}{a}\right] d\left(\frac{x}{a}\right) = \frac{1}{2} + \frac{\sinh 2\sqrt{2}\lambda\pi}{4\sqrt{2}\pi\lambda}$$

$$\int_0^1 \left(\frac{\pi x}{2a}\right)\sin\frac{\pi x}{2a}\cos\frac{\pi x}{2a}\, d\left(\frac{x}{a}\right) = \frac{1}{4}$$

$$\int_0^1 \sin\frac{\pi x}{2a}\sinh\frac{\sqrt{2}\pi\lambda x}{a}\, d\left(\frac{x}{a}\right) = \frac{1}{\pi i}\int_0^{\frac{\pi}{2}}\left[\cosh(2\sqrt{2}\lambda+i)\theta - \cosh(2\sqrt{2}\lambda-i)\theta\right] d\theta$$

$$= \frac{1}{\pi i}\left[\frac{\sinh(2\sqrt{2}\lambda+i)\frac{\pi}{2}}{2\sqrt{2}\lambda+i} - \frac{\sinh(2\sqrt{2}\lambda-i)\frac{\pi}{2}}{2\sqrt{2}\lambda-i}\right] = \frac{1}{\pi}\,\frac{4\sqrt{2}\lambda\cosh\sqrt{2}\lambda\pi}{(1+8\lambda^2)}$$

$$\int_0^1 \left(\frac{\pi x}{2a}\right)^2\cos^2\frac{\pi x}{2a}\, d\left(\frac{x}{a}\right) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\theta^2\left[1+\cos 2\theta\right] d\theta = \left[\frac{\pi^2}{24}-\frac{1}{4}\right]$$

$$\int_0^1 \frac{\pi x}{2a} \cos\frac{\pi x}{2a} \sinh\frac{\sqrt{2}\pi\lambda x}{a}\, d\left(\frac{x}{a}\right) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \theta \left[\sinh(2\sqrt{2}\lambda + i)\theta + \sinh(2\sqrt{2}\lambda - i)\theta \right] d\theta$$

$$= \frac{1}{\pi}\left[\frac{\frac{\pi}{2}}{2\sqrt{2}\lambda + i} \cosh(2\sqrt{2}\lambda + i)\frac{\pi}{2} + \frac{\frac{\pi}{2}}{2\sqrt{2}\lambda - i} \cosh(2\sqrt{2}\lambda - i)\frac{\pi}{2} \right]$$

$$- \frac{1}{\pi}\left[\frac{1}{(2\sqrt{2}\lambda + i)^2} \sinh(2\sqrt{2}\lambda + i)\frac{\pi}{2} + \frac{1}{(2\sqrt{2}\lambda - i)^2} \sinh(2\sqrt{2}\lambda - i)\frac{\pi}{2} \right]$$

$$= \frac{1}{\pi}\left[\frac{\pi \sinh\sqrt{2}\pi\lambda}{1+8\lambda^2} - \frac{8\sqrt{2}\lambda \cosh\sqrt{2}\lambda\pi}{(1+8\lambda^2)^2} \right]$$

$$\int_0^1 \sinh^2\frac{\sqrt{2}\pi\lambda x}{a}\, d\left(\frac{x}{a}\right) = \frac{\sinh 2\sqrt{2}\lambda\pi}{4\sqrt{2}\pi\lambda} - \frac{1}{2}$$

$$\frac{1}{2ab}\int_0^a\int_0^b \varepsilon_2^2\,dx\,dy = \left(\frac{a}{R}\right)^4\left(\frac{f}{4}\right)^2\lambda^4\left[\frac{1}{4}\left(\frac{f\pi^2}{32}\right)^2 + 2\frac{(1-8\lambda^2)^2}{(1+8\lambda^2)^4} + \frac{1}{8}\left(\frac{f\pi^2}{16}\right)^2\right.$$

$$+\frac{2(1-8\lambda^2)}{(1+8\lambda^2)^2}\left\{\frac{1}{1+8\lambda^2} - \frac{64\lambda^2\cosh\sqrt{2}\lambda\pi}{(1+8\lambda^2)^3(\sqrt{2}\lambda\pi\sinh\sqrt{2}\pi\lambda)}\right\} + \frac{f\pi^2}{32}\left\{-\frac{40}{9\pi}\frac{1}{(1+8\lambda^2)} + \frac{32\lambda^2}{(1+2\lambda^2)(1+8\lambda^2)^2\pi}\right\}$$

$$+\frac{1}{4}\left\{\left(\frac{\pi^2}{24}+\frac{1}{4}\right)\frac{16}{(1+8\lambda^2)^2}\right\} - \frac{2}{1+8\lambda^2}\cdot\frac{32\lambda^2}{(1+8\lambda^2)^2}\left\{\frac{2}{1+8\lambda^2} + \frac{2(1-8\lambda^2)}{(1+8\lambda^2)^2}\frac{\cosh\sqrt{2}\lambda\pi}{(\sqrt{2}\lambda\pi\sinh\sqrt{2}\pi\lambda)}\right\}$$

$$\left.+\frac{1}{4}\frac{(32\lambda^2)^2}{(1+8\lambda^2)^4(\sqrt{2}\lambda\sinh\sqrt{2}\pi\lambda)^2}\left\{\frac{1}{2} + \frac{\sinh 2\sqrt{2}\lambda\pi}{4\sqrt{2}\lambda\pi}\right\} + \frac{1}{8}\left\{\frac{f\pi^2}{32}\frac{1-8\lambda^2}{1+8\lambda^2}\right\}^2\right] + \frac{1}{2}\left(\frac{\sigma}{E}\right)^2$$

$$\frac{1}{4ab}\int_0^a\int_0^b \gamma^2\,dx\,dy = \left(\frac{a}{R}\right)^2\left(\frac{f}{4}\right)^2\lambda^2\left[\frac{64\lambda^4}{(1+8\lambda^2)^4} + \frac{16\lambda^2}{(1+8\lambda^2)^2}\left\{-\frac{2\lambda^2}{1+8\lambda^2} - \frac{128\lambda^4\cosh\sqrt{2}\lambda\pi}{(1+8\lambda^2)^3(\sqrt{2}\lambda\pi\sinh\sqrt{2}\pi\lambda)}\right\}\right.$$

$$+\frac{1}{2}\frac{64\lambda^4}{(1+8\lambda^2)^2}\left(\frac{\pi^2}{24}-\frac{1}{4}\right) + \frac{128\lambda^4}{(1+8\lambda^2)^3}\left\{\frac{1}{(1+8\lambda^2)} - \frac{16\lambda^2\cosh\sqrt{2}\lambda\pi}{(1+8\lambda^2)^2(\sqrt{2}\lambda\pi\sinh\sqrt{2}\lambda\pi)}\right\}$$

$$\left.+\frac{\lambda^2}{4}\frac{(32\lambda^2)^2}{(1+8\lambda^2)^4(\sqrt{2}\lambda\sinh\sqrt{2}\lambda\pi)^2}\left\{-\frac{1}{2} + \frac{\sinh 2\sqrt{2}\lambda\pi}{4\sqrt{2}\lambda\pi}\right\} + \frac{1}{4}\left(\frac{f\pi^2}{8}\frac{\lambda^2}{1+8\lambda^2}\right)^2\right]$$