

钱学森

力学手稿

5

钱学森

Application of Telegraphic Transformer
Two Dimensional Flow

The equations of two dimensional motion of compressible fluids without rotation, assuming that the pressure is only a function of density, can be reduced to a single non-linear equation of the velocity potential. In the supersonic case the problem is solved by Prandtl, Meyer and Gurewitsch by means of the powerful method of characteristics. But the essential difficulty of the problem lies in the subsonic case especially when the velocity is near to the velocity of sound. To solve this problem it is necessary that the disturbance superimposed on the parallel streamlines is sufficiently small. The reference should be made to an example of this method applied to the flow of thin airfoil due to Prandtl. But the presence of stagnation points in the case of the airfoil makes the application of the linearized theory questionable at least near this region because when the



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出版前言

2011年12月11日是西安交通大学杰出校友钱学森先生的百年诞辰。为缅怀钱学森学长,学习他的科学思想和卓越风范,展示其丰功伟绩和人格魅力,西安交通大学举办了“纪念钱学森诞辰100周年”系列活动:作为制片方之一,参与西部电影集团摄制传记故事片《钱学森》;与中央电视台合作,出品纪录片《实验班的故事——沿着钱学森走过的路》;扩建钱学森生平业绩展馆,向校内外开放;举办钱学森科学与教育思想研讨会;出版发行《钱学森力学手稿》、《钱学森年谱(初编)》、《钱学森第六次产业革命思想探微丛书》等。

钱学森先生在美国深造和工作期间留下大量珍贵手稿,这些手稿真实展示了钱学森先生博大精深的学识、开拓求实的精神和严谨奋进的作风,是钱老勇攀科学高峰和严谨治学的集中体现。这里,我们将部分原稿整理汇集成册,出版《钱学森力学手稿》,作为钱老百年诞辰的献礼。

《钱学森力学手稿》共10卷,包含两部分内容。第一部分是草稿,包括扁壳、球壳和圆柱壳屈曲分析的公式推导和数值演算。在研究圆柱壳轴压屈曲问题时,为了求得圆柱壳体的临界压力,在有关的五百多页草稿中,对多达二十多种可能的屈曲模

态逐一进行公式推演和数值计算,最终才找到满意的并在论文中采用的屈曲模态。仔细观察草稿中的数据列表,每个数字有效位数都长达八位,在手摇机械式计算机作为主要计算工具的年代,这串串数字凝聚着多少现今难以想象的艰辛劳动。

第二部分是手稿,以航空航天工程为核心,涵盖空气动力学、固体力学、火箭技术、工程控制论和物理力学等领域的部分学术论文手稿、打印稿和讲义。

《钱学森力学手稿》是在西安交通大学校领导的大力支持下,由西安交通大学航天航空学院沈亚鹏教授整理完成。图书出版过程中得到了西安交通大学党委宣传部、校友关系发展部、图书馆、航天航空学院等的积极协助,在此深表感谢。

Preliminary Calculation of
Circular Cylinder (IV)

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We have the equation

486

$$\frac{\partial^2 v}{\partial x^2} + 2 \frac{\partial^2 v}{\partial y^2} = - \left\{ \frac{\partial w}{\partial y} \left(\frac{\partial^2 w}{\partial x^2} + 2 \frac{\partial^2 w}{\partial y^2} \right) + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x \partial y} \right\}$$

$$\frac{w}{R} = \frac{1}{2} \left(\frac{a}{R} \right)^2 \left[1 - \left(\frac{x}{a} \right)^2 - \frac{f}{2} \cos \frac{\pi x}{2a} \left(1 + \cos \frac{\pi y}{b} \right) \right]$$

$$\frac{w}{R} = \frac{1}{2} \left(\frac{a}{R} \right)^2 \left[1 - \left(\frac{x}{a} \right)^2 \right]$$

$$\frac{\partial w}{\partial x} = \left(\frac{a}{R} \right) \left[- \left(\frac{x}{a} \right) + \frac{f}{8} \pi \sin \frac{\pi x}{2a} \left(1 + \cos \frac{\pi y}{b} \right) \right]$$

$$\frac{\partial w}{\partial y} = \left(\frac{a}{R} \right) \left[+ \frac{f}{4} \pi \lambda \cos \frac{\pi x}{2a} \sin \frac{\pi y}{b} \right]$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{R} \left[-1 + \frac{f}{16} \pi^2 \cos \frac{\pi x}{2a} \left(1 + \cos \frac{\pi y}{b} \right) \right]$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{1}{R} \left[- \frac{f}{8} \pi^2 \lambda \sin \frac{\pi x}{2a} \sin \frac{\pi y}{b} \right]$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{1}{R} \left[+ \frac{f}{4} \pi^2 \lambda^2 \cos \frac{\pi x}{2a} \cos \frac{\pi y}{b} \right]$$

$$R \left[\frac{\partial^2 v}{\partial x^2} + 2 \frac{\partial^2 v}{\partial y^2} \right] = - \left(\frac{a}{R} \right) \left(\frac{f}{4} \pi \lambda \right) \left[\cos \frac{\pi x}{2a} \sin \frac{\pi y}{b} \left\{ -1 + \frac{f \pi^2}{16} \cos \frac{\pi x}{2a} \left(1 + \cos \frac{\pi y}{b} \right) \right. \right. \right. \\ \left. \left. \left. + \frac{f \pi^2}{2} \lambda^2 \cos \frac{\pi x}{2a} \cos \frac{\pi y}{b} \right\} \right. \right. \\ \left. \left. - \frac{\pi}{2} \sin \frac{\pi x}{2a} \sin \frac{\pi y}{b} \left\{ - \left(\frac{x}{a} \right) + \frac{f \pi}{8} \sin \frac{\pi x}{2a} \left(1 + \cos \frac{\pi y}{b} \right) \right\} \right] \right]$$

$$= - \left(\frac{a}{R} \right) \frac{f}{4} \pi \lambda \left[- \cos \frac{\pi x}{2a} \sin \frac{\pi y}{b} + \frac{f \pi^2}{64} (1 + \cos \frac{\pi x}{a}) (2 \sin \frac{\pi y}{b} + \sin \frac{2 \pi y}{b}) \right. \\ \left. + \left(\frac{\pi x}{2a} \right) \sin \frac{\pi x}{2a} \sin \frac{\pi y}{b} + \frac{f \pi^2}{64} (-1 + \cos \frac{\pi x}{a}) (2 \sin \frac{\pi y}{b} + \sin \frac{2 \pi y}{b}) \right. \\ \left. + \frac{f \pi^2}{8} \lambda^2 (1 + \cos \frac{\pi x}{a}) \sin \frac{2 \pi y}{b} \right]$$

$$\begin{aligned}
 R \left[\frac{\partial^2 v}{\partial x^2} + 2 \frac{\partial^2 v}{\partial y^2} \right] &= - \left(\frac{R}{a} \right) \frac{\pi \lambda}{4} \left[\left(\frac{\pi \lambda}{ga} \right) \sin \frac{\pi x}{2a} \sin \frac{\pi y}{b} - \cos \frac{\pi x}{2a} \sin \frac{\pi y}{b} + \frac{\pi^2}{16} \cos \frac{\pi x}{2a} \sin \frac{\pi y}{b} \right. \\
 &\quad \left. + \frac{\pi^2}{32} \cos \frac{\pi x}{2a} \sin \frac{2\pi y}{b} + \frac{\pi^2}{8} \lambda^2 \sin \frac{\pi y}{b} + \frac{\pi^2}{8} \lambda^2 \cos \frac{\pi x}{2a} \sin \frac{2\pi y}{b} \right] \\
 &= - \left(\frac{R}{R} \right) \frac{\pi \lambda}{4} \left[\left(\frac{\pi \lambda}{ga} \right) \sin \frac{\pi x}{2a} \sin \frac{\pi y}{b} - \cos \frac{\pi x}{2a} \sin \frac{\pi y}{b} + \frac{\pi^2}{16} \lambda^2 \sin \frac{\pi y}{b} + \frac{\pi^2}{16} (1+2\lambda^2) \cos \frac{\pi x}{2a} \sin \frac{\pi y}{b} \right. \\
 &\quad \left. + \frac{\pi^2}{32} \cos \frac{\pi x}{2a} \sin \frac{2\pi y}{b} \right]
 \end{aligned}$$

Investigate the particular solution of the following eqn.

$$B = - \frac{1}{\left(\frac{\pi}{a} \right)^2} \frac{4}{(1+\delta\lambda^2)}$$

$$A = - \frac{1}{\left(\frac{\pi}{a} \right)^2} \frac{8}{(1+\delta\lambda^2)^2}$$

Put

$$v = X \sin \frac{\pi y}{b}$$

$$\frac{d^2 X}{dx^2} - 2 \left(\frac{\pi}{b} \right)^2 X = \left(\frac{\pi}{ga} \right) \sin \frac{\pi x}{2a}$$

Let

$$X = A \cos \frac{\pi x}{2a} + B \left(\frac{\pi x}{2a} \right) \sin \frac{\pi x}{2a}$$

$$\begin{aligned}
 \frac{d^2 X}{dx^2} &= \left(\frac{\pi}{a} \right)^2 \left\{ \frac{(2B-A)}{4} \cos \frac{\pi x}{2a} - B \left(\frac{\pi x}{2a} \right) \sin \frac{\pi x}{2a} \right\} \\
 - 2 \left(\frac{\pi}{b} \right)^2 X &= \left(\frac{\pi}{a} \right)^2 \left\{ -2\lambda^2 A \cos \frac{\pi x}{2a} - 2\lambda^2 B \left(\frac{\pi x}{2a} \right) \sin \frac{\pi x}{2a} \right\}
 \end{aligned}$$

$$\therefore \left(\frac{\pi}{a} \right)^2 \left\{ \frac{2B-A}{4} - 2\lambda^2 A \right\} = 0, \quad \left(\frac{\pi}{a} \right)^2 \left\{ -\frac{B}{4} - 2\lambda^2 B \right\} = 1$$

$$\begin{aligned} \frac{v}{R} = & \left(\frac{10^3}{R} \right) \frac{f}{4} \frac{1}{\pi} \lambda \left[\frac{f}{(1+fl)^2} \cos \frac{\pi x}{2a} \sin \frac{\pi y}{b} + \frac{f}{(1+fl)^2} \left(\frac{\pi x}{2a} \right) \sin \frac{\pi x}{2a} \sin \frac{\pi y}{b} - \frac{f}{(1+fl)^2} \cos \frac{\pi x}{2a} \sin \frac{\pi y}{b} \right. \\ & + \frac{fl^2}{8} \lambda^2 \frac{1}{\pi} \sin \frac{2\pi y}{b} + \frac{fl^2}{16} \frac{1+2fl^2}{1+fl^2} \cos \frac{\pi x}{a} \sin \frac{\pi y}{b} + \frac{fl^2}{32} \frac{1}{1+fl^2} \cos \frac{\pi x}{a} \sin \frac{2\pi y}{b} \\ & \left. + a_0 \left(\frac{\pi y}{b} \right) + a_2 \cos \sqrt{2} \frac{\pi \lambda x}{a} \sin \frac{\pi y}{b} \right] \end{aligned}$$

$$\begin{aligned} = & \left(\frac{10^3}{R} \right) \frac{f}{4} \frac{1}{\pi} \lambda \left[\frac{4(1-fl^2)}{(1+fl^2)^2} \cos \frac{\pi x}{2a} \sin \frac{\pi y}{b} + \frac{f}{(1+fl)^2} \left(\frac{\pi x}{2a} \right) \sin \frac{\pi x}{2a} \sin \frac{\pi y}{b} + \frac{fl^2}{64} \sin \frac{2\pi y}{b} \right. \\ & \left. + \frac{fl^2}{16} \cos \frac{\pi x}{a} \sin \frac{\pi y}{b} + \frac{fl^2}{32} \frac{1}{(1+fl^2)} \cos \frac{\pi x}{a} \sin \frac{2\pi y}{b} + a_0 \left(\frac{\pi y}{b} \right) + a_2 \cos \sqrt{2} \frac{\pi \lambda x}{a} \sin \frac{\pi y}{b} \right] \end{aligned}$$

$$\begin{aligned} \frac{\partial v}{\partial x} = & \left(\frac{10^3}{R} \right) \frac{f}{4} \lambda \left[- \frac{2(1-fl^2)}{(1+fl^2)^2} \sin \frac{\pi x}{2a} \sin \frac{\pi y}{b} + \frac{2}{(1+fl)^2} \sin \frac{\pi x}{2a} \sin \frac{\pi y}{b} + \frac{2}{(1+fl)^2} \left(\frac{\pi x}{2a} \right) \cos \frac{\pi x}{2a} \sin \frac{\pi y}{b} \right. \\ & \left. - \frac{fl^2}{16} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} - \frac{fl^2}{32} \frac{1}{(1+fl^2)} \sin \frac{\pi x}{a} \sin \frac{2\pi y}{b} + a_2 \sqrt{2} \sin \sqrt{2} \frac{\pi \lambda x}{a} \sin \frac{\pi y}{b} \right] \\ \frac{\partial v}{\partial y} = & \left(\frac{10^3}{R} \right) \frac{f}{4} \lambda \left[- 2 \left(\frac{\pi x}{2a} \right) \cos \frac{\pi x}{2a} \sin \frac{\pi y}{b} + \frac{fl^2}{32} \sin \frac{\pi x}{a} \left(2 \sin \frac{\pi y}{b} + \sin \frac{2\pi y}{b} \right) \right] \end{aligned}$$

48

$$\begin{aligned} \left(\frac{q}{R}\right)^2 \frac{1}{4} \lambda^2 a_0 &= -\frac{\sigma}{E} - \left(\frac{q}{R}\right)^2 \frac{1}{4} \lambda^2 \left[-\frac{16}{\pi} \frac{1}{(1+\lambda^2)^2} + \frac{\pi^2}{32} + \frac{16}{\pi} \frac{1}{(1+\lambda^2)^2} \right] \\ &= -\frac{\sigma}{E} - \left(\frac{q}{R}\right)^2 \frac{1}{4} \lambda^2 \left(\frac{\pi^2}{32} \right) \end{aligned}$$

$$\boxed{\frac{\Delta \rho}{ab E} = -\left(\frac{\sigma}{E}\right)^2 - \left(\frac{q}{R}\right)^2 \left(\frac{1}{4}\right)^2 \left(\frac{\pi^2}{8}\right)}$$

$$\varepsilon_1 = \frac{1}{2} \left\{ \left(\frac{\partial u}{\partial x} \right)^2 - \left(\frac{\partial u}{\partial y} \right)^2 \right\} = \left(\frac{q}{R} \right)^2 \frac{1}{4} \left\{ -\left(\frac{\pi x}{2a} \right) \sin \frac{\pi x}{2a} \left(1 + \cos \frac{\pi x}{a} \right) + \frac{\pi^2}{128} \left(1 - \cos \frac{\pi x}{a} \right)^2 + 4 \cos \frac{\pi x}{a} + \cos \frac{2\pi x}{a} \right\}$$

$$\begin{aligned} &= \left(\frac{q}{R} \right)^2 \frac{1}{4} \left\{ -\left(\frac{\pi x}{2a} \right) \sin \frac{\pi x}{2a} + \frac{3\pi^2}{128} \left(1 - \cos \frac{\pi x}{a} \right) \right. \\ &\quad + \left\{ -\left(\frac{\pi x}{2a} \right) \sin \frac{\pi x}{2a} + \frac{\pi^2}{32} \left(1 - \cos \frac{\pi x}{a} \right) \cos \frac{\pi x}{a} \right. \\ &\quad \left. \left. + \frac{\pi^2}{128} \left(1 - \cos \frac{\pi x}{a} \right) \cos \frac{2\pi x}{a} \right\} \right\} \end{aligned}$$

$$\frac{1}{2ab} \int_0^a \int_0^b \varepsilon_1^2 dx dy = \left(\frac{q}{R} \right)^4 \left(\frac{1}{4} \right)^2 \left\{ \frac{3}{4} \left[\left(\frac{\pi}{2a} \right)^2 + \frac{1}{2} \left(\frac{1}{32} \right)^2 \right] \left(\frac{\pi x}{2} \right)^2 + \frac{3}{4} \int_0^1 \left(\frac{\pi x}{2a} \right)^2 \sin^2 \frac{\pi x}{2a} dx \right\}$$

$$= \frac{5}{128} \left(\frac{\pi^2}{a} \right)^2 \int_0^1 \left(\frac{\pi x}{2a} \right) \sin \frac{\pi x}{2a} \left(1 - \cos \frac{\pi x}{a} \right) dx \left\{ \right.$$

$$\left. = \left(\frac{q}{R} \right)^4 \left(\frac{1}{4} \right)^2 \left[\frac{105}{8(128)^2} \left(\frac{\pi^2}{a} \right)^2 - \frac{5}{128} \left(\frac{\pi}{a} + \frac{10}{9\pi} \right) \left(\frac{\pi^2}{a} \right) + \left(\frac{\pi^2}{32} + \frac{3}{16} \right) \right] \right\}$$

$$\frac{\lambda^2}{2} = \left(\frac{q}{R}\right)^2 \frac{f}{4} \lambda^2 \left[\frac{4(1-f\lambda^2)}{(1+f\lambda^2)^2} \cos \frac{\pi x}{2a} \cos \frac{\pi y}{b} + \frac{f}{(1+f\lambda^2)} \left(\frac{\pi x}{2a} \right) \sin \frac{\pi x}{2a} \cos \frac{\pi y}{b} + \frac{f\pi^2}{32} \cos \frac{2\pi x}{b} \right. \\ \left. + \frac{f\pi^2}{16} \cos \frac{\pi x}{a} \cos \frac{\pi y}{b} + \frac{f\pi^2}{16} \frac{1}{(1+f\lambda^2)} \cos \frac{\pi x}{a} \cos \frac{2\pi y}{b} - \frac{f\pi^2}{32} + a_2 \cos \sqrt{\frac{2\pi x}{a}} \cos \frac{\pi y}{b} \right] - \frac{\sigma}{E}$$

$$\frac{f(2a)^2}{2(\frac{b}{2})^2} = \left(\frac{q}{R}\right)^2 \frac{f}{4} \lambda^2 \left[\frac{f\pi^2}{32} (1 + \cos \frac{\pi x}{a}) (1 - \cos \frac{2\pi y}{b}) \right] = \left(\frac{q}{R}\right)^2 \frac{f}{4} \lambda^2 \left[\frac{f\pi^2}{32} + \frac{f\pi^2}{32} \cos \frac{\pi x}{a} - \frac{f\pi^2}{32} \cos \frac{2\pi y}{b} \right. \\ \left. - \frac{f\pi^2}{32} \cos \frac{\pi x}{a} \cos \frac{2\pi y}{b} \right]$$

$$\varepsilon_2 = \left(\frac{q}{R}\right)^2 \frac{f}{4} \lambda^2 \left[\frac{4(1-f\lambda^2)}{(1+f\lambda^2)^2} \cos \frac{\pi x}{2a} \cos \frac{\pi y}{b} + \frac{f}{(1+f\lambda^2)} \left(\frac{\pi x}{2a} \right) \sin \frac{\pi x}{2a} \cos \frac{\pi y}{b} + \frac{f\pi^2}{16} \cos \frac{\pi x}{a} \cos \frac{\pi y}{b} + \frac{f\pi^2}{32} \cos \frac{2\pi x}{b} \right. \\ \left. + \frac{f\pi^2}{32} \frac{(1-f\lambda^2)}{(1+f\lambda^2)} \cos \frac{\pi x}{a} \cos \frac{2\pi y}{b} + a_2 \cos \sqrt{\frac{2\pi x}{a}} \cos \frac{\pi y}{b} \right] - \frac{\sigma}{E}$$

$$= \left(\frac{q}{R}\right)^2 \frac{f}{4} \lambda^2 \left[\frac{f\pi^2}{32} \cos \frac{\pi x}{a} \right. \\ \left. + \left\{ \frac{4(1-f\lambda^2)}{(1+f\lambda^2)^2} \cos \frac{\pi x}{2a} + \frac{f\pi^2}{16} \cos \frac{\pi x}{a} + \frac{f}{(1+f\lambda^2)} \left(\frac{\pi x}{2a} \right) \sin \frac{\pi x}{2a} \right\} \cos \frac{\pi y}{b} \right. \\ \left. + \left\{ \frac{f\pi^2}{32} \frac{(1-f\lambda^2)}{(1+f\lambda^2)} \cos \frac{\pi x}{a} \cos \frac{2\pi y}{b} \right\} - \frac{\sigma}{E} \right]$$

$$\begin{aligned}
\frac{1}{2ab} \int_0^b \int_0^b \mathcal{E}_z^2 dx dy &= \left(\frac{q}{R}\right)^2 \left(\frac{b}{4}\right)^2 \lambda^4 \left[\frac{1}{4} \left(\frac{b^2}{32}\right)^2 + \frac{1}{8} \left\{ \frac{4(1-b^2)}{(1+bl^2)^2} \right\}^2 + \frac{1}{8} \left(\frac{b^2}{16}\right)^2 \right] \\
&+ \frac{2(1-b^2)}{(1+bl^2)^2} \int_0^b \cos \frac{\pi x}{2a} \left\{ \frac{4}{1+bl^2} \left(\frac{\pi x}{2a}\right) \sin \frac{\pi x}{2a} + q_2 \cosh \frac{\sqrt{2}\pi \lambda x}{a} \right\} d\left(\frac{x}{a}\right) \\
&+ \frac{b^2}{32} \int_0^b \cos \frac{\pi x}{a} \left\{ \frac{4}{1+bl^2} \left(\frac{\pi x}{2a}\right) \sin \frac{\pi x}{2a} + q_2 \cosh \frac{\sqrt{2}\pi \lambda x}{a} \right\} d\left(\frac{x}{a}\right) \\
&+ \frac{1}{4} \int_0^b \left\{ \frac{4}{1+bl^2} \left(\frac{\pi x}{2a}\right) \sin \frac{\pi x}{2a} + q_2 \cosh \frac{\sqrt{2}\pi \lambda x}{a} \right\}^2 d\left(\frac{x}{a}\right) \\
&+ \frac{1}{8} \left\{ \frac{b^2}{32} \frac{1-b^2}{1+bl^2} \right\}^2 + \frac{1}{2} \left(\frac{\sigma}{E}\right)^2
\end{aligned}$$

$$\begin{aligned}
\frac{1}{4ab} \int_0^b \int_0^b \mathcal{H}_z^2 dx dy &= \left(\frac{q}{R}\right)^2 \left(\frac{b}{4}\right)^2 \lambda^2 \left[\frac{1}{4} \left\{ \frac{16b^2}{(1+bl^2)^2} \right\}^2 + \frac{16b^2}{(1+bl^2)^2} \int_0^b \sin \frac{\pi x}{2a} \left\{ -\frac{bl^2}{1+bl^2} \left(\frac{\pi x}{2a}\right) \cos \frac{\pi x}{2a} + \frac{q_2 \lambda}{\sqrt{2}} \sinh \frac{\sqrt{2}\pi \lambda x}{a} \right\} d\left(\frac{x}{a}\right) \right. \\
&+ \frac{1}{2} \int_0^b \left\{ -\frac{bl^2}{(1+bl^2)} \left(\frac{\pi x}{2a}\right) \cos \frac{\pi x}{2a} + \frac{q_2 \lambda}{\sqrt{2}} \sinh \frac{\sqrt{2}\pi \lambda x}{a} \right\}^2 d\left(\frac{x}{a}\right) \\
&\left. + \frac{1}{4} \left(\frac{b^2}{8} \frac{b^2}{1+bl^2} \right)^2 \right]
\end{aligned}$$

$$\int_0^1 \left(\frac{\pi x}{2a} \right) \sin \frac{\pi x}{2a} \cos \frac{\pi x}{2a} d\left(\frac{x}{a}\right) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \theta \sin \theta d\theta = \frac{1}{4\pi} \int_0^{\pi} x \sin x dx = \frac{1}{4\pi} \left[\sin x - x \cos x \right]_0^{\pi} = \frac{4}{\pi}$$

$$\int_0^1 \cos \frac{\pi x}{2a} \cosh \frac{\sqrt{2}\pi x}{a} d\left(\frac{x}{a}\right) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \left[\cosh(\sqrt{2}\lambda + i)\theta + \cosh(\sqrt{2}\lambda - i)\theta \right] d\theta$$

$$= \frac{1}{\pi} \left[\frac{\sinh(\sqrt{2}\lambda + i)\frac{\pi}{2}}{2\sqrt{2}\lambda + i} + \frac{\sinh(\sqrt{2}\lambda - i)\frac{\pi}{2}}{2\sqrt{2}\lambda - i} \right] = \frac{2}{\pi} \frac{\cosh \sqrt{2}\lambda \pi}{(1 + \theta^2)}$$

$$\int_0^1 \left(\frac{\pi x}{2a} \right) \sin \frac{\pi x}{2a} \cos \frac{\pi x}{a} d\left(\frac{x}{a}\right) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \theta [\sin 3\theta - \sin \theta] d\theta = \frac{1}{\pi} \left[\frac{1}{9} \{ \sin 3\theta - 3\theta \cos 3\theta \} - \{ \sin \theta - \theta \cos \theta \} \right]_0^{\frac{\pi}{2}}$$

$$= -\frac{10}{9\pi}$$

$$\int_0^1 \cosh \frac{\sqrt{2}\pi x}{a} \cos \frac{\pi x}{a} d\left(\frac{x}{a}\right) = \frac{1}{2\pi} \int_0^{\frac{\pi}{2}} \left[\cosh(\sqrt{2}\lambda + i)\theta + \cosh(\sqrt{2}\lambda - i)\theta \right] d\theta$$

$$= \frac{1}{2\pi} \left[\frac{\sinh(\sqrt{2}\lambda + i)\pi}{\sqrt{2}\lambda + i} + \frac{\sinh(\sqrt{2}\lambda - i)\pi}{\sqrt{2}\lambda - i} \right] = -\frac{1}{\pi} \frac{\sqrt{2}\lambda \sinh \sqrt{2}\lambda \pi}{1 + \lambda^2}$$

$$\int_0^1 \left(\frac{\pi x}{2a} \right)^2 \sin \frac{\pi x}{2a} d\left(\frac{x}{a}\right) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \theta^2 [1 - \cos 2\theta] d\theta = \frac{1}{\pi} \left[\frac{1}{3} \left(\frac{\pi}{2} \right)^3 - \frac{1}{8} \left[2x \cos x + (x^2 - 2) \sin x \right]_0^{\frac{\pi}{2}} \right]$$

$$= \frac{1}{\pi} \left[\frac{1}{24} \pi^3 + \frac{1}{4} \pi \right] = \left[\frac{\pi^2}{24} + \frac{1}{4} \right]$$

$$\begin{aligned}
 \int_0^1 \frac{\left(\frac{\pi x}{2a}\right) \operatorname{Im} \frac{\pi x}{2a} \coth \frac{\sqrt{2} \pi x}{a} d\left(\frac{x}{a}\right)}{\pi i} &= \frac{1}{\pi i} \int_0^{\frac{\pi}{2}} \theta \left[\sinh(2\sqrt{2}\lambda + i)\theta - \sinh(2\sqrt{2}\lambda - i)\theta \right] d\theta \\
 &= \frac{1}{\pi i} \left[\frac{\frac{\pi}{2}}{2\sqrt{2}\lambda + i} \coth(2\sqrt{2}\lambda + i)\frac{\pi}{2} - \frac{\frac{\pi}{2}}{2\sqrt{2}\lambda - i} \coth(2\sqrt{2}\lambda - i)\frac{\pi}{2} \right. \\
 &\quad \left. - \left\{ \frac{1}{(2\sqrt{2}\lambda + i)^2} \sinh(2\sqrt{2}\lambda + i)\frac{\pi}{2} - \frac{1}{(2\sqrt{2}\lambda - i)^2} \sinh(2\sqrt{2}\lambda - i)\frac{\pi}{2} \right\} \right] \\
 &= \frac{1}{\pi} \left[\frac{2\sqrt{2}\pi\lambda \sinh \sqrt{2}\lambda \pi}{1 + \lambda^2} + \frac{2(1 - \lambda^2) \coth \sqrt{2}\lambda \pi}{(1 + \lambda^2)^2} \right]
 \end{aligned}$$

$$\int_0^1 \frac{\coth^2 \frac{\sqrt{2} \pi x}{a} d\left(\frac{x}{a}\right)}{2} = \frac{1}{2} \int_0^1 \left[1 + \coth \frac{2\sqrt{2} \pi x}{a} \right] d\left(\frac{x}{a}\right) = \frac{1}{2} + \frac{\sinh 2\sqrt{2}\lambda \pi}{4\sqrt{2} \pi \lambda}$$

$$\int_0^1 \frac{\left(\frac{\pi x}{2a}\right) \operatorname{Im} \frac{\pi x}{2a} \coth \frac{\pi x}{2a} d\left(\frac{x}{a}\right)}{2} = \frac{1}{4}$$

$$\begin{aligned}
 \int_0^1 \frac{\operatorname{Im} \frac{\pi x}{2a} \sinh \frac{\sqrt{2} \pi x}{a} d\left(\frac{x}{a}\right)}{2} &= \frac{1}{\pi i} \int_0^{\frac{\pi}{2}} \left[\cosh(2\sqrt{2}\lambda + i)\theta - \cosh(2\sqrt{2}\lambda - i)\theta \right] d\theta \\
 &= \frac{1}{\pi i} \left[\frac{\sinh(2\sqrt{2}\lambda + i)\frac{\pi}{2}}{2\sqrt{2}\lambda + i} - \frac{\sinh(2\sqrt{2}\lambda - i)\frac{\pi}{2}}{2\sqrt{2}\lambda - i} \right] = \frac{1}{\pi} \frac{4\sqrt{2}\lambda \coth \sqrt{2}\lambda \pi}{(1 + \lambda^2)}
 \end{aligned}$$

$$\int_0^1 \frac{\left(\frac{\pi x}{2a}\right)^2 \coth \frac{\pi x}{2a} d\left(\frac{x}{a}\right)}{2} = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \theta^2 [1 + \coth \theta] d\theta = \left[\frac{\pi^2}{24} - \frac{1}{4} \right]$$

$$\begin{aligned}
 \int_0^1 \frac{\pi \lambda}{2a} \frac{\cosh \frac{\pi \lambda}{2a} \sinh \frac{\sqrt{2} \pi \lambda}{a} d\left(\frac{\lambda}{a}\right)}{2\sqrt{2}\lambda + i} &= \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \left[\sinh(2\sqrt{2}\lambda + i)\theta + \sinh(2\sqrt{2}\lambda - i)\theta \right] d\theta \\
 &= \frac{1}{\pi} \left[\frac{\frac{\pi}{2}}{2\sqrt{2}\lambda + i} \cosh(2\sqrt{2}\lambda + i)\frac{\pi}{2} + \frac{\frac{\pi}{2}}{2\sqrt{2}\lambda - i} \cosh(2\sqrt{2}\lambda - i)\frac{\pi}{2} \right] \\
 &\quad - \frac{1}{\pi} \left[\frac{1}{(2\sqrt{2}\lambda + i)^2} \sinh(2\sqrt{2}\lambda + i)\frac{\pi}{2} + \frac{1}{(2\sqrt{2}\lambda - i)^2} \sinh(2\sqrt{2}\lambda - i)\frac{\pi}{2} \right] \\
 &= \frac{1}{\pi} \left[\frac{\pi \cosh \sqrt{2} \pi \lambda}{1 + 8\lambda^2} - \frac{\sqrt{2} \lambda \cosh \sqrt{2} \pi \lambda}{(1 + 8\lambda^2)^2} \right]
 \end{aligned}$$

$$\int_0^1 \frac{\sinh \frac{\sqrt{2} \pi \lambda}{a} d\left(\frac{\lambda}{a}\right)}{2\sqrt{2}\lambda + i} = \frac{\sinh 2\sqrt{2}\lambda \sqrt{2}}{4\sqrt{2}\pi \lambda} - \frac{1}{2}$$

$$\begin{aligned}
\frac{1}{32b} \int_0^a \int_0^b \varepsilon^2 dx dy &= \left(\frac{a}{b}\right)^4 \left(\frac{b}{a}\right)^2 \lambda^4 \left[\frac{1}{4} \frac{H\lambda^2}{32} + 2 \frac{(1-\delta\lambda^2)^2}{(1+\delta\lambda^2)^4} + \frac{1}{8} \left(\frac{H\lambda^2}{16}\right)^2 \right. \\
&+ \frac{2(1-\delta\lambda^2)}{(1+\delta\lambda^2)^2} \left\{ \frac{1}{14\delta\lambda^2} - \frac{64\lambda^2 \cos \sqrt{2}\lambda\pi}{(1+\delta\lambda^2)^3 (\sqrt{2}\lambda\pi \sinh \sqrt{2}\lambda\pi)} \right\} + \frac{H\pi^2}{32} \left\{ -\frac{42}{9\pi} \frac{1}{(1+\delta\lambda^2)} + \frac{32\lambda^2}{(1+\delta\lambda^2)(1+\delta\lambda^2)^2 \pi} \right\} \\
&+ \frac{1}{4} \left\{ \left(\frac{\pi^2}{24} + \frac{1}{4}\right) \frac{16}{(1+\delta\lambda^2)^2} \right\} - \frac{2}{1+\delta\lambda^2} \cdot \frac{32\lambda^2}{(1+\delta\lambda^2)^2} \left\{ \frac{2}{14\delta\lambda^2} + \frac{2(1-\delta\lambda^2)}{(1+\delta\lambda^2)^2} \frac{\cos \sqrt{2}\lambda\pi}{(\sqrt{2}\lambda\pi \sinh \sqrt{2}\lambda\pi)} \right\} \\
&+ \frac{1}{4} \frac{(32\lambda^2)^2}{(1+\delta\lambda^2)^4 (\sqrt{2}\lambda \sinh \sqrt{2}\lambda\pi)^2} \left\{ \frac{1}{2} + \frac{\sinh 2\sqrt{2}\lambda\pi}{4\sqrt{2}\lambda\pi} \right\} + \frac{1}{8} \left\{ \frac{H\pi^2}{32} \frac{1-\delta\lambda^2}{1+\delta\lambda^2} \right\}^2 + \frac{1}{2} \left(\frac{H}{E}\right)^2 \Bigg\} \\
\hline
\frac{1}{4ab} \int_0^a \int_0^b \delta^2 dx dy &= \left(\frac{a}{b}\right)^2 \left(\frac{b}{a}\right)^2 \lambda^2 \left[-\frac{64\lambda^4}{(1+\delta\lambda^2)^4} + \frac{16\lambda^2}{(1+\delta\lambda^2)^2} \right] - \frac{2\lambda^2}{1+\delta\lambda^2} - \frac{128\lambda^4 \cos \sqrt{2}\lambda\pi}{(1+\delta\lambda^2)^3 (\sqrt{2}\lambda\pi \sinh \sqrt{2}\lambda\pi)} \Bigg\} \\
&+ \frac{1}{2} \frac{64\lambda^4}{(1+\delta\lambda^2)^2} \left(\frac{\pi^2}{24} - \frac{1}{4}\right) + \frac{128\lambda^4}{(1+\delta\lambda^2)^3} \left\{ -\frac{1}{(1+\delta\lambda^2)} - \frac{16\lambda^2 \cos \sqrt{2}\lambda\pi}{(1+\delta\lambda^2)^3 (\sqrt{2}\lambda\pi \sinh \sqrt{2}\lambda\pi)} \right\} \\
&+ \frac{\lambda^2}{4} \frac{(32\lambda^2)^2}{(1+\delta\lambda^2)^4 (\sqrt{2}\lambda \sinh \sqrt{2}\lambda\pi)^2} \left\{ -\frac{1}{2} + \frac{\sinh 2\sqrt{2}\lambda\pi}{4\sqrt{2}\lambda\pi} \right\} + \frac{1}{4} \left(\frac{H\pi^2}{8} \frac{\lambda^2}{1+\delta\lambda^2}\right)^2 \Bigg]
\end{aligned}$$