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Intuitionistic Fuzzy Information Aggregation: Theory and Applications

Zeshui Xu Xiaoqiang Cai

(直觉模糊信息集成理论及应用)

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Preface

Since it was introduced by Zadeh in 1965, the fuzzy set theory has been widely applied in various fields of modern society. Central to the fuzzy set is the extension of the characteristic function taking the value of 0 or 1 to the membership function which can take any value from the closed interval [0,1]. However, the membership function is only a single-valued function, which cannot be used to express the evidences of support and objection simultaneously in many practical situations. In the processes of cognition of things, people may not possess a sufficient level of knowledge of the problem domain, due to the increasing complexity of the socio-economic environments. In such cases, they usually have some uncertainty in providing their preferences over the objects considered, which makes the results of cognitive performance exhibit the characteristics of affirmation, negation, and hesitation. For example, in a voting event, in addition to support and objection, there is usually abstention which indicates hesitation and indeterminacy of the voter to the object. As the fuzzy set cannot be used to completely express all the information in such a problem, it faces a variety of limits in actual applications.

Atanassov (1983) extends the fuzzy set characterized by a membership function to the intuitionistic fuzzy set (IFS), which is characterized by a membership function, a non-membership function and a hesitancy function. As a result, the IFS can describe the fuzzy characters of things more detailedly and comprehensively, which is found to be more effective in dealing with vagueness and uncertainty. Over the last few decades, the IFS theory has been receiving more and more attention from researchers and practitioners, and has been applied to various fields, including decision making, logic programming, medical diagnosis, pattern recognition, robotic systems, fuzzy topology, machine learning and market prediction, etc.

The IFS theory is undergoing continuous in-depth study as well as continuous expansion of the scope of its applications. As such, it has been found that effective aggregation and processing of intuitionistic fuzzy information becomes increasingly important. Information processing tools, including aggregation techniques for intuitionistic fuzzy information, association measures, distance measures and similarity measures for IFSs, have broad prospects for actual applications, but pose many interesting yet challenging topics for research.

In this book, we will give a thorough and systematic introduction to the latest research results in intuitionistic fuzzy information aggregation theory and its applications to various fields such as decision making, medical diagnosis and pattern recognition, etc. The book is organized as follows:

Chapter 1 introduces the aggregation techniques for intuitionistic fuzzy information. We first define the concept of intuitionistic fuzzy number (IFN), and based on score function and accuracy function, give a ranking method for IFNs. We then define the operational laws of IFNs, and introduce a series of operators for aggregating intuitionistic fuzzy information. These include the intuitionistic fuzzy averaging operator, intuitionistic fuzzy bonferroni means, and intuitionistic fuzzy aggregation operators based on Choquet integral, to name just a few. The desirable properties of these operators are described in detail, and their applications to multi-attribute decision making are also discussed.

Chapter 2 mainly introduces the aggregation techniques for interval-valued intuitionistic fuzzy information. We first define the concept of interval-valued intuitionistic fuzzy number (IVIFN), and introduce some basic operational laws of IVIFNs. We then define the concepts of score function and accuracy function of IVIFNs, based on which a simple method for ranking IVIFNs is presented. We also introduce a number of operators for aggregating interval-valued intuitionistic fuzzy information, including the interval-valued intuitionistic fuzzy averaging operator, the interval-valued intuitionistic fuzzy geometric operator, the interval-valued intuitionistic fuzzy aggregation operators based on Choquet integral, and many others. The interval-valued intuitionistic fuzzy aggregation operators are applied to the field of decision making, and some approaches to multi-attribute decision making based on interval-valued intuitionistic fuzzy information are developed.

Chapter 3 introduces three types of measures: association measures, distance measures, and similarity measures for IFSs and interval-valued intuitionistic fuzzy sets (IVIFSs).

Chapter 4 introduces decision making approaches based on intuitionistic preference relation. We first define preference relations, then introduce the concepts of interval-valued intuitionistic fuzzy positive and negative ideal points. We also utilize aggregation tools to establish models for multi-attribute decision making. Approaches to multi-attribute decision making in interval-valued intuitionistic fuzzy environments are also developed. Finally, consistency analysis on group decision making with intuitionistic preference relations is given.

Chapter 5 introduces multi-attribute decision making with IFN/IVIFN attribute values and known or unknown attribute weights. We introduce the concepts such as relative intuitionistic fuzzy ideal solution, relative uncertain intuitionistic fuzzy ideal solution, modules of IFNs and IVIFNs, etc. We then establish projection models to measure the similarity degree between each alternative and the relative intuitionistic fuzzy ideal solution and the similarity degree between each alternative and the relative uncertain intuitionistic fuzzy ideal solution, by which the best alternative can be obtained.

Chapter 6 introduces aggregation techniques for dynamic intuitionistic fuzzy information and methods for weighting time series. We introduce the concepts of intuitionistic fuzzy variable and uncertain intuitionistic fuzzy variable. We describe dynamic intuitionistic averaging operators, based on which dynamic intuitionistic fuzzy multi-attribute decision making and uncertain dynamic intuitionistic fuzzy multi-attribute decision making problems are tackled.

Chapter 7 considers multi-attribute group decision making problems in which the attribute values provided by experts are expressed in IFNs, and the weight information about both the experts and the attributes is to be determined. We introduce two nonlinear optimization models, from which exact formulas can be obtained to derive the weights of experts. To facilitate group consensus, we introduce a nonlinear optimization model based on individual intuitionistic fuzzy decision matrices. A simple procedure is used to rank the alternatives. The results are also extended to interval-valued intuitionistic fuzzy situations.

This book is suitable for practitioners and researchers working in the fields of fuzzy mathematics, operations research, information science and management science and engineering, etc. It can also be used as a textbook for postgraduate and senior-year undergraduate students.

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Zeshui Xu, Xiaoqiang Cai
Hong Kong
September, 2011

Contents

Chapter 1

Intuitionistic Fuzzy Information Aggregation

The fuzzy set theory has been extensively applied in various fields of modern society (Chen et al., 2005) since its introduction by Zadeh (1965) in 1960s. Central to the fuzzy set is the extension from the characteristic function taking the value of 0 or 1 to the membership function which can take any value from the closed interval [0,1]. However, the membership function is only a single-valued function, which cannot be used to express the support and objection evidences simultaneously in many practical situations.

In cognition of things, people may not possess a precise or sufficient level of knowledge of the problem domain, due to the complexity of the socio-economic environment. In such cases, they usually have some uncertainty in providing their preferences over the objects considered, which makes the results of cognitive performance exhibit the characteristics of affirmation, negation and hesitation. For example, in a voting problem, in addition to "support" and "objection", there is usually "abstention" which indicates the hesitation and indeterminacy of the voter regarding the object. Because a fuzzy set cannot be used to completely express all the information in such problems, its applicability is often limited in many practical situations.

Atanassov (1986; 1983) generalizes Zadeh's fuzzy set theory with the concept of intuitionistic fuzzy set (IFS), which is characterized by a membership function, a non-member function, and a hesitancy (indeterminancy) function. It is argued that IFS can describe the fuzzy characters of things more detailedly and comprehensively, and is therefore more useful in dealing with vagueness and uncertainty than the classical fuzzy set theory. Over the last few decades, researchers have paid great attention to investigation of the IFS theory, and achieved fruitful results (Atanassov, 1999; Bustince et al., 2007). Atanassov (1986) and De et al. (2000) introduce several basic operations on IFSs, including "intersection", "union", "supplement", and "power". However, as the study of the IFS theory expands in both depth and scope, effective aggregation and handling of intuitionistic fuzzy information has become necessary and increasingly important. These basic operations on IFSs have been far from meeting the actual needs.

In recent years, Xu (2010c; 2007e), Xu and Xia (2011), Xu and Yager (2011; 2006), and Zhao et al. (2010) have focused on the subject of aggregation techniques for intuitionistic fuzzy information. They have defined the concept of intuitionistic fuzzy number and introduced, based on the score function and the accuracy function, a ranking method for intuitionistic fuzzy numbers. They have further defined operational laws of intuitionistic fuzzy numbers, and introduced a series of operators for aggregating intuitionistic fuzzy information, including the intuitionistic fuzzy averaging operator, intuitionistic fuzzy weighted averaging operator, intuitionistic fuzzy ordered weighted averaging operator, intuitionistic fuzzy hybrid averaging operator, intuitionistic fuzzy geometric operator, intuitionistic fuzzy weighted geometric operator, intuitionistic fuzzy ordered weighted geometric operator, intuitionistic fuzzy hybrid geometric operator, intuitionistic fuzzy bonferroni means, generalized intuitionistic fuzzy aggregation operators, intuitionistic fuzzy aggregation operators based on Choquet integral, induced generalized intuitionistic fuzzy aggregation operators, etc. They have also applied these operators to the field of multi-attribute decision making.

1.1 Intuitionistic Fuzzy Sets

We first introduce the concept of Zadeh's fuzzy set:

Definition 1.1.1 (Zadeh, 1965) Let X be a fixed set. Then

$$F = \{\langle x,\ \mu_F(x)\rangle \,|\, x \in X\ \} \tag{1.1}$$

is called a fuzzy set, where μ_F is the membership function of F, $\mu_F : X \to [0,\ 1]$, and $\mu_F(x)$ indicates the membership degree of the element x to F, which is a single value belonging to the unit closed interval $[0,\ 1]$.

Atanassov (1986; 1983) generalizes Zadeh's fuzzy set with the concept of intuitionistic fuzzy set (IFS) as defined below:

Definition 1.1.2 (Atanassov, 1986; 1983) An IFS is an object having the following form:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \,|\, x \in X\} \tag{1.2}$$

which is characterized by a membership function:

$$\mu_A : X \to [0,1], \quad x \in X \ \to\ \mu_A(x) \in [0,1] \tag{1.3}$$

and a non-membership function:

$$\nu_A : X \to [0,1], \quad x \in X \ \to\ \nu_A(x) \in [0,1] \tag{1.4}$$

with the condition:

$$0 \leqslant \mu_A(x) + \nu_A(x) \leqslant 1, \quad \text{for all} \quad x \in X \tag{1.5}$$

where $\mu_A(x)$ and $\nu_A(x)$ represent, respectively, the membership degree and the non-membership degree of x in A.

Moreover, for each IFS A in X, if

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x), \quad \text{for all} \quad x \in X \tag{1.6}$$

then $\pi_A(x)$ is called an indeterminancy degree of x to A.

Szmidt and Kacprzyk (2000) call $\pi_A(x)$ an intuitionistic index of x in A. It is a hesitancy degree of x to A. Obviously,

$$0 \leqslant \pi_A(x) \leqslant 1, \quad \text{for all} \quad x \in X \tag{1.7}$$

In particular, if

$$\pi_A(x) = 1 - \mu_A(x) - [1 - \mu_A(x)] = 0, \quad x \in X \tag{1.8}$$

then A reduces to Zadeh's fuzzy set. Thus, fuzzy sets are the special cases of IFSs.

For convenience, Xu (2007e) calls $\alpha = (\mu_\alpha,\ \nu_\alpha)$ an intuitionistic fuzzy number (IFN) or an intuitionistic fuzzy value (IFV), where

$$\mu_\alpha \in [0,1], \quad \nu_\alpha \in [0,1], \quad \mu_\alpha + \nu_\alpha \leqslant 1 \tag{1.9}$$

and let Θ be the set of all IFNs. Clearly, $\alpha^+ = (1,\ 0)$ is the largest IFN, and $\alpha^- = (0,\ 1)$ is the smallest IFN.

Each IFN $\alpha = (\mu_\alpha, \nu_\alpha)$ has a physical interpretation. For example, if $\alpha = (0.5,\ 0.3)$, then we can see that $\mu_\alpha = 0.5$ and $\nu_\alpha = 0.3$. It can be interpreted as "the vote for resolution is 5 in favor, 3 against, and 2 abstentions".

For any IFN $\alpha = (\mu_\alpha,\ \nu_\alpha)$, the score of α can be evaluated by the score function s (Chen and Tan, 1994) as shown below:

$$s(\alpha) = \mu_\alpha - \nu_\alpha \tag{1.10}$$

where $s(\alpha) \in [-1,\ 1]$.

From Eq.(1.10), we can see that the score $s(\alpha)$ of the IFN α is directly related to the difference between the membership degree μ_α and the non-membership degree ν_α. The greater the difference $\mu_\alpha - \nu_\alpha$, the larger the score $s(\alpha)$, and then the larger the IFN α. In particular, if $s(\alpha) = 1$, then the IFN α takes the largest value $(1,\ 0)$; If $s(\alpha) = -1$, then α takes the smallest value $(0,\ 1)$.

Example 1.1.1 Let $\alpha_1 = (0.7, 0.2)$ and $\alpha_2 = (0.5, 0.3)$ be two IFNs. Then by Eq. (1.10), we can get the scores of α_1 and α_2 respectively:

$$s(\alpha_1) = 0.7 - 0.2 = 0.5, \quad s(\alpha_2) = 0.5 - 0.3 = 0.2$$

Since $s(\alpha_1) > s(\alpha_2)$, we have $\alpha_1 > \alpha_2$.

Gau and Buehrer (1993) define the concept of vague set. Bustince and Burillo (1996) point out that vague sets are IFSs. Chen and Tan (1994) utilize the max and min operations and the score function to develop an approach to multi-attribute decision making based on vague sets. However, in some special cases, this approach cannot be used to distinguish two IFNs.

Example 1.1.2 Let $\alpha_1 = (0.6,\ 0.2)$ and $\alpha_2 = (0.7,\ 0.3)$ be two IFNs. Then by Eq. (1.10), we have

$$s(\alpha_1) = 0.6 - 0.2 = 0.4, \quad s(\alpha_2) = 0.7 - 0.3 = 0.4$$

Since $s(\alpha_1) = s(\alpha_2)$, we cannot tell the difference between α_1 and α_2 by using the score function.

Hong and Choi (2000) define an accuracy function:

$$h(\alpha) = \mu_\alpha + \nu_\alpha \tag{1.11}$$

where $\alpha = (\mu_\alpha, \nu_\alpha)$ is an IFN, h is the accuracy function of α, and $h(\alpha)$ is the accuracy degree of α. The larger $h(\alpha)$, the higher the accuracy degree of the IFN α.

From Eqs.(1.6) and (1.11), the relationship between the hesitancy degree and the accuracy degree of the IFN α can be shown as follows:

$$\pi_\alpha + h(\alpha) = 1 \tag{1.12}$$

Hence, the smaller the hesitancy degree π_α, the bigger the accuracy degree $h(\alpha)$.

By Eq.(1.11), we can calculate the accuracy degrees of the IFNs α_1 and α_2 in Example 1.1.2:

$$h(\alpha_1) = 0.6 + 0.2 = 0.8, \quad h(\alpha_2) = 0.7 + 0.3 = 1$$

Then $h(\alpha_2) > h(\alpha_1)$, i.e., the accuracy degree of the IFN α_2 is higher than that of the IFN α_1.

Hong and Choi (2000) show that the relation between the score function s and the accuracy function h is similar to the relation between the mean and the variance in statistics. It is well known that an efficient estimator is a measure of the variance of an estimate's sampling distribution in statistics, i.e., the smaller the variance, the better the performance of the estimator. Based on this idea, it is meaningful and appropriate to stipulate that the higher the accuracy degree $h(\alpha)$, the better the IFN α. Consequently, α_2 is larger than α_1.

Motivated by the above analysis, Xu and Yager (2006) develop a method for comparison and ranking of two IFNs, which is based on the score function s and the accuracy function h as defined below:

Definition 1.1.3 (Xu and Yager, 2006) Let $\alpha_1 = (\mu_{\alpha_1}, \nu_{\alpha_1})$ and $\alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2})$ be two IFNs, $s(\alpha_1) = \mu_{\alpha_1} - \nu_{\alpha_1}$ and $s(\alpha_2) = \mu_{\alpha_2} - \nu_{\alpha_2}$ the scores of the IFNs α_1 and α_2 respectively, and $h(\alpha_1) = \mu_{\alpha_1} + \nu_{\alpha_1}$ and $h(\alpha_2) = \mu_{\alpha_2} + \nu_{\alpha_2}$ the accuracy degrees of the IFNs α_1 and α_2 respectively. Then

- If $s(\alpha_1) < s(\alpha_2)$, then the IFN α_1 is smaller than the IFN α_2, denoted by $\alpha_1 < \alpha_2$.
- If $s(\alpha_1) = s(\alpha_2)$, then

(1) If $h(\alpha_1) = h(\alpha_2)$, the IFNs α_1 and α_2 are equal, i.e., $\mu_{\alpha_1} = \mu_{\alpha_2}$ and $\nu_{\alpha_1} = \nu_{\alpha_2}$, denoted by $\alpha_1 = \alpha_2$;

(2) If $h(\alpha_1) < h(\alpha_2)$, the IFN α_1 is smaller than the IFN α_2, denoted by $\alpha_1 < \alpha_2$;

(3) If $h(\alpha_1) > h(\alpha_2)$, the IFN α_1 is larger than α_2, denoted by $\alpha_1 > \alpha_2$.

Hong and Choi (2000) further strengthen the decision making method of Chen and Tan (1994). They utilize the score function, the accuracy function, and the max and min operations to develop another technique for handling multi-attribute decision making with intuitionistic fuzzy information. However, the main problem of the aforementioned techniques using the minimum and maximum operations to carry the combination process is the loss of information, and hence a lack of precision in the final results. Therefore, "how to aggregate a collection of IFNs without any loss of information" is an interesting research topic (Xu, 2007e).

Up to now, many operators have been proposed for aggregating information in various decision making environments (Calvo et al., 2002; Xu 2007g; 2004e; Xu and Da, 2003b; Yager and Kacprzyk, 1997). Four of the most common operators for aggregating arguments are the weighted averaging operator (Harsanyi, 1955), weighted geometric operator (Saaty, 1980), ordered weighted averaging operator (Yager, 1988) and ordered weighted geometric operator (Chiclana et al., 2001b; Xu and Da, 2002a), which are defined as follows respectively:

Definition 1.1.4 (Harsanyi, 1955) Let WA : $(Re)^n \rightarrow Re$, and a_j $(j = 1, 2, \cdots, n)$ be a collection of real numbers. If

$$\mathrm{WA}_\omega(a_1, a_2, \cdots, a_n) = \sum_{j=1}^{n} \omega_j a_j \tag{1.13}$$

then the function WA is called a weighted averaging (WA) operator, where Re is the set of all real numbers, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$ is the weight vector of a_j $(j =$

$1, 2, \cdots, n)$, with $\omega_j \in [0, 1]$ $(j = 1, 2, \cdots, n)$ and $\sum_{j=1}^{n} \omega_j = 1$.

Definition 1.1.5 (Saaty, 1980) Let $\text{WG} : (Re)^{+n} \to (Re)^{+}$. If

$$\text{WG}_\omega(a_1, a_2, \cdots, a_n) = \prod_{j=1}^{n} a_j^{\omega_j} \tag{1.14}$$

then the function WG is called a weighted geometric (WG) operator, where $(Re)^{+}$ is the set of all positive real numbers, and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$ is the exponential weighting vector of a_j $(j = 1, 2, \cdots, n)$, with $\omega_j \in [0, 1]$ $(j = 1, 2, \cdots, n)$ and $\sum_{j=1}^{n} \omega_j = 1$.

Both the WA and WG operators first weight all the given arguments a_j $(j = 1, 2, \cdots, n)$, and then aggregate these weighted arguments. The difference between these two operators is that the WG operator is much more sensitive to the given arguments. Especially in the case where there is an argument taking the value of zero, the aggregated value of these arguments by using the WG operator must be zero no matter what the other given arguments are.

Definition 1.1.6 (Yager, 1988) Let $\text{OWA} : (Re)^{n} \to Re$. If

$$\text{OWA}_w(a_1, a_2, \cdots, a_n) = \sum_{j=1}^{n} w_j b_j \tag{1.15}$$

where $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ is the weighting vector associated with the function OWA, with $w_j \in [0, 1]$, $j = 1, 2, \cdots, n$, $\sum_{j=1}^{n} w_j = 1$, and b_j is the j-th largest of a_j $(j = 1, 2, \cdots, n)$, then the function OWA is called an ordered weighted averaging (OWA) operator.

Definition 1.1.7 (Chiclana et al., 2001b; Xu and Da, 2002a) Let $\text{OWG} : (Re)^{+n} \to (Re)^{+}$. If

$$\text{OWG}_w(a_1, a_2, \cdots, a_n) = \prod_{j=1}^{n} b_j^{w_j} \tag{1.16}$$

where $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ is the exponential weighting vector associated with the function OWG, with $w_j \in [0, 1]$, $j = 1, 2, \cdots, n$, $\sum_{j=1}^{n} w_j = 1$, and b_j is the j-th largest of a_j $(j = 1, 2, \cdots, n)$, then the function OWG is called an ordered weighted geometric (OWG) operator.

The fundamental characteristic of the OWA and OWG operators is that they first rearrange all the given arguments in descending order, and then weight the ordered positions of the arguments. These two operators aggregate the ordered arguments together with the weights of their ordered positions. Obviously, the argument a_i is not associated with the particular weight w_i. Instead, the weight w_i is associated with the particular ordered position i of the arguments. Thus, w_i is also called a position weight. The OWG operator is developed on the basis of the OWA operator and the geometric mean.

The above four operators are generally suitable to aggregate the arguments taking the values of real numbers. They have been extended to accommodate uncertain or fuzzy linguistic environments, see (Bordogna et al., 1997; Delgado et al., 1993; Herrera et al., 2005; 1996b; Herrera and Martinez, 2000a; 2000b; Xu, 2008a; 2007g; 2006a; 2006b; 2006c; 2006g; 2004a; 2004d; Xu and Da, 2002b; Yager, 2004c; 2003a; 2003b; 1996; 1995; Zhang and Xu, 2005). Xu (2010c; 2007e), Xu and Xia (2011), Xu and Yager (2011; 2006), and Zhao et al. (2010) have further generalized them to accommodate intuitionistic fuzzy environments and investigated the aggregation techniques for intuitionistic fuzzy information.

1.2 Operational Laws of Intuitionistic Fuzzy Numbers

Theorem 1.2.1 (Xu, 2007e) Let $\alpha_1 = (\mu_{\alpha_1},\ \nu_{\alpha_1})$ and $\alpha_2 = (\mu_{\alpha_2},\ \nu_{\alpha_2})$ be two IFNs. Then

$$\alpha_1 \leqslant \alpha_2 \Leftarrow \mu_{\alpha_1} \leqslant \mu_{\alpha_2} \quad \text{and} \quad \nu_{\alpha_1} \geqslant \nu_{\alpha_2} \tag{1.17}$$

Proof Since $s(\alpha_1) = \mu_{\alpha_1} - \nu_{\alpha_1}$, $s(\alpha_2) = \mu_{\alpha_2} - \nu_{\alpha_2}$, $\mu_{\alpha_1} \leqslant \mu_{\alpha_2}$ and $\nu_{\alpha_1} \geqslant \nu_{\alpha_2}$, we have

$$\begin{aligned} s(\alpha_1) - s(\alpha_2) &= (\mu_{\alpha_1} - \nu_{\alpha_1}) - (\mu_{\alpha_2} - \nu_{\alpha_2}) \\ &= (\mu_{\alpha_1} - \mu_{\alpha_2}) + (\nu_{\alpha_2} - \nu_{\alpha_1}) \end{aligned}$$

If $\mu_{\alpha_1} = \mu_{\alpha_2}$ and $\nu_{\alpha_1} = \nu_{\alpha_2}$, then $\alpha_1 = \alpha_2$; Otherwise $s(\alpha_1) - s(\alpha_2) < 0$, i.e., $s(\alpha_1) < s(\alpha_2)$. Hence, by Definition 1.1.3, we have $\alpha_1 < \alpha_2$. The proof is completed.

Goguen (1967) defines an L-fuzzy set on X as an $X \to L$ mapping, which is a generalization of the concept of fuzzy set. It covers the fuzzy set as a special case when $L = [0,\ 1]$, where L is a complete lattice equipped with an operator satisfying certain conditions. For example, Deschrijver and Kerre (2003b) define a complete lattice as a partially ordered set $(L,\ \leqslant_L)$ such that every nonempty subset of L has a supremum and an infimum.

A traditional relation on the lattice $(L,\ \leqslant_L)$, defined by

$$\alpha_1 \leqslant_L \alpha_2 \Leftrightarrow \mu_{\alpha_1} \leqslant \mu_{\alpha_2} \quad \text{and} \quad \nu_{\alpha_1} \geqslant \nu_{\alpha_2} \tag{1.18}$$

is also applied to the operations of IFSs (Atanassov, 1986; Cornelis et al., 2004).

However, in some situations, Eq.(1.18) cannot be used to compare IFNs. For example, let $\alpha_1 = (\mu_{\alpha_1}, \nu_{\alpha_1}) = (0.2, 0.4)$ and $\alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2}) = (0.4, 0.5)$, where $\mu_{\alpha_1} = 0.2 < \mu_{\alpha_2} = 0.4$ and $\nu_{\alpha_1} = 0.4 < \nu_{\alpha_2} = 0.5$. Then it is impossible to know which one is larger by using Eq.(1.18). But in this case, we can use Definition 1.1.3 to compare them. In fact, since

$$s(\alpha_1) = 0.2 - 0.4 = -0.2, \quad s(\alpha_2) = 0.4 - 0.5 = -0.1$$

it follows from Definition 1.1.3 that $\alpha_1 < \alpha_2$.

Another well-known relation on the lattice $(L, \leqslant_L)$, defined by

$$\alpha_1 \prec_L \alpha_2 \Leftrightarrow \mu_{\alpha_1} \leqslant \mu_{\alpha_2} \quad \text{and} \quad \nu_{\alpha_1} \leqslant \nu_{\alpha_2} \tag{1.19}$$

does not conform to the implication of IFS (Atanassov, 1986).

Atanassov (1986) and De et al. (2000) introduce some basic operations on IFSs, which not only ensure that the operational results are also IFSs, but also are useful in the calculus of linguistic variables in an intuitionistic fuzzy environment:

Definition 1.2.1 (Atanassov, 1986) Let a set X be fixed, and let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$, $A_1 = \{\langle x, \mu_{A_1}(x), \nu_{A_1}(x)\rangle \mid x \in X\}$ and $A_2 = \{\langle x, \mu_{A_2}(x), \nu_{A_2}(x)\rangle \mid x \in X\}$ be three IFSs. Then the following operations are valid:

(1) $\bar{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}$;

(2) $A_1 \cap A_2 = \{\langle x, \min\{\mu_{A_1}(x), \mu_{A_2}(x)\}, \max\{\nu_{A_1}(x), \nu_{A_2}(x)\}\rangle \mid x \in X\}$;

(3) $A_1 \cup A_2 = \{\langle x, \max\{\mu_{A_1}(x), \mu_{A_2}(x)\}, \min\{\nu_{A_1}(x), \nu_{A_2}(x)\}\rangle \mid x \in X\}$;

(4) $A_1 + A_2 = \{\langle x, \mu_{A_1}(x) + \mu_{A_2}(x) - \mu_{A_1}(x)\,\mu_{A_2}(x), \nu_{A_1}(x)\,\nu_{A_2}(x)\rangle \mid x \in X\}$;

(5) $A_1 \cdot A_2 = \{\langle x, \mu_{A_1}(x)\,\mu_{A_2}(x), \nu_{A_1}(x) + \nu_{A_2}(x) - \nu_{A_1}(x)\,\nu_{A_2}(x)\rangle \mid x \in X\}$.

De et al. (2000) further give another two operations of IFSs:

(6) $nA = \{\langle x, 1-(1-\mu_A(x))^n, (\nu_A(x))^n\rangle \mid x \in X\}$;

(7) $A^n = \{\langle x, (\mu_A(x))^n, 1-(1-\nu_A(x))^n\rangle \mid x \in X\}$,

where n is a positive integer.

Motivated by the above operations, Xu (2007e), Xu and Yager (2006) define some basic operational laws of IFNs, which will be used in the remainder of this book:

Definition 1.2.2 (Xu, 2007e; Xu and Yager, 2006) Let $\alpha = (\mu_\alpha, \nu_\alpha)$, $\alpha_1 = (\mu_{\alpha_1}, \nu_{\alpha_1})$ and $\alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2})$ be IFNs. Then

(1) $\bar{\alpha} = (\nu_\alpha, \mu_\alpha)$;

(2) $\alpha_1 \wedge \alpha_2 = (\min\{\mu_{\alpha_1}, \mu_{\alpha_2}\}, \max\{\nu_{\alpha_1}, \nu_{\alpha_2}\})$;

(3) $\alpha_1 \vee \alpha_2 = (\max\{\mu_{\alpha_1}, \mu_{\alpha_2}\}, \min\{\nu_{\alpha_1}, \nu_{\alpha_2}\})$;

(4) $\alpha_1 \oplus \alpha_2 = (\mu_{\alpha_1} + \mu_{\alpha_2} - \mu_{\alpha_1}\mu_{\alpha_2}, \nu_{\alpha_1}\nu_{\alpha_2})$;

(5) $\alpha_1 \otimes \alpha_2 = (\mu_{\alpha_1}\mu_{\alpha_2}, \nu_{\alpha_1} + \nu_{\alpha_2} - \nu_{\alpha_1}\nu_{\alpha_2})$;

(6) $\lambda\alpha = (1-(1-\mu_\alpha)^\lambda, \nu_\alpha^\lambda)$, $\lambda > 0$;