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# Interest Rate Models: An Infinite Dimensional Stochastic Analysis Perspective

利率模型



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*To Lenny Gross*

# Preface

The level of complexity of the bond market is higher than for the equity markets: one simple reason is contained in the fact that the underlying instruments on which the derivatives are written are more sophisticated than mere shares of stock. As a consequence, the mathematical models needed to describe their time evolution will have to be more involved. Indeed on each given day $t$, instead of being given by a single number $S_t$ as the price of one share of a common stock, the term structure of interest rates is given by a curve determined by a finite discrete set of values. This curve is interpreted as the sampling of the graph of a function $T \hookrightarrow P(t, T)$ of the date of maturity of the instrument. In particular, whenever we have to deal with stock models involving ordinary or stochastic differential equations or finite dimensional dynamical systems, we will have to deal with stochastic partial differential equations or infinite dimensional systems!

The main goal of the book is to present, in a self-contained manner, the empirical facts needed to understand the sophisticated mathematical models developed by the financial mathematics community over the last decade. So after a very elementary introduction to the mechanics of the bond market, and a thorough statistical analysis of the data available to any curious spectator without any special inside track information, we gradually introduce the mathematical tools needed to analyze the stochastic models most widely used in the industry. Our point of view has been strongly influenced by recent works of Cont and his collaborators and the Ph.D. of Filipović. They merge the original proposal of Musiela inviting us to rewrite the HJM model as a stochastic partial differential equation, together with Björk's proposal to recast the HJM model in the framework of stochastic differential equations in a Banach space. The main thrust of the book is to present this approach from scratch, in a rigorous and self-contained manner.

**Quick Summary.** The first part comprises two chapters. The first one is very practical. Starting from scratch, it offers a lowbrow presentation of the bond markets. This chapter is of a descriptive nature, and it can be skipped by readers familiar with the mechanics of these markets or those who are more interested in mathematical models. The presentation is self-contained and no statistical prerequisites are needed despite the detailed discussion of Principal Component Analysis (PCA for short) and curve estimation. On the other hand, the discussion of the factor models found in the following chapter assumes some familiarity with Itô's stochastic calculus and the rudiments of the Black–Scholes pricing theory. This first part of the book constitutes a good introduction to the fixed income markets at the level of a Master in Quantitative Finance.

The second part is a course on infinite dimensional analysis. If it weren't for the fact that the choice of topics was motivated by the presentation of the stochastic partial differential and random field approaches to fixed income models, this part could be viewed as stand-alone text. Prompted by issues raised at the end of Part I, we introduce the theory of infinite dimensional Itô processes, and we develop the tools of infinite dimensional stochastic analysis (including Malliavin calculus) for the purpose of the general fixed income models we study in Part III of the book.

The last part of the book resumes the analysis of fixed income market models where it was left off at the end of Part I. The dynamics of the term structure are recast as a stochastic system in a function space, and the results of Part II are brought to bear to analyze these infinite dimensional dynamical systems both from the geometric and probabilistic point of views. Old models are revisited and new financial results are derived and explained in the light of infinitely many sources of randomness.

**Acknowledgments.** The first version of the manuscript was prepared as a set of lecture notes for a graduate seminar given by the first-named author during the summer of 2000 at Princeton University. Subsequently, the crash course on the mechanics of the bond market was prepared in December of 2000 for the tutorial presented in Los Angeles at the IPAM on the 3rd, 4th and 5th of January 2001. Rough drafts of the following chapters were added in preparation for short courses given in Paris in January 2001 and Warwick in March of the same year. RC would like to thank Jaksa Cvitanic, Nizar Touzi, and David Elworthy respectively, for invitations to offer these short courses. MT acknowledges support during writing of this book from the National Science Foundation in the form of a VIGRE postdoctoral fellowship. He would also like to thank Thaleia Zariphopoulou for inviting him to give a course on this material at the University of Texas at Austin during the fall of 2003.

Finally, we would like to dedicate this book to Leonard Gross. The footprints of his seminal work on infinite dimensional stochastic analysis can be found all over the text: the depth of his contribution to this corner of mathematics cannot be emphasized enough. And to make matter even more

personal, RC would like to acknowledge an unrepayable personal debt to L. Gross for being an enlightening teacher, an enjoyable advisor, a role model for his humorous perspective on academia and life in general, and for being a trustworthy friend.

Princeton, NJ, October 2005 *René A. Carmona*
Cambridge, UK, October 2005 *Michael R. Tehranchi*

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# Part I

# The Term Structure of Interest Rates

# 1

# Data and Instruments of the Term Structure of Interest Rates

The size and the level of sophistication of the market of fixed income securities increased dramatically over the last twenty years and it became a prime test bed for financial institutions and academic research. The fundamental object to model is the term structure of interest rates, and we shall approach it via the prices of Treasury bond issues. Models for these prices are crucial for several reasons including the pricing of derivatives such as swaps, quantifying and managing financial risk, and setting monetary policy. We mostly restrict ourselves to Treasury issues to avoid credit issues and the likelihood of default.
We consider some of the fundamental statistical challenges of the bond markets after presenting a crash course on the mechanics of interest rates and the fixed income securities through which we gradually introduce concepts of increasing level of sophistication. This also gives us a chance to introduce the notation and the terminology used throughout the book.

## 1.1 Time Value of Money and Zero Coupon Bonds

We introduce the *time value of money* by valuing the simplest possible fixed income instrument. Like for all the other financial instruments considered in this book, we define it by specifying its cash flow. In the present situation, the instrument provides a single payment of a fixed amount (the principal or nominal value $X$) at a given date in the future. This date is called the maturity date. If the time to maturity is exactly $n$ years, the present value of this instrument is:

$$P(X,n) = \frac{1}{(1+r)^n}X. \tag{1.1}$$

This formula gives the present value of a nominal amount $X$ due in $n$ years time. Such an instrument is called a *discount bond* or a *zero coupon bond* because the only cash exchange takes place at the end of the life of the instrument, i.e. at the date of maturity. The positive number $r$ is referred to as the (yearly) *discount rate* or *spot interest rate* for time to maturity $n$ since it is the interest rate which is applicable today (hence the terminology *spot*) on an $n$-year loan. Formula (1.1) is the simplest way to quantify the adage: *one dollar is worth more today than later!*

### 1.1.1 Treasury Bills

Zero coupon bonds subject to pricing formula (1.1) do exist. Examples include Treasury bills (T-bills for short) which are securities issued by the US government with a time to maturity of one year or less. A noticeable difference with the other securities discussed later is the fact that *they do not carry coupon payments.*

Let us consider for example the case of an investor who buys a \$ 100000 13-week T-bill at a 6% yield (rate). The investor pays (approximately) \$ 98500 at the inception of the contract, and receives the nominal value \$ 100000 at maturity 13 weeks later. Since $13 = 52/4$ weeks represent one quarter, and since 6% is understood as an annual rate, the discount is computed as $100000 \times .06/4 = 1500$.

So in order to price a 5.1% rate T-bill which matures in 122 days, we first compute the discount rate:

$$\delta = 5.1 \times (122/360) = 1.728$$

which says that the investor receives a discount of \$ 1.728 per \$ 100 of nominal value. Consequently, the price of a T-bill with this (annual) rate and time to maturity should be:

$$\$\ 100 - \delta = 100 - 1.728 = 98.272$$

per \$ 100 of nominal value.

Rates, yields, spreads, etc. are usually quoted in *basis points.* There are 100 basis points in one percentage point. The Treasury issues bills with times to maturity of 13 weeks, 26 weeks and 52 weeks. These bills are called *three-month bills, six-month bills* and *one-year bills,* although these names are accurate only at their inception. Thirteen-week bills and twenty-six week bills are auctioned off every Monday while the fifty-two week bills are auctioned off once a month.

The T-bill market is high volume, and liquidity is not an issue. We reproduce below the market quotes for the US government Treasury bills for Friday September 2nd, 2005 (source: eSpeed/Cantor Fitzgerald).

Treasury Bills

| Maturity | Days to Mat. | Bid | Ask | Chg | Asked Yield |
|---|---|---|---|---|---|
| Sep 08 05 | 2 | 3.25 | 3.24 | +0.06 | 3.29 |
| Sep 15 05 | 9 | 3.46 | 3.45 | -0.01 | 3.50 |
| Sep 22 05 | 16 | 3.32 | 3.31 | -0.01 | 3.36 |
| Sep 29 05 | 23 | 3.32 | 3.31 | +0.02 | 3.36 |

| | | | | | |
|---|---|---|---|---|---|
| Oct 06 05 | 30 | 3.27 | 3.26 | +0.01 | 3.31 |
| Oct 13 05 | 37 | 3.27 | 3.26 | +0.02 | 3.32 |
| Oct 20 05 | 44 | 3.27 | 3.26 | +0.01 | 3.32 |
| Oct 27 05 | 51 | 3.26 | 3.25 | -0.01 | 3.31 |
| Nov 03 05 | 58 | 3.31 | 3.30 | +0.02 | 3.36 |
| Nov 10 05 | 65 | 3.32 | 3.31 | .... | 3.38 |
| Nov 17 05 | 72 | 3.34 | 3.33 | +0.01 | 3.40 |
| Nov 25 05 | 80 | 3.36 | 3.35 | +0.01 | 3.42 |
| Dec 01 05 | 86 | 3.38 | 3.37 | +0.01 | 3.44 |
| Dec 08 05 | 93 | 3.41 | 3.40 | +0.02 | 3.48 |
| Dec 15 05 | 100 | 3.41 | 3.40 | +0.02 | 3.48 |
| Dec 22 05 | 107 | 3.39 | 3.38 | -0.02 | 3.46 |
| Dec 29 05 | 114 | 3.42 | 3.41 | -0.01 | 3.50 |
| Jan 05 06 | 121 | 3.44 | 3.43 | +0.02 | 3.52 |
| Jan 12 06 | 128 | 3.45 | 3.44 | +0.01 | 3.53 |
| Jan 19 06 | 135 | 3.46 | 3.45 | +0.01 | 3.54 |
| Jan 26 06 | 142 | 3.47 | 3.46 | +0.01 | 3.56 |
| Feb 02 06 | 149 | 3.48 | 3.47 | .... | 3.57 |
| Feb 09 06 | 156 | 3.48 | 3.47 | -0.01 | 3.57 |
| Feb 16 06 | 163 | 3.50 | 3.49 | +0.01 | 3.60 |
| Feb 23 06 | 170 | 3.49 | 3.48 | -0.01 | 3.59 |
| Mar 02 06 | 177 | 3.51 | 3.50 | .... | 3.61 |

The first column gives the date of maturity of the bill, while the second column gives the number of days to maturity. See our discussion of day count conventions later in this chapter. The third and fourth columns give the bid and ask prices in decimal form. Compare with the bid and ask columns of the quotes we give below for Treasury notes and bonds on the same day, and the ensuing discussion.

### 1.1.2 Discount Factors and Interest Rates

Since the nominal value $X$ appears merely as a plain multiplicative factor in formula (1.1), it is convenient to assume that its value is equal to 1, and effectively drop it from the notation. This leads to the notion of discount factor. Discount factors can be viewed as quantities used at a given point in time to obtain the present value of future cash flows. At a given time $t$, the discount factor $P_{t,m}$ with time to maturity $m$, or maturity date $T = t + m$, is given by the formula:

$$P_{t,m} = \frac{1}{(1 + r_{t,m})^m} \tag{1.2}$$

where $r_{t,m}$ is the yield or yearly spot interest rate in force at time $t$ for this time to maturity. We assumed implicitly that the time to maturity $T - t$ is a whole number $m$ of years. Definition (1.2) can be rewritten in the form:

$$\log(1 + r_{t,m}) = -\frac{1}{m} \log P_{t,m}$$

and considering the fact that $\log(1 + x) \sim x$ when $x$ is small, the same definition gives the approximate identity:

$$r_{t,m} \sim -\frac{1}{m} \log P_{t,m}$$

which becomes an exact equality if we use continuous compounding. This formula justifies the terminology discount rate for $r$. Considering payments occurring in $m$ years time, the spot rate $r_{t,m}$ is the single rate of return used to discount all the cash flows for the discrete periods from time $t$ to time $t+m$. As such, it appears as some sort of composite of interest rates applicable over shorter periods. Moreover, this formula offers a natural generalization to continuous time models with continuous compounding of the interest. As we shall see later, this extension reads:

$$P(t,T) = e^{-(T-t)r(t,T)}. \tag{1.3}$$

where $P(t,t+m) = P_{t,m}$ and $r(t,t+m) = r_{t,m}$.

The discount factor is a very useful quantity. Indeed, according to the above discussion, the present value of any future cash flow can be computed by multiplying its nominal value by the appropriate value of the discount factor. We use the notation $P(t,T)$ to indicate that is the price at time $t$ of a zero coupon bond with maturity $T$.

The information contained in the graph of the discount factor as a function of the maturity $T$ (i.e the so-called discount function) is often repackaged in quantities which better quantify the returns associated with purchasing future cash flows at their present value. These quantities go under the names of *spot-interest-rate* curve, *par-yield* curve, and *implied forward rate* curve. This chapter is devoted to the introduction of these quantities in the discrete time setting, and to the definition of their analogs in the continuous time limit. The latter is a mathematical convenience which makes it possible to use the rules of the differential and integral calculus. It is somehow unrealistic because money is lent for discrete periods of time, but when these periods are short, the continuous time limit models become reasonable. We shall discuss later in the next chapter how to go from discrete data to continuous time models and vice versa.

## 1.2 Coupon Bearing Bonds

Now that we know what a zero coupon bond is, it is time to introduce the notion of coupon bearing bond. If a zero coupon bond was involving only one payment, what is called a bond (or a coupon bearing bond), is a regular stream of future cash flows of the same type. To be more specific, a *coupon bond* is a series of payments amounting to $C_1$, $C_2$, ... , $C_m$, at times $T_1$, $T_2$, ... , $T_m$, and a terminal payment $X$ at the maturity date $T_m$. The coupon payments are in arrears in the sense that at date $T_j$, the coupon payment is the reward for the interests accrued up until $T_j$. As before, $X$ is called the nominal value, or the face value, or the principal value of the bond. According to the above discussion of the discount factors, the bond price at time $t$ should be given by the formula:

$$B(t) = \sum_{t \leq T_j} C_j P(t,T_j) + XP(t,T_m). \tag{1.4}$$