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(Abstract Part)

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Foreword

Advanced algebra is an important basic course for students majoring in Mathematics, with matrix, vector, linear space and linear transformation as its main research objects. It plays a significant role in cultivating students' abilities in abstract thinking and logical reasoning, and lays a solid foundation for math majors' learning of some following courses. Since 2001, especially after the issuing of the 4th document and 46th document by the Department of Higher Education, Ministry of Education, in order to meet the nation's needs for cultivating innovating talents, almost all the colleges in China, led and encouraged actively and steadily by the Ministry of Education, have launched bilingual courses in relevant majors which are becoming increasingly rational and mature. But generally speaking, bilingual teaching in China is still in its empirical and exploratory stage with many problems waiting to be studied and solved, such as the relationship between bilingual teaching and native language teaching as well as between bilingual teaching and native culture, the relationship between bilingual teaching and foreign language teaching; the goals and standards for bilingual teaching, the course selection for bilingual teaching, the construction of teaching faculty and textbooks for bilingual teaching, the selection of the teaching modes and so on. In recent years, we have conducted research projects such as "The Bilingual Teaching of Science Courses in Western Colleges of China and the Research and Practice of Cultivating Bilingual Teaching Faculty" and "The Research and Experiment of Bilingual Teaching of Mathematical Curriculum in Colleges". Besides, we have selected and used some original English textbooks in the teaching practice of advanced algebra. Admittedly, these foreign textbooks do have some advantages in comprehensive design, integration of various kinds of teaching resources and in providing solutions for the overall teaching conduct. Furthermore, exposed to an authentic English teaching environment, students can better grasp the era background of the mathematical objects they are learning and the advanced teaching concepts, and appreciate the mathematical thinking of foreign scholars to present and analyze problems as well as their strategies, methods and application skills of solving problems. They can also check whether they have mastered the new knowledge and can put what they have learned into practice through the English teaching environment. Nevertheless, it has to be noticed that there exists a huge gap between English textbooks and current Chinese textbooks in terms of teaching contents, teaching systems, teaching modes and teaching habits. We are constantly faced with the issue of integrating the textbooks' internationalization with Chinese traditional education culture. Specifically, revisions and rearrangements have to be made to English textbooks due to limited teaching periods to better conform to the math course syllabus in terms of teaching contents and methods. This textbook is the result of conscientious research by our autonomous region-level teaching team in dealing with this situation and is one of the research fruits of the construction process of Algebra as a quality course of the autonomous region level.

Bilingual teaching practice has shown that freshmen in local universities are poor in the reception of professional English teaching. Therefore, it seems more appropriate for them to learn advanced algebra in a bilingual environment from the second semester of their first academic year. Following such a principle, the textbook only covers the abstract part. Four years' teaching practice has won the students' praise that the chapter arrangement of the book is reasonable and logical with typical examples in each chapter for analysis that cover the most general ideas and usual methods in advanced algebra.

By drawing on the teaching team's rich teaching experience, their research accomplishments in teaching reform and excellent foreign textbooks, this book has the following characteristics.

First, each chapter has a diagram at the very beginning to generalize the knowledge structure for readers to have an overall and clear idea of each chapter as soon as possible.

Second, combined with the teaching contents, some historical background and some mathematicians' lives are introduced, so that mathematical culture and history are absorbed to enhance the interests of students in math learning, expand their knowledge horizon and cultivate their mathematical literacy.

Third, numerous classic examples are selected to help with students' understanding and knowledge analogy, some of which are important conclusions themselves of this course. Through learning by examples, students can not only understand the abstract concepts and contents as well as the interrelationship between knowledge chains more easily, but also receive and digest the knowledge better and grasp mathematical ideas and methods of this course more clearly.

Fourth, nearly 20 exercises are listed in each section considering the fact that advanced algebra is one of the compulsory subjects for math majors in the Entrance Examination for Postgraduates and is included in the National Math Competition for College Students. The first 10 exercises are designed for beginners to have a deeper understanding of the abstract concepts of advanced algebra, thus covering the main contents of each chapter. And the last last 10 exercises are for those who aspire to be a postgraduate or want to participate in the competition. About 750 exercises are included in the book, some of which are taken from the Entrance Examination for Postgraduates and the National Math Competition for College Students. At the end of each chapter, there is a test to check teachers' teaching and students' learning as a periodical conclusion and review.

Fifth, this book is originally intended for math majors learning advanced algebra. However, since most of the contents are related to linear algebra, it can also be a reference for students majoring in engineering or economic management in their study of linear algebra. To meet the needs of undergraduates and postgraduates majoring in science who are good at self-study and wish to deepen their understanding in algebra, elective contents are added and marked with "※", upon which other beginners do not have to rack their brains.

This book would not have been made possible without the support and encouragement from

the leaders of Guangxi Normal College and its office of teaching affairs. It is funded by the National Natural Science Foundation Project(10961007), Guangxi Natural Science Foundation Project (2011GXNSFA018144,2010GXNSFB013048), Guangxi Education Department Research Project (200911MS145) and Guangxi New Century Higher Education Teaching Reform Project (2008C035, 2012JGA162). We have referred to some books and textbooks published in China and abroad as listed in this book to express our deepest thanks and gratitude.

This book is the result of cooperative design and collaboration of our autonomous region-level teaching team. Specifically, Ren Beishang and Liu Liming are in charge of the first three chapters and the seventh chapter; Su Huadong, Huang Qianxia and Yang Liying, the fourth and fifth chapters; Huang Qianxia and Ren Beishang, the sixth chapter; and Ren Beishang and Li Birong are responsible for the editing of the whole book with Liu Liming, Li Birong and Feng Jiajia as reviewers.

We need to extend our heartfelt gratitude to postgraduates Liu Xiaotong, Wang Jinjin, Wang Fu and undergraduates Chen Qiumei, Peng Yan, Li Hua, Wang Haifei, Ma Yiying and Que Xiaoming for their conscientiousness in the computer input. Our special thanks also go to undergraduates Dai Yarong, Zhao Ruju and Liu Junwei who have not only participated in the work of computer input but also written and checked drafts for some of the solutions for the exercises. Last but not least, we want to express our gratitude to all the students from grade 2003 to grade 2010 majoring in mathematics and applied mathematics in the School of Mathematical Sciences of Guangxi Normal College. It's our hope that bilingual teaching and the use of this textbook have left a wonderful memory in their college study with their increased passion in learning and in mathematics.

The process of writing this book is also a process of our learning. Owing to the limitation of our teaching ability and especially our English, mistakes are inevitable. Comments and criticism from experts and readers will be greatly appreciated.

Editor

February 2012

前　言

高等代数是数学专业的一门重要的基础课程。它以矩阵、向量、线性空间和线性变换作为主要的研究对象，对培养学生的抽象思维能力、逻辑推理能力，以及数学专业的若干后续课程的学习都起着非常重要的作用。

自 2001 年以来，尤其是教育部教高[2001]4 号文件和 46 号文件下发后，为了适应当前我国高校培养创新人才的需要，在教育部积极稳妥的引领和鼓励下，国内高校已在相关专业开展双语教学课程，并趋于理性和成熟。但总体来看，双语教学在国内还处在实验和探索的起步阶段，有很多问题亟待研究和解决，如双语教学与母语教学以及本国传统文化的关系、双语教学与外语教学的关系、双语教学的目标和标准、双语教学的课程选择、双语教学的师资建设和教材建设、双语教学的模式选择等。近年来，我们陆续开展了“西部高等院校理科专业课程群双语教学及‘双语’师资队伍培养和建设的研究与实践”、“高师院校数学专业课程双语教学的研究与实验”等课题的研究。配合项目的研究，我们在高等代数课程教学实践环节中，也曾挑选和试用了多部英文原版教材。外版教材确实在立体化配套、多种教学资源的整合及为课程提供整体教学解决方案等方面有不少可供借鉴之处。同时学生在原汁原味的专业英语教学活动中，能更近距离地了解所学的数学对象的时代背景、先进的教学理念；领略国外学者提出和分析问题的思想及解决问题的思路、方式和应用技巧；在实践专业英语学习的同时，检验自身通过外语学习和掌握新知识的能力。但一个不容忽视的问题是，外版教材与我国现行的教学内容、教学体系、教学模式和习惯存在着巨大的差异。我们自始至终面临着教材的国际化与本民族的文化教育传统相融合的问题。具体而言，由于课时所限而不得不对外文原版教材不断进行改编或调整，试图在教学内容和教学方式上更符合该课程教学大纲的要求。这本高等代数教材就是我们自治区级教学团队经过多年的教学实践，针对上述项目潜心研讨及自治区级精品课程“代数学”建设过程的研究成果之一。

双语教学实践表明，地方普通高校新生的专业英语的接受和消化能力较弱。因此，高等代数这门第一学年所开设的课程可能从下册再开展双语教学活动更适宜些。本着这样的原则，本教材只涵盖了高等代数课程抽象部分的内容。从近四年的连续使用情况来看，学生普遍反映教材章节安排自然合理，逻辑清晰，每一章都选择了一批有一定层次的典型例题进行分析讲解，例题的内容涵盖了相应章节中高等代数最常用的数学思想和方法。

本书的编者们吸取了多年的教学实践经验、教改研究成果和国外原版优秀教材的长处，本教材有以下特点。

(1) 每章前面都设置了知识脉络图解，以框图的方式概括了本章的知识结构，提纲挈领，一目了然。

(2) 结合教材的内容分别介绍了有关的历史和有关数学家的生平，将数学文化与数学历史渗透在教材中，以提高学生的学习兴趣，拓展学生的知识视野和培养学生的数学素养。

(3) 在每一章我们都选择了大批具有典型意义的例题，帮助学生举一反三，触类旁通，其中有一些就是本课程的重要结论。通过例题的学习，学生不仅可以更容易地理解抽象的数学概念和内容，疏通各知识链条环环相扣的彼此联系，而且更便于加深对课堂内容的吸纳和消化，从中掌握本

课程的数学思想和数学方法。

(4) 考虑到高等代数也是数学专业硕士研究生入学考试的一门必考科目及全国大学生数学竞赛的内容之一,本教材中几乎每节都配备了约 20 道习题。前 10 道一般为基本训练题,主要是为了加深初学者对高等代数中诸多抽象概念的理解,故基本覆盖了该章节的主要内容。后面约 10 题则作为有志报考硕士研究生和参与全国大学生数学竞赛的学生的学习补充和提高训练。本书共配备了大约 750 道题,其中吸纳了部分硕士研究生入学考试及全国大学生数学竞赛的题目。每章最后还配置了一份自测试卷,方便任课教师、学生对自身的教、学作阶段性的小结和梳理。

(5) 本书是为学习高等代数课程的数学专业学生编写的,然而事实上高等代数的大部分内容都涵盖了线性代数课程的内容,所以本书也适合大学理工科与经济管理学科等相关专业学生在学习线性代数课程时作为参考书。为了满足自学能力较强和希望对代数学加深了解的本科生、理科研究生的学习愿望,还在部分章节中增设选修内容并加上了"※"号,其他初学者可不必为之大伤脑筋。

本书的编写始终得到广西师范学院及教务处领导的支持和鼓励,还得到下列各基金项目的资助:国家自然科学基金项目(10961007)、广西自然科学基金项目(2011GXNSFA018144,2010GXNSFB013048)、广西教育厅科研项目(200911MS145)、新世纪广西高等教育教学改革工程立项项目(2008C035,2012JGA162)。在编写过程中,我们也参阅了部分中外代数教材,在此特向原书作者致谢。

本教材是我们自治区级教学团队共同策划、分工协作的成果。具体分工如下:任北上、刘立明编写了第 1、2、3、7 章;苏华东、黄倩霞、杨立英编写了第 4、5 章;黄倩霞、任北上编写了第 6 章。最后由任北上、李碧荣统稿,刘立明、李碧荣、冯家佳审校。

我们要感谢硕士研究生刘晓瞳、王锦锦、王弗及本科生陈秋梅、彭燕、黎华、汪海飞、马艺瑛、阙晓明等同学在本书的编写过程中为书稿的电脑录入所付出的努力。感谢近年来我们的几位本科生——戴亚荣、赵汝菊和刘君伟,她们不仅参与了录入工作,而且编写并检验了一些习题解答的草稿。最后我们还要感谢广西师范学院数学科学学院数学与应用数学专业 2003 级至 2010 级的所有学生,但愿我们的双语教学工作和这本教材的使用能为他们的大学学习生涯留下一段美好的记忆,更希望没有削弱他们学习和热爱数学的热情。

编写本书的过程事实上也是我们不断学习的过程。由于教学能力及教学水平,尤其是英语水平有限,因此书中必定会有一些不妥之处,恳请专家和读者批评指正。

编　者

2012 年 2 月

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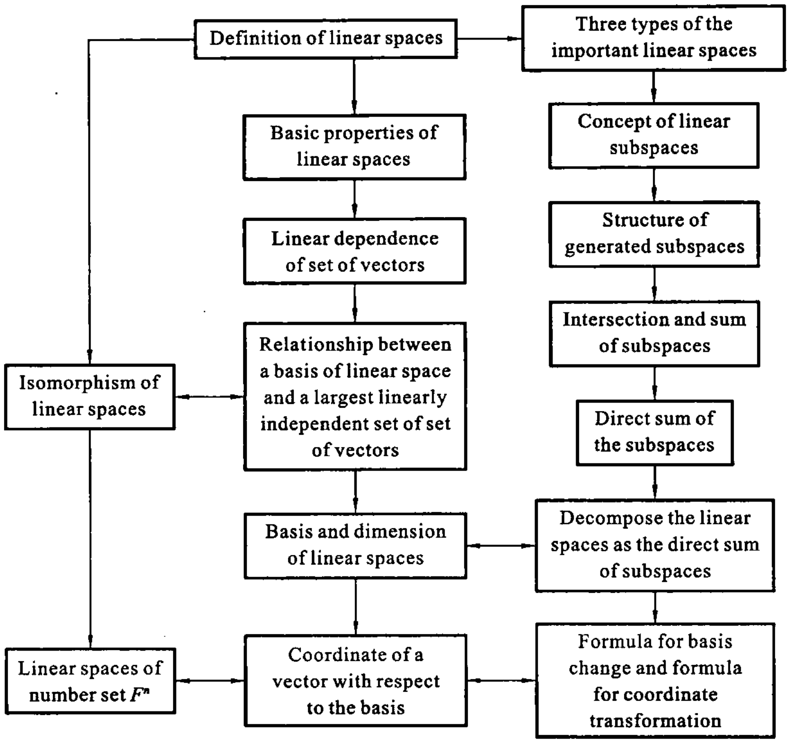
Interdependence of Chapter 1



1.1 Basic Concept (基本概念)

1.1.1 Integers (整数)

We already know **nature number**, **integers**, **rational numbers**, **real numbers** and **complex numbers** from middle school. Some basic properties of integers are discussed here.

An important property of the integers which we will often use is the so-called **Well Ordering Principle.** Since this property cannot be proved from the usual properties of arithmetic, we will take it as an axiom.

Well Ordering Principle Every nonempty set of positive integers contains a smallest member.

We denote the set of integers by $\mathbf{Z}$ and $a \in \mathbf{Z}$ means that a is an integer. The concept of divisibility plays a fundamental role in the theory of integers. We say a nonzero integer b **divides** an integer a, or that b is a **divisor** (or **factor**) of the integer a if there is an integer q such that $a=bq$. In this case, we sometimes write $b|a$(reads "b divides a") to denote this fact. If $b|a$, we say that a is a **multiple** of b. A **prime number**, denoted by p, is a positive integer greater than 1 whose only positive divisor is 1 and itself. Obviously, for any integer m, $\pm 1, \pm m$ are divisors of m, and they are called the **trivial factors** of m.

As our first application of the Well Ordering Principle, we establish a fundamental property of integer we will use often.

Division Algorithm Let a and b be integers with $b\neq 0$. Then there exist unique integers q and r with the property that $a=bq+r$ where $0\leqslant r<|b|$.

Proof. We first assume that $b>0$ and begin with the existence portion of the theorem. Consider the set $S=\{a-bk \mid k$ is an integer and $a-bk\geqslant 0\}$. If $0\in S$, then there is an integer k such that $a-bk=0$ and we may obtain the desired result with $q=k, r=0$. Now assume $0\notin S$. Since $S\neq\varnothing$, we may apply the Well Ordering Principle to conclude that S has a smallest member, say $r=a-bq$ for some integer $q\in \mathbf{Z}$. Then $a=bq+r$ and $r\geqslant 0$, so all that remains to be proved is that $r<b$. In fact, if $r>b$, then

$$a-b(q+1)=a-bq-b=r-b>0,$$

so that $a-b(q+1)\in S$. But $a-b(q+1)<a-bq=r$ and $r=a-bq$ is the smallest member of S, a contradiction. If $r=b$, it means that there exists an integer $q+1$ such that $a-b(q+1)=0$, i. e., $0\in S$, which is a contradiction. So, $r<b$.

To establish the uniqueness of q and r, let us suppose that there are integers q_1, q_2, r_1 and r_2 such that $a=bq_1+r_1$ with $0\leqslant r_1<b$ and $a=bq_2+r_2$ with $0\leqslant r_2<b$. For convenience, we may also suppose that $r_2\geqslant r_1$. Then we can obtain the equality $b(q_1-q_2)=r_2-r_1$. So, b divides r_2-r_1 and $0\leqslant r_2-r_1<r_2<b$. It follows that $r_2-r_1=0$ and therefore $r_2=r_1$ and $q_2=q_1$.

Similarly, we can complete the portion of the theorem in the case of $b<0$.

The integer q in the division algorithm is usually called the **quotient** upon dividing a by b; the integer r is called the **remainder** of a divided by b.

If an integer c is a divisor for both a and b, we say that c is a **common divisor** of a and b.

Definition 1.1 If d is a positive common divisor of a and b such that any common divisor c of a and b is a factor of d, we call d the **greater common divisor** of a and b and denote this integer by $d=\gcd(a,b)$ or simply $d=(a,b)$. When $(a,b)=1$, we say a and b are **relatively prime.**

We define the greater common divisor of $a_1, a_2, \cdots, a_m$ similarly and denote it as $\gcd(a_1, a_2, \cdots, a_m)$ or simply $(a_1, a_2, \cdots, a_m)$. It can be proved that

$$(a_1, a_2, \cdots, a_m)=((a_1, a_2), a_3, \cdots, a_m).$$

The following property of the greater common divisor of two integers plays a critical role in algebra. The proof provides an application of the division algorithm and our second application of the Well Ordering Principle. This result is Proposition 1 in Book Seven of Euclid's *Elements*, written about 300 B. C.

Theorem 1.1 For any integers a and b, there exist integers s and t such that $(a,b)=as+bt$. Moreover, when a and b are both nonzero integers, (a,b) is the smallest positive integer of the form $as+bt$.

Proof. If $a=b=0$, then $(a,b)=0$ and we are done. Now let $ab\neq 0$ and consider the set $S=\{am+bn \mid m, n$ are integers and $am+bn>0\}$. Since S is obviously nonempty (if some choice of m and n makes $am+bn<0$, then replace m and n by $-m$ and $-n$), the Well Ordering Principle asserts that S has a smallest member, say, $d=as+bt$. We claim that $d=\gcd(a,b)$. To verify this claim, use the division algorithm to write $a=dq+r$ where $0\leqslant r<d$. If $r>0$, then $r=a-dq=a-(as+bt)q=a-asq-btq=a(1-sq)+b(-tq)\in S$, contradicting the fact that d is the smallest member of S. So, $r=0$ and d divides a. Analogously (or better yet, by symmetry), d divides b as well. This proves that d is a common divisor of a and b. Now suppose d_1 is another common divisor of a and b and write $a=d_1h$ and $b=d_1k$. Then $d=as+bt=(d_1h)s+(d_1k)t=d_1(hs+kt)$ so that d_1 is a divisor of d. The proof of the last portion of this theorem is obvious.

Theorem 1.2(Euclid's Lemma) If p is a prime, then $p\mid ab$ implies $p\mid a$ or $p\mid b$.

Proof. Suppose that p does not divide a, then $(p,a)=1$. This shows that there exist integers s and t such that $1=as+pt$. Moreover, $b=abs+pbt$, and since p divides the right-hand side of the equation, we must show that p divides b.

Our next property shows that the primes are the building blocks for all integers. We will often use this property without explicitly saying so.

Theorem 1.3(Fundamental Theorem of Arithmetic) Every integer greater than 1 is a prime or a product of primes. This product is unique, except for the order in which the factors appear. Thus, if $n=p_1p_2\cdots p_r$ and $n=q_1q_2\cdots q_s$ where the $p's$ and $q's$ are primes, then $r=s$ and, after renumbering the $q's$, we have $p_i=q_i$ for all i.

Proof. Let $n>1$ be an integer. We prove first that n is a product of some primes. If n is a prime, then we are done. For general case we make use of induction. Suppose that any integer m with $1<m<n$ is a product of primes and that n is not a prime. Then there exists a non-trivial factor of n. That is $n=n_1n_2$ with $1<n_1,n_2<n$. The divisors n_1,n_2 are both products of primes by our hypothesis, and so is n. Secondly, suppose that $a=p_1p_2\cdots p_r=q_1q_2\cdots q_s$. Since p_1 is a prime and $p_1\mid a$, p_1 divides some q_i. We may suppose that p_1 divides q_1 by suitable arrangement of the order of q_i. Therefore $p_1=(p_1,q_1)=q_1$, since both p_1 and q_1 are primes. Also we have $p_2\cdots p_r=q_2\cdots q_s$. Similarly, by suitable arrangement of the order of q_i, we get $p_2=q_2,\cdots$, and finally $r=s$.

Another concept that frequently arises is that of the least common multiple of two integers.

Definition 1.2 If m is a positive common multiple of a and b such that any common multiple c of a and b is a multiple of m, we call m the **least common multiple** of a and b and denote this integer by $m=\mathrm{lcm}(a,b)$ or simply $m=[a,b]$.

1.1.2 Mappings (映射)

Definition 1.3 A **mapping** f from a set A to a set B is a correspondence rule that assigns to each element a of A exactly one element b of B. The set A is called the **domain** of f and B is called the **codomain** of f. If f assigns b to a, then b is called the **image** of a under f, and a is called an **inverse image** of b under f.

We use the shorthand $f: A \to B$ to mean that f is a mapping from A to B. We will write $f(a)=b$ to indicate that f carries a to b.

Remark. A correspondence rule f from set A to set B is a mapping if f satisfies the following two conditions:

(1) For each a in A, there exists an element b in B such that $f(a)=b$;

(2) If $f(a)=b$ and $f(a)=b_1$, then $b=b_1$.

The second condition is called "single-valueless". In specifying a definition one often says that "the map is well-defined" when one is assured that condition (2) holds.

Definition 1.4 Let $f: A \to B$ be a mapping, and $A_1 \subseteq A, B_1 \subseteq B$. The subset of B comprised of all the images of elements of A_1 is called the image of A_1 under f and denoted by $f(A_1)$. The subset of A containing all the inverse images of elements of B_1 is called the inverse image of B_1 under f and denoted by $f^{-1}(B_1)$.

Example 1.1 Given sets $A=\{a,b,c,d,e,g,h\}$ and $B=\{1,2,3,4,5,6,7,8\}$, let $f: A \to B$ be a mapping defined by $f(a)=2, f(b)=3, f(c)=5, f(d)=6, f(e)=8, f(g)=5, f(h)=8$. If $A_1=\{a,b,c\}, A_2=\{d,e,g,h\}, B_1=\{1,3,5,8\}$ and $B_2=\{1,4,7\}$. Then $f(A_1)=\{2,3,5\}, f(A_2)=\{5,6,8\}, f^{-1}(B_1)=\{b,c,e,g,h\}$ and $f^{-1}(B_2)=\varnothing$.

Definition 1.5 Let $f: A \to B$ and $g: B \to C$ be mappings. The **composition** gf is the mapping from A to C defined by $(gf)(a)=g(f(a))$ for all a in A.

In calculus courses, the composition of g with f is written $(g \circ f)(x)$ and is defined as $g(f(x))$. When we compose mappings, we omit the "circle".

There are several kinds of mappings that occur often enough to be given names.

Definition 1.6 Suppose $f: A \to B$ is a mapping.

(1) f is called **one-to-one** (or **injective**) if distinct elements of A have distinct images in B, that is, if $f(a_1)=f(a_2)$ implies $a_1=a_2$ (or if $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$).

(2) f is said to be **onto** B (or **surjective**) if each element of B is the image of at least one element of A. In symbols, f is onto if for every b in B there exists at least one a in A such that $f(a)=b$. That is to say, $f(A)=B$.

(3) f is called **one-to-one and onto** (or **bijective**) if f is both injective and surjective. In symbols, f is bijective if for each b in B there is exactly one a in A such that $f(a)=b$.

For any set S, we can obtain the **identity mapping** $1_s: S \to S$ defined by $1_s(x)=x$ for each x in S. Obviously, identity mapping is one-to-one and onto.

Remark. Suppose $f: A \to B$ and $g: C \to D$ are mappings, if the following conditions are satisfied, then we say that $f=g$.

(1) $A=C$;

(2) $B=D$;

(3) $f(a)=g(a)$ for any $a \in A$.

The next theorem summarizes the facts about mappings we will need.

Theorem 1.4 Given mappings $f: A \to B, g: B \to C$, and $h: C \to D$. Then

(1) $h(gf)=(hg)f$.

(2) If f and g are both one-to-one, then gf is one-to-one.

(3) If f and g are both onto, then gf is onto.

(4) If f and g are both bijective, then gf is bijective.

(5) If gf is one-to-one, then f is one-to-one. If gf is onto, then g is onto.

Proof. We prove only part(1). The remaining parts are left as exercises. Let $a \in A$. Then $(h(gf))(a)=h((gf)(a))=h(g(f(a)))=(hg)(f(a))=((hg)f)(a)$. So $h(gf)=(hg)f$.

Example 1.2 Let **Z** denote the set of integers, **R** the set of real numbers, and **N** the set of nature numbers. The following table illustrates the properties of one-to-one and onto.

Domain	Codomain	Rule	1−1	onto
Z	**Z**	$x \to x^3$	Y	N
R	**R**	$x \to x^3$	Y	Y
Z	**N**	$x \to \|x\|$	N	Y
Z	**Z**	$x \to x^2$	N	N

Theorem 1.5 The mapping $f: A \to B$ is bijective iff there exists a mapping $g: B \to A$ such that $gf=1_A$ and $fg=1_B$.

Proof. Suppose $f: A \to B$ is bijective, for each b in B, surjectivity of f implies there is a in A such that $f(a)=b$. Hence condition(1) in the definition of mapping from B to A holds for the rule g of correspondence $g(b)=a$. Condition(2) holds for g by the injectivity of f, since if $g(b)=a$ and $g(b)=a_1$ for g, then $f(a)=b$ and $f(a_1)=b$, so $a=a_1$. Hence we have the mapping $g: B \to A$. If a in A and $f(a)=b$, by construction of g we know $g(b)=a$. This fact implies that $g(f(a))=g(b)=a$. Thus $gf=1_A$. If b in B and $g(b)=a$, this means that we always have $f(a)=b$. Analogously, we can obtain $fg=1_B$.

Conversely, suppose $f: A \to B$ and $g: B \to A$ satisfy $gf=1_A$ and $fg=1_B$. For each b in B, let $g(b)=a$. Then $f(a)=f(g(b))=fg(b)=1_B(b)=b$, hence f is a surjective. Next suppose $f(a_1)=f(a_2)$ for $a_1, a_1 \in A$. Then

$$a_1=gf(a_1)=g(f(a_1))=g(f(a_2))=gf(a_2)=a_2,$$

and f is injective.

Remark. If $f: A \to B$ is bijective, then the mapping $g: B \to A$ satisfying conditions $gf=1_A$ and $fg=1_B$ in Theorem 1.4 is said to be the **inverse mapping** of f and denoted by $g=f^{-1}$.

Definition 1.7 Let $f: A \to B$ be a map and $A_1 \subsetneqq A$ a subset of A. We can make a new map, say φ, from A_1 to B in the sense that $\varphi(a)=f(a)$ for any $a \in A_1$. It is customary to replace φ by $f|_{A_1}$ and we call it the **restriction** of f to A_1.

Example 1.3 Let $f: \mathbf{Z} \to \mathbf{Z}$ be a map given by $f(n)=3n$ for every $n \in \mathbf{Z}$. If $2\mathbf{Z}$ define a subset of all even numbers of **Z**, then

(1) $f|_{2\mathbf{Z}}(-4)=-12$, $f|_{2\mathbf{Z}}(3)$ is meaningless;

(2) $f|_{2\mathbf{Z}}^{-1}(18)=\{6\}$, $f|_{2\mathbf{Z}}^{-1}(9)=\varnothing$;

(3) If $A=\{2,4,6\} \subseteq 2\mathbf{Z}, B=\{2,3,4,5,6\} \subseteq \mathbf{Z}$, then $f|_{2\mathbf{Z}}(A)=\{6,12,18\}$ and $f|_{2\mathbf{Z}}^{-1}(B)=\{2\}$.

Remark. The difference between f and $f|_{A_1}$ as in Definition 1.7 is that $f(a)$ is always