ALM 22

Advanced Lectures in Mathematics

Differential Geometry: Under the Influence of S.-S. Chern

在陈省身先生影响下的微分几何

Editors: Yibing Shen • Zhongmin Shen • Shing-Tung Yau

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ZAI CHENXINGSHEN XIANSHENG YINGXIANG XIA DE WEIFEN JIHE

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Preface

This book is dedicated to S.-S. Chern, one of the greatest mathematicians of the twentieth century. S.-S. Chern is a leader in the field of differential geometry. He has made seminal contributions to many areas such as web geometry, integral geometry, complex geometry, Riemannian geometry and Finsler geometry. He is well-known for the Chern-Simons theory, the Chern-Weil theory and Chern classes. His brilliant research and teaching have exerted a deep and lasting influence on mathematics.

This book consists of survey papers by mathematicians around the world, in particular those in China. They review the current status of the fields S.-S. Chern had worked on and discuss the future directions as well. This book is suitable for graduate students and young mathematicians. Every reader can find valuable information about related fields.

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Gauss-Bonnet-Chern Formulae and Related Topics for Curved Riemannian Manifolds

Jianguo Cao* and Hongyan Tang†

Abstract

In this paper, we survey recent results on Gauss-Bonnet-Chern formulae and related issues for closed Riemannian manifolds with variable curvature. Among other things, we address the following problem: "*if M^{2n} is an oriented $2n$-dimensional closed manifold with non-positive curvature, then is it true that its Euler number $\chi(M^{2n})$ satisfies the inequality $(-1)^n\chi(M^{2n}) \geq 0$?*" We will present some partial answer to this question in Kähler case. In addition, we discuss some related results to characterize a curved manifolds via various geometric invariants, along the line of Professor Chern.

2000 Mathematics Subject Classification: 58C20.

Keywords and Phrases: Gauss-Bonnet-Chern; Euler number; d-sublinear.

1 Curvature, Gauss-Bonnet-Chern formulae and Euler numbers

One of important contributions of Professor Chern to global differential geometry is the celebrated Gauss-Bonnet-Chern formula. Let us recall the classical Gauss-Bonnet formula for closed surfaces. Suppose that (M^2, g) is an oriented closed Riemmanian surface with curvature sec_g and the Euler number $\chi(M^2)$. It follows from Gauss-Bonnet formula that the total Gauss curvature is equal to the Euler number:

$$\int_{M^2} sec_g dA = 2\pi\chi(M^2). \tag{1}$$

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Professor Chern was able to extend Gauss-Bonnet formula to higher dimensional manifolds. For instance, let us consider an oriented closed Kähler manifold (M^{2n}, g) of complex dimension $n = \dim_{\mathbb{C}}(M^{2n})$. For such a Kähler manifold (M^{2n}, g), there is a top-dimensional Chern-form $c_n(M^{2n}) = c_n(T^{(1,0)}M^{2n})$ which is equal to the Euler class $e(M^{2n})$ (see [8] page 273, [31] page 155). When (M^{2n}, g) is a Kähler closed manifold, the Gauss-Bonnet-Chern formula reads as

$$\int_{M^{2n}} c_n(M^{2n}) = \int_{M^{2n}} e(M^{2n}) = \beta_n \chi(M^{2n}), \tag{2}$$

where $\chi(M^{2n})$ is the Euler number of M^{2n} and β_n is a constant only depends on the dimension $2n$.

It follows from Poincaré duality theorem for the cohomology ring of a closed manifold that any odd-dimensional manifold M^{2n+1} has zero Euler number (compare with [31] page 308). Thus, we only need to apply the Gauss-Bonnet-Chern formula to even-dimensional closed manifolds. For the Kähler manifolds, the construction of top-dimensional Chern class is related to the determinant of the so-called Chern curvature matrix-valued differential forms, which we now describe.

Indeed, if ∇ is the Chern connection of a Kähler metric g, then its curvature tensor R_g is defined by $R_g(X,Y)Z = -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z$. The sectional curvature of the metric g is given by

$$sec_g(X,Y) = \frac{\langle R(X,Y)X, Y\rangle}{\|X \wedge Y\|^2}. \tag{3}$$

The Chern curvature tensor $R = R_g$ has the property

$$\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle \text{ and } R_g(X,Y)Z = -R_g(Y,X)Z. \tag{4}$$

Therefore, there is a matrix-valued Chern curvature 2-form given by

$$\langle \Omega(X \wedge Y), Z \wedge W\rangle = \langle R(X,Y)Z, W\rangle. \tag{5}$$

Our top-dimensional Chern $2n$-form is related to determinant of the Chern curvature form Ω is given by

$$c_n = c_n(M^{2n}, g) = \det\left[\frac{1}{2\pi i}\Omega_{k,\bar{j}}\right](dz_1 \wedge d\bar{z}_1) \wedge \cdots \wedge (dz_n \wedge d\bar{z}_n)$$

where $\Omega_{k,\bar{j}} = \Omega(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_j})$ and $\{z_1, \cdots, z_n\}$ is a holomorphic local coordinate chart of M^{2n}, (see [31] page 299).

Open Problem 1.1. ([39] Problem # 8) *Let (M^{2n}, g) be a closed and oriented Riemannian manifold with positive sectional curvature. Is it true that the Euler number $\chi(M^{2n})$ is positive?*

Let us now turn our attention to the case of negative curvature. If $M^{2n} = \mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a compact quotient of the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$. It is well-known that $\mathbb{H}^n_{\mathbb{C}}$ has constant holomorphic sectional curvature. Thus, its curvature matrix Ω is similar to $-I_n$, where I_n is an $n \times n$ identity matrix. It is clear that

$$\det(-I_n) = (-1)^n. \tag{6}$$

Therefore, it is reasonable to ask the following question similar to Problem 1.1:

Open Problem 1.2. ([39] Problem # 10, Hopf conjecture) *Let (M^{2n}, g) be a closed and oriented Riemannian manifold with negative sectional curvature. Does the Euler number $\chi(M^{2n})$ satisfy inequality $(-1)^n\chi(M^{2n}) > 0$?*

The open problems 1.1–1.2 above were solved by Professor Chern himself in dimensions 2 and 4 (see [15]). We discuss other cases in upcoming sections.

2 Chern-Hopf problem and positive sectional curvature

To our surprise, in 1976, Robert Geroch provided a purely algebraic example, in dimension six, of a curvature tensor having nonnegative sectional curvatures and negative Gauss-Bonnet-Chern integrand. Let us explain why Geroch can construct such an example of incomplete Riemannian metric.

In fact, for Chern curvature form Ω, we need the linear space $\bigwedge^2(TM)$ spanned by all $\{X \wedge Y\}$, where X and Y are tangent vectors. The elements that can be expressed as $X \wedge Y$ are called *simple*.

There are many *non-simple* elements in $\bigwedge^2(TM)$. For instance, if $\dim(M^{2n}) \geq 4$ and if $\{X, Y, Z, U\}$ are linearly independent, then $(X \wedge Y + Z \wedge U)$ is a *non-simple* element of $\bigwedge^2(TM)$.

If sectional curvature is positive and $X \wedge Y \neq 0$ is a simple element, then

$$\langle \Omega(X \wedge Y), X \wedge Y \rangle > 0. \tag{7}$$

Positive sectional curvature only implies that the Chern Curvature form Ω is positive on simple elements of $\bigwedge^2(TM)$. However, positive sectional curvature does *not* imply

$$\langle \Omega(X \wedge Y + Z \wedge W), X \wedge Y + Z \wedge U \rangle > 0, \tag{8}$$

for non-simple element $X \wedge Y + Z \wedge W$, where we assume that $\{X, Y, Z, U\}$ are linearly independent.

Thus, it is natural to consider Chern-Hopf problem for positively curved manifolds with various additional assumption. One special class of positively curved Riemannian manifolds is Kähler manifolds with non-negative sectional curvature. Y.-T. Siu and S.-T. Yau used harmonic maps to prove the following conjecture of Frankel: Every compact Kähler manifold M of positive bisectional curvature is biholomorphic to the complex projective space.

Theorem 2.1. (Siu-Yau, [34]) (i) *Suppose that (M^{2n}, g) is a compact Kähler manifold with positive bisectional curvature. Then M^{2n} is biholomorphic to a complex projective space.*

(ii) (cf. [32]) *Let (M^{2n}, g) be a compact simply connected Kähler manifold whose metric g has nonnegative bisectional curvature and whose second Betti number is one. Then either M^{2n} is biholomorphic to complex projective space* *or (M^{2n}, g) is biholomorphically isometric to an irreducible compact Hermitian symmetric space of rank ≥ 2.*

Let M^{2n} be a complex manifold with a complex structure J. A metric g is said to be Kähler if its corresponding Levi-Civita connection ∇ is a Chern connection, i.e.,

$$\nabla_X(JY) = J(\nabla_X Y) \tag{9}$$

for all real tangent vectors $\{X, Y\}$. For each real tangent vector $X \in T(M^{2n})$, there is a corresponding complex vector of type $(1, 0)$:

$$\tilde{X} = \frac{1}{\sqrt{2}}[X + \sqrt{-1}JX]. \tag{10}$$

The map $X \to \tilde{X}$ is an isometry from $T(M^{2n})$ to $T^{(1,0)}(M^{2n})$.

If $\{X, JX, Y, JY\}$ are four orthonormal real vectors contained in $T_z(M^{2n})$, then using Bianchi identity we can verify that the bisectional curvature of $\{\tilde{X}, \tilde{Y}\}$ has the following property

$$\begin{aligned}
&\langle \Omega(\tilde{X} \wedge \bar{\tilde{Y}}), \tilde{X} \wedge \bar{\tilde{Y}} \rangle \\
&= \langle R(\tilde{X}, \bar{\tilde{Y}})\tilde{X}, \bar{\tilde{Y}} \rangle \\
&= \langle R(X, Y)X, Y \rangle + \langle R(X, JY)X, JY \rangle.
\end{aligned}$$

Among other things, the proof of Theorem 2.1 used the detailed study of the harmonic maps and minimal 2-spheres when the Kähler metric has positive bisectional curvature. The first author also studied a class of *real* 4-manifolds with non-negative sectional curvature, using the minimal surface technique (cf. [9]).

Without the Kähler assumption (9), Brendle and Schoen recently provided a classification (up to diffeomorphism) of Riemannian manifolds with weakly $\frac{1}{4}$-pinched sectional curvature in the following point-wise sense: for any point $p \in M$ and all 2-planes $\pi_1, \pi_2 \subset T_pM$,

$$0 \leq K(\pi_1) \leq 4K(\pi_2). \tag{11}$$

Theorem 2.2. (Brendle-Schoen, [5]) *Let (M^n, g) be a compact Riemannian manifold of dimension $n \geq 4$ with weakly $\frac{1}{4}$-pinched sectional curvature. Assume that (M^n, g) is not locally symmetric; then M^n is diffeomorphic to a spherical space form.*

If we replace the condition on the lower curvature bound by a lower diameter bound, then we have the following curvature-diameter rigidity theorem.

Theorem 2.3. (Gromoll-Grove [25], Wilking [37], Cao-Tang [10]) *A simply connected compact Riemannian manifold (M^n, g) of dimension $n \geq 2$ with sectional curvature ≥ 1 and diameter greater than or equal to $\frac{\pi}{2}$(i.e., $Diam(M^n, g) \geq \frac{\pi}{2}$, M^n is either homeomorphic to a sphere, or isometric to a compact rank one symmetric space.)*

Using the critical point theory of distance function, Gromov found a uniform upper bound for all complete Riemannian manifold with non-negative sectional curvature. Indeed, Gromov ([26]) provided in each dimension an a priori bound for the Betti numbers of a closed connected Riemannian manifold M^n in terms only of the invariant $\min K \cdot D^2$, where K is the sectional curvature of M^n and D is the diameter of M^n. When (M^n, g) has non-negative sectional curvature, we observe that $\min K \cdot D^2 \geq 0$. A simple proof of this result can be found in Yamaguchi's recent paper ([38]).

3 Negative curvature and a reduction by the Atiyah-Singer index theory

In this section, we consider Euler number of closed manifolds with negative sectional curvature.

The so-called Chern-Hopf inequality $(-1)^n \chi(M^{2n}) > 0$ has been verified in the case of Kähler manifold (M^{2n}, g) with negative sectional curvature for all n by Gromov [27] and Stern [35] (the work in [35] also used results of Greene and Wu; see [24], page 183–215). Other related results on the Euler numbers of negatively curved pinched manifolds were obtained by Donnelly, Fefferman and Xavier (see [19], [20]).

A related conjecture asserts that, if the sectional curvature of a Riemannian manifold (M^{2n}, g) is assumed to be only *non-positive*, then the Euler number must satisfy $(-1)^n \chi(M^{2n}) \geq 0$. Again, this second conjecture is known to be true in dimensions two and four [15]. The aim of this paper is to review its validity for all n in the Kähler case, thus complementing the above result of Gromov:

Theorem 3.1. (Cao-Xavier [11], Jost-Zuo [29]) *Let (M^{2n}, g) be a compact Riemannian manifold of non-positive curvature. If M^{2n} is homotopy equivalent to a Kähler manifold, then the Euler number of M^{2n} satisfies $(-1)^n \chi(M^{2n}) \geq 0$.*

Remark. If $F : M_1^n \to M_2^n$ is a homotopy equivalence map between two closed manifolds, and if one of them has non-positive curvature, then a theorem from Farrell and Jones [22] implies that the manifolds are actually homeomorphic, where $\dim M_i^n \geq 6$ (the case $\dim(M^n) = 4$ of Theorem 3.1 follows from [15]).

For the convenience of readers, we briefly outline the proof of Theorem 3.1.

In fact, Dodziuk [18] and Singer [39](page 672) have proposed to settle the Chern-Hopf inequality $(-1)^n \chi(M^{2n}) \geq 0$ using the Atiyah index theorem for coverings (see [2]). In fact, The de Rham-Hodge theory for a compact Riemannian manifold states that the space of harmonic k-forms is isomorphic to the k-th real cohomology group $H^k(M^{2n})$ and hence is a homotopy invariant.

In our case, we observe that if (M^{2n}, g) has non-positive sectional curvature, then we consider the universal covering space $\tilde{M}^{2n}$ with lifted metric $\tilde{g}$. Since M^{2n} has non-positive sectional curvature, the Cartan-Hadarmard theorem implies that M^{2n} is diffeomorphic to $\mathbb{R}^{2n}$. Suppose that $\Gamma = \pi_1(M^{2n})$ is the fundamental group of M^{2n}. Then Γ acts on $(\tilde{M}^{2n}, \tilde{g})$ by isometries. In this case, Dudziuk [17] derived a generalization of these results for a non-compact manifold $\tilde{M}^{2n}$ which is a covering space of a compact manifold with covering group Γ. For this generalization, one takes harmonic L^2-forms; and similarly one replaces the usual simplicial cohomology by cohomology based on L^2-cochains. Both spaces are now Γ-modules and Dodziuk demonstrated that they are isomorphic Γ-modules and homotopy invariants of the pair (M^{2n}, Γ). In particular the real-valued Betti numbers $\{\dim_\Gamma[H^k_{L^2}(\tilde{M}^{2n})]\}$ introduced by Atiyah (cf. [2]) are homotopy invariants.

Theorem 3.2. (Dodziuk [17]) *Suppose that (M^{2n}, g) is a closed Riemannian manifold with non-positive curvature, Γ is the fundamental group of M^{2n} and that $(\tilde{M}^{2n}, \tilde{g})$ is the universal cover with the lifted metric $\tilde{g}$. Then the following is true:* (1) *The equivalence class of the representation of Γ on L^2-cohomology ring $H^*_{L^2}(\tilde{M}^{2n})$ is a homotopy invariant of M^{2n}.*

(2) *Consequently, the normalized dimensions $\{\dim_\Gamma[H^k_{L^2}(\tilde{M}^{2n})]\}$ introduced by Atiyah* ([2]) *are homotopy invariants.*

(3) *Moreover, the Euler number of M^{2n} is given by*

$$\chi(M^{2n}) = \sum_{k=0}^{2n} (-1)^k \dim_\Gamma[H^k_{L^2}(\tilde{M}^{2n})].$$

In order to establish the inequality $(-1)^n\chi(M^{2n}) \geq 0$, by Theorem 3.2 (3) above, it is sufficient to prove a vanishing theorem for L^2 harmonic k-forms, $k \neq n$, on the universal covering of M^{2n}. The vanishing of these L^2 Betti numbers implies, by Atiyah-Dodziuk's result, that $(-1)^n\chi(M^{2n}) \geq 0$. The strict inequality $(-1)^n\chi(M^{2n}) > 0$ follows provided one can establish the existence of non-trivial L^2 harmonic n-forms on the universal cover.

This approach outlined above was first successfully carried out by Mark Stern (a former student of Professor Yau) and Gromov (cf. [35], [27]), when the manifold in question is Kähler and is homotopy equivalent to a compact manifold with *strictly* negative sectional curvatures. The central idea in Gromov's approach is the notion of d-bounded differential forms. As the terminology suggests, these are differential forms that are exterior derivatives of *bounded* forms, where boundedness is to be understood in the Riemannian sense. For instance, Gromov [27] asserted that a bounded closed k-form, $k \geq 2$, on a complete simply-connected manifold whose sectional curvatures are bounded above by a negative constant is automatically d-bounded.

A Kähler manifold is termed *Kähler-hyperbolic* if the Kähler form of its universal cover is d-bounded. Armed with the concept of Kähler-hyperbolicity, Stern and Gromov went on to prove the desired vanishing and existence theorems on the universal cover, thus establishing the inequality $(-1)^n\chi(M^{2n}) > 0$ when the Kähler manifold M^{2n} is homotopy equivalent to a compact manifold with negative curvature.

Although Kähler hyperbolicity is a powerful idea in the case of strictly negative curvature, it is clearly inadequate for the problem "$K \leq 0 \Longrightarrow (-1)^n\chi(M^{2n}) \geq 0$". This can be seen by taking a compact Kähler manifold M^{2n} satisfying $K \leq 0$ and $\chi(M^{2n}) = 0$ (e.g., a flat complex torus or, more generally, the product of such a torus and a Kähler manifold of negative curvature). The Kähler form on the universal cover of M^{2n} is not d-bounded. Otherwise, by Gromov's work one would have the correct vanishing and existence theorems of L^2 harmonic forms and, by Atiyah-Dodziuk's theorem, the Euler number of M^{2n} would actually be non-zero.

The first author and Fred Xavier introduced (cf. [11]) a condition which is weaker than d-boundedness and can be applied to Kähler manifolds of *non-positive* curvature:

Definition 3.3. *A differential form α on a complete non-compact Riemannian manifold is called d-sublinear if there exist a differential form β and a number $c > 0$ such that $d\beta = \alpha$ and $|\beta(x)| \leq c(1 + \rho(x, x_0))$, where $\rho(x, x_0)$ stands for the Riemannian distance between x and a base point x_0.*

The concept of d-sublinearity is both natural and flexible. For instance, we have following results related to d-sublinearity.

Theorem 3.4. *Let M^n be a complete simply-connected manifold of non-positive sectional curvature and α a bounded closed k-form on M^n, $k \geq 1$. Then α is d-sublinear.*

Theorem 3.5. *Let N^{2n} be a complete non-compact Kähler manifold of complex dimension n whose Kähler form is d-sublinear. If $k \neq n$, then any L^2 harmonic k-form on N^{2n} is identically zero.*

Other results on L^2-cohomology can be also found in Anderson [1], Donnelly and Fefferman [19], Donnelly and Xavier [20], Elworthy and Rosenberg [21] and Lott [30].

Extending Gromov's terminology, we propose the notion of Kähler parabolicity.

Definition 3.6. *A Kähler manifold is Kähler-parabolic if the Kähler form on its universal cover is d-sublinear but not d-bounded.*

Accordingly, our results imply that $(-1)^n\chi(M^{2n}) \geq 0$ if the compact manifold M^{2n} is Kähler-parabolic, with strict inequality holding if M^{2n} is Kähler-hyperbolic ([27]).

In upcoming sections, we will outline the proof of our vanishing Theorem 3.5. In addition, We will discuss the (weighted)-L^p cohomology theory of curved spaces.

4 Vanishing theorem and controlled Poincaré lemma

Throughout this section (M^m, g) will be a complete simply-connected manifold of non-positive curvature. Let also α be a bounded smooth closed k-form on M. Since M^m is diffeomorphic to $\mathbb{R}^m$ there exists a form β such that $d\beta = \alpha$. The purpose of this section is to show that β can be chosen to have sublinear growth, in the sense of the definition given in Section 3 above. The proof will follow from a controlled version of Poincaré lemma by a comparison argument.

4.1 Non-positive curvature, d-sublinear condition and controlled Poincaré lemma

Fix $p \in M$ and denote by $\exp_p : T_pM \to M$ the exponential map based at p.

Lemma 4.1. *Consider the maps $\tau_t : M \to M$, given by $x \longmapsto \exp_p(t \exp_p^{-1}(x))$, where $0 \le t \le 1$. Then*

$$|(\tau_t)_*\xi| \le t|\xi| \tag{12}$$

for every tangent vector ξ.

Proof. Let $\sigma : [0,1] \to M^n$ be the geodesic segment joining p to x, $\xi \in T_xM^n$ and $y = (\exp_p)^{-1}(x) \in T_pM^n$. By a straightforward computation one has

$$\begin{aligned}(\tau_t)_*\xi &= (d \exp_p)_{t(\exp_p)^{-1}(x)}[td(\exp_p^{-1})_{(x)}\xi] \\ &= (d \exp_p)_{ty}\{t[d(\exp_p)_y]^{-1}\xi\}.\end{aligned}$$

It is now manifest from the above formula that

$$J(t) := (\tau_t)_*\xi \tag{13}$$

is the Jacobi field along σ satisfying $J(0) = 0,\ J(1) = \xi$. On the other hand, since the sectional curvatures are non-positive, the function $f(s) := |J(s)|$ is convex ([3]). In particular, for $s \ge t$, one has

$$f(s) \ge f(t) + (s-t)\frac{f(t) - f(0)}{t - 0} = \frac{s}{t}f(t).$$

Setting $s = 1$ one has $f(t) \le tf(1)$ and the result follows from (13). □

Recall that if α is a k-form and Z is a vector field, then $(\alpha\lfloor_Z)$ is the $(k-1)$-form given by

$$(\alpha\lfloor_Z)(\xi_1, \cdots, \xi_{k-1}) = \alpha(Z, \xi_1, \cdots, \xi_{k-1}).$$

For the sake of completeness we give a proof of the following elementary result.

Lemma 4.2. *Let Ψ be a closed k-form in $\mathbb{R}^m$. Then the $(k-1)$-form Φ defined by*

$$\Phi(x) = r\int_0^1 [(\tau_t)^*(\Psi\lfloor_{\frac{\partial}{\partial r}})](x)dt \tag{14}$$

satisfies $d\Phi = \Psi$; here $\frac{\partial}{\partial r} = \sum_{i=1}^m \frac{x_i}{r}\frac{\partial}{\partial x_i}$, $r = (\sum_{i=1}^m x_i^2)^{1/2}$ and $\tau_t(x) = tx$.

Proof. By the standard proof of the Poincaré lemma ([33], p.130), Φ can be taken to be

$$\Phi(x) = \sum_{i_1<\cdots<i_k}\sum_{j=1}^k(-1)^{j-1}x_{i_j}\Big(\int_0^1 t^{k-1}\Psi_{i_1\cdots i_k}(tx)dt\Big)dx_{i_1}\wedge\cdots\wedge\widehat{dx_{i_j}}\wedge\cdots\wedge dx_{i_k},$$

where $\Psi = \sum_{i_1<\cdots<i_k}\Psi_{i_1\cdots i_k}dx_{i_1}\wedge\cdots\wedge dx_{i_k}$.

In particular, one has

$$\begin{aligned}\Psi(x) &= \sum_{i_1<\cdots<i_k}\sum_{j=1}^{k} x_{i_j}\Big(\int_0^1 t^{k-1}\Psi_{i_1\cdots i_k}(tx)dt\Big)(dx_{i_1}\wedge\cdots\wedge dx_{i_k})\lfloor_{\frac{\partial}{\partial x_{i_j}}}\\ &= r\sum_{i_1<\cdots<i_k}\Big(\int_0^1 t^{k-1}\Psi_{i_1\cdots i_k}(tx)dt\Big)(dx_{i_1}\wedge\cdots\wedge dx_{i_k})\lfloor_{\frac{\partial}{\partial r}}\\ &= r\int_0^1 t^{k-1}(\Psi\lfloor_{\frac{\partial}{\partial r}})(tx)dt\\ &= r\int_0^1 [(\tau_t)^*(\Psi\lfloor_{\frac{\partial}{\partial r}})](x)dt,\end{aligned}$$

as desired. □

Proof of Theorem 3.4 (controlled Poincaré Lemma). Let $(x_1,\cdots,x_n)$ be Euclidean coordinates in T_pM and consider the pull-back metric h of the metric g under $exp_p : T_pM \to M$. Observe that there are now two ways to interpret the map τ_t. The first interpretation comes from Lemma 4.1 with (M,g) being replaced by (T_pM,h); alternatively, one can think of τ_t as the self-map of T_pM, $(x_1,\cdots,x_n)\longmapsto t(x_1,\cdots,x_n)$, that appears in the Poincaré lemma (Lemma 4.2). It is an easy and yet basic observation that these two ways of thinking about τ_t give rise to the same map.

Let Φ be given by Lemma 4.2 and observe that, by Lemma 4.1,

$$|(\tau_t)^*\varphi(x)|_h \le Ct^{k-1}|\varphi(\tau_t(x))|_h,\quad k\ge 1, \tag{15}$$

holds for any $(k-1)$-form φ on T_pM; here $|\cdot|_h$ is any one of the equivalent norms induced by h. Since $|\frac{\partial}{\partial r}|=1$, it follows from (15) and Lemma 4.2 that

$$\begin{aligned}|\Phi(x)|_h &\le r\int_0^1 |[(\tau_t)^*(\Psi\lfloor_{\frac{\partial}{\partial r}})](x)|_h dt\\ &\le Cr\int_0^1 t^{k-1}|\Psi\lfloor_{\frac{\partial}{\partial r}}(tx)|_h dt\\ &\le C_1 r\int_0^1 t^{k-1}|\Psi(tx)|_h dt\\ &\le C_1 r\,\sup_{0\le t\le 1}|\Psi(tx)|_h.\end{aligned}$$

In particular,

$$|\Phi(x)|_h \le C_1\rho_h(0,x)\sup|\Psi|_h \le C_2\rho_h(0,x).$$

Hence Φ is d-sublinear and the proof of Theorem 3.4 is complete. □

4.2 The d-sublinear condition implies vanishing theorems

We begin the proof of our Vanishing Theorem 3.5 by recalling some basic facts in Hodge theory and Kähler geometry. If M^m is an oriented complete Riemannian manifold, let δ be the adjoint operator of d acting on the space of L^2 k-forms. Denote by $\Omega^k_{(2)}(M^m)$ and $\mathcal{H}^k_{(2)}(M^m)$ the spaces of L^2 k-forms and L^2 harmonic