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Lars Hörmander

The Analysis of Linear Partial Differential Operators IV

线性偏微分算子分析

第4卷

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Preface

to Volumes III and IV

The first two volumes of this monograph can be regarded as an expansion and updating of my book "Linear partial differential operators" published in the Grundlehren series in 1963. However, volumes III and IV are almost entirely new. In fact they are mainly devoted to the theory of linear differential operators as it has developed after 1963. Thus the main topics are pseudo-differential and Fourier integral operators with the underlying symplectic geometry. The contents will be discussed in greater detail in the introduction.

I wish to express here my gratitude to many friends and colleagues who have contributed to this work in various ways. First I wish to mention Richard Melrose. For a while we planned to write these volumes together, and we spent a week in December 1980 discussing what they should contain. Although the plan to write the books jointly was abandoned and the contents have been modified and somewhat contracted, much remains of our discussions then. Shmuel Agmon visited Lund in the fall of 1981 and generously explained to me all the details of his work on long range scattering outlined in the Goulaouic-Schwartz seminars 1978/79. His ideas are crucial in Chapter XXX. When the amount of work involved in writing this book was getting overwhelming Anders Melin lifted my spirits by offering to go through the entire manuscript. His detailed and constructive criticism has been invaluable to me; I as well as the readers of the book owe him a great debt. Bogdan Ziemian's careful proofreading has eliminated numerous typographical flaws. Many others have also helped me in my work, and I thank them all.

Some material intended for this monograph has already been included in various papers of mine. Usually it has been necessary to rewrite these papers completely for the book, but selected passages have been kept from a few of them. I wish to thank the following publishers holding the copyright for granting permission to do so, namely:
Marcel Dekker, Inc. for parts of [41] included in Section 17.2;
Princeton University Press for parts of [38] included in Chapter XXVII;
D. Reidel Publishing Company for parts of [40] included in Section 26.4;
John Wiley & Sons Inc. for parts of [39] included in Chapter XVIII.
(Here [N] refers to Hörmander [N] in the bibliography.)

Finally I wish to thank the Springer-Verlag for all the support I have received during my work on this monograph.

Djursholm in November, 1984 Lars Hörmander

Contents

Introduction

to Volumes III and IV

A great variety of techniques have been developed during the long history of the theory of linear differential equations with variable coefficients. In this book we shall concentrate on those which have dominated during the latest phase. As a reminder that other earlier techniques are sometimes available and that they may occasionally be preferable, we have devoted the introductory Chapter XVII mainly to such methods in the theory of second order differential equations. Apart from that Volumes III and IV are intended to develop systematically, with typical applications, the three basic tools in the recent theory. These are the theory of pseudo-differential operators (Chapter XVIII), Fourier integral operators and Lagrangian distributions (Chapter XXV), and the underlying symplectic geometry (Chapter XXI). In the choice of applications we have been motivated mainly by the historical development. In addition we have devoted considerable space and effort to questions where these tools have proved their worth by giving fairly complete answers.

Pseudo-differential operators developed from the theory of singular integral operators. In spite of a long tradition these played a very modest role in the theory of differential equations until the appearance of Calderón's uniqueness theorem at the end of the 1950's and the Atiyah-Singer-Bott index theorems in the early 1960's. Thus we have devoted Chapter XXVIII and Chapters XIX, XX to these topics. The early work of Petrowsky on hyperbolic operators might be considered as a precursor of pseudo-differential operator theory. In Chapter XXIII we discuss the Cauchy problem using the improvements of the even older energy integral method given by the calculus of pseudo-differential operators.

The connections between geometrical and wave optics, classical mechanics and quantum mechanics, have a long tradition consisting in part of heuristic arguments. These ideas were developed more systematically by a number of people in the 1960's and early 1970's. Chapter XXV is devoted to the theory of Fourier integral operators which emerged from this. One of its first applications was to the study of asymptotic properties of eigenvalues (eigenfunctions) of higher order elliptic operators. It is therefore discussed in Chapter XXIX here together with a number of later developments which give beautiful proofs of the power of the tool. The study by Lax of the propagation of singularities of solutions to the Cauchy problem was one of the forerunners of the theory. We prove such results using only pseudo-differential operators in Chapter XXIII. In Chapter XXVI the propagation of singularities is discussed at great length for operators of principal type. It is the only known approach to general existence theorems for such operators. The completeness of the results obtained has been the reason for the inclusion of this chapter and the following one on subelliptic operators. In addition to Fourier integral operators one needs a fair amount of symplectic geometry then. This topic, discussed in Chapter XXI, has deep roots in classical mechanics but is now equally indispensible in the theory of linear differential operators. Additional symplectic geometry is provided in the discussion of the mixed problem in Chapter XXIV, which is otherwise based only on pseudo-differential operator theory. The same is true of Chapter XXX which is devoted to long range scattering theory. There too the geometry is a perfect guide to the analytical constructs required.

The most conspicuous omission in these books is perhaps the study of analytic singularities and existence theory for hyperfunction solutions. This would have required another volume - and another author. Very little is also included concerning operators with double characteristics apart from a discussion of hypoellipticity in Chapter XXII. The reason for this is in part shortage of space, in part the fact that few questions concerning such operators have so far obtained complete answers although the total volume of results is large. Finally, we have mainly discussed single operators acting on scalar functions or occasionally determined systems. The extensive work done on for example first order systems of vector fields has not been covered at all.

Chapter XXV. Lagrangian Distributions and Fourier Integral Operators

Summary

In Section 18.2 we introduced the space of conormal distributions associated with a submanifold Y of a manifold X. This is a natural extension of the classical notion of multiple layer on Y. All such distributions have their wave front sets in the normal bundle of Y which is a conic Lagrangian manifold. In Section 25.1 we generalize the notion of conormal distribution by defining the space of Lagrangian distributions associated with an arbitrary conic Lagrangian $\Lambda \subset T^*(X) \setminus 0$. This is the space of distributions u such that there is a fixed bound for the order of $P_1 \dots P_N u$ for any sequence of first order pseudo-differential operators $P_1, \dots, P_N$ with principal symbols vanishing on Λ. This implies that $WF(u) \subset \Lambda$. Symbols can be defined for Lagrangian distributions in much the same way as for conormal distributions. The only essential difference is that the symbols obtained are half densities on the Lagrangian tensored with the Maslov bundle of Section 21.6.

In Section 25.2 we introduce the notion of Fourier integral operator; this is the class of operators having Lagrangian distribution kernels. As in the discussion of wave front sets in Section 8.2 (see also Section 21.2) it is preferable to associate a Fourier integral operator with the canonical relation $\subset (T^*(X) \setminus 0) \times (T^*(Y) \setminus 0)$ obtained by twisting the Lagrangian with reflection in the zero section of $T^*(Y)$. We prove that the adjoint of a Fourier integral operator associated with the canonical relation C is associated with the inverse of C, and that the composition of operators associated with C_1 and C_2 is associated with the composition $C_1 \circ C_2$ when the compositions are defined. Precise results on continuity in the $H_{(s)}$ spaces are proved in Section 25.3 when the canonical relation is the graph of a canonical transformation. We also study in some detail the case where the canonical relation projects into $T^*(X)$ and $T^*(Y)$ with only fold type of singularities.

The real valued C^∞ functions vanishing on a Lagrangian $\subset T^*(X) \setminus 0$ form an ideal with $\dim X$ generators which is closed under Poisson brackets. We define general Lagrangian ideals by taking complex valued functions instead. With suitable local coordinates in X they always have a local system

of generators of the form

$$x_j - \partial H(\xi)/\partial \xi_j, \quad j=1,\dots,n,$$

just as in the real case. The ideal is called positive if $\operatorname{Im} H \leqq 0$. This condition is crucial in the analysis and turns out to have an invariant meaning. Distributions associated with positive Lagrangian ideals are studied in Section 25.4. The corresponding Fourier integral operators are discussed in Section 25.5. The results are completely parallel to those of Sections 25.1, 25.2 and 25.3 apart from the fact that for the sake of brevity we do not extend the notion of principal symbol.

25.1. Lagrangian Distributions

According to Definition 18.2.6 the space $I^m(X, Y; E)$ of conormal distribution sections of the vector bundle E is the largest subspace of ${}^\infty H^{\mathrm{loc}}_{(-m-n/4)}(X, E)$, $n=\dim X$, which is left invariant by all first order differential operators tangent to the submanifold Y. It follows from Theorem 18.2.12 that it is even invariant under all first order pseudo-differential operators from E to E with principal symbol vanishing on the conormal bundle of Y. The definition is therefore applicable with no change to any Lagrangian manifold:

Definition 25.1.1. Let X be a C^∞ manifold and $\Lambda \subset T^*(X) \setminus 0$ a C^∞ closed conic Lagrangian submanifold, E a C^∞ vector bundle over X. Then the space $I^m(X, \Lambda; E)$ of Lagrangian distribution sections of E, of order m, is defined as the set of all $u \in \mathscr{D}'(X, E)$ such that

$$L_1 \dots L_N u \in {}^\infty H^{\mathrm{loc}}_{(-m-n/4)}(X, E) \tag{25.1.1}$$

for all N and all properly supported $L_j \in \Psi^1(X; E, E)$ with principal symbols L_j^0 vanishing on Λ.

The following lemma allows us to localize the study of $I^m(X, \Lambda; E)$.

Lemma 25.1.2. *If $u \in I^m(X, \Lambda; E)$ then $WF(u) \subset \Lambda$, and $Au \in I^m(X, \Lambda; E)$ if $A \in \Psi^0(X; E, E)$. Conversely, $u \in I^m(X, \Lambda; E)$ if for every $(x_0, \xi_0) \in T^*(X) \setminus 0$ one can find $A \in \Psi^0(X; E, E)$ properly supported and non-characteristic at (x_0, ξ_0) such that $Au \in I^m(X, \Lambda; E)$.*

Proof. If $(x_0, \xi_0) \notin \Lambda$ we can choose $L_1, \dots, L_N$ in (25.1.1) non-characteristic in a conic neighborhood Γ and conclude that $u \in H^{\mathrm{loc}}_{(s)}$ in Γ if $s < N - m - n/4$. Hence $WF(u) \cap \Gamma = \emptyset$. To prove the second statement we observe that

$$L_1 \dots L_N A u = L_1 \dots L_{N-1} A L_N u - L_1 \dots L_{N-1} [A, L_N] u.$$

Here $[A, L_N] \in \Psi^0(X; E, E)$ and $L_N u \in I^m(X, \Lambda; E)$ by Definition 25.1.1. By induction with respect to N we conclude that

$$L_1 \dots L_N A u \in {}^\infty H^{\text{loc}}_{(-m-n/4)}(X, E)$$

for all properly supported $A \in \Psi^0(X; E, E)$ and $L_j \in \Psi^1(X; E, E)$ with principal symbols vanishing on Λ. To prove the converse we choose B according to Lemma 18.1.24 so that $(x_0, \xi_0) \notin WF(BA - I)$. Thus $(x_0, \xi_0) \notin WF(BAu - u)$, and since $BAu \in I^m(X, \Lambda; E)$ it follows that

$$L_1 \dots L_N u \in {}^\infty H^{\text{loc}}_{(-m-n/4)} \quad \text{at } (x_0, \xi_0)$$

if $L_1, \dots, L_N$ satisfy the conditions in Definition 25.1.1. Hence (25.1.1) is fulfilled so $u \in I^m(X, \Lambda; E)$.

Remark. So far we have not used that Λ is Lagrangian. However, if (25.1.1) is fulfilled we have $[L_j, L_k]^N u \in {}^\infty H^{\text{loc}}_{(-m-n/4)}(X, E)$ for any N, so $WF(u)$ is contained in the characteristic set of $[L_j, L_k]$ by the first part of the proof. Hence $WF(u)$ cannot contain an arbitrary point in Λ unless Λ is involutive. The hypothesis that Λ is Lagrangian means that Λ is minimal with this property, or alternatively that we have a maximal set of conditions (25.1.1) which do not imply that u is smooth.

Lemma 25.1.2 reduces the study of distributions $u \in I^m(X, \Lambda; E)$ to the case where $WF(u)$ is contained in a small closed conic neighborhood Γ_0 of some point $(x_0, \xi_0) \in \Lambda$, and supp u is close to x_0. In that case Definition 25.1.1 is applicable even if Λ is just defined in an open conic neighborhood Γ_1 of Γ_0, for only the restriction of the principal symbol of L_j to Γ_1 is relevant. More generally, given a conic Lagrangian submanifold Λ of the open cone $\Gamma_1 \subset T^*(X) \setminus 0$ we shall say that $u \in I^m(X, \Lambda; E)$ at $(x_0, \xi_0) \in \Gamma_1$ if there is an open conic neighborhood $\Gamma_0 \subset \Gamma_1$ of (x_0, ξ_0) such that $Au \in I^m(X, \Lambda; E)$ for all properly supported $A \in \Psi^0$ with $WF(A) \subset \Gamma_0$; it suffices to know this for some such A which is non-characteristic at (x_0, ξ_0).

In view of Theorem 21.2.16 we may thus assume now that $X = \mathbb{R}^n$ and that $\Lambda = \{(H'(\xi), \xi); \xi \in \mathbb{R}^n \setminus 0\}$ where H is a real valued function in $C^\infty(\mathbb{R}^n \setminus 0)$ which is homogeneous of degree 1. We may also assume that E is the trivial bundle, which is then omitted from the notation.

Proposition 25.1.3. *If $u \in I^m_{\text{comp}}(\mathbb{R}^n, \Lambda)$, $\Lambda = \{(H'(\xi), \xi);\ \xi \in \mathbb{R}^n \setminus 0\}$, then $\hat{u}(\xi) = e^{-iH(\xi)} v(\xi)$, $|\xi| > 1$, where $v \in S^{m-n/4}(\mathbb{R}^n)$. Conversely, the inverse Fourier transform of $e^{-iH} v$ is in $I^m(\mathbb{R}^n, \Lambda)$ if $v \in S^{m-n/4}(\mathbb{R}^n)$.*

Proof. Choose $\chi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 in a neighborhood of 0 and define h by $\hat{h} = \chi \hat{H}_0$ where $H_0 = (1-\chi)H$. Then $\hat{H}_0 - \hat{h} \in \mathscr{S}$ (see the proof of Theorem 7.1.22), so $H_0 - h \in \mathscr{S}$. Thus $h \in S^1$ has the principal symbol H, so it suffices to prove the result with H replaced by h. Set $h_j(\xi) = \partial h(\xi)/\partial \xi_j$. The operator $h_j(D)$ is convolution with the inverse Fourier transform of h_j so it is properly supported. Hence

(25.1.2) $$D^\beta \prod (x_j - h_j(D))^{\alpha_j} u \in {}^\infty H_{(-m-n/4)} \quad \text{if } |\beta| = |\alpha|$$

for $[x_j - h_j(D), D_k] = i\delta_{jk}$ so commuting the factors D^β we obtain a sum of products of operators of the form $(x_j - h_j(D))D_k$ to which (25.1.1) is applicable. Recalling the definition of ${}^\infty H_{(-m-n/4)}$ we obtain

$$\int_{R/2<|\xi|<2R} |\xi^\beta \prod (-D_j - h_j(\xi))^\alpha \hat{u}(\xi)|^2 \, d\xi \leqq C_\alpha R^{2(m+n/4)}; \qquad R>1, \ |\beta| = |\alpha|.$$

With the notation $\hat{u}(\xi) = e^{-ih(\xi)} v(\xi)$ this means that

$$\int_{R/2<|\xi|<2R} |\xi|^{2|\alpha|} |D^\alpha v(\xi)|^2 \, d\xi \leqq C_\alpha R^{2(m+n/4)}.$$

If $v_R(\xi) = v(R\xi)/R^{m-n/4}$ then

$$\int_{\frac{1}{2}<|\xi|<2} |D^\alpha v_R(\xi)|^2 \, d\xi \leqq C_\alpha$$

which by Lemma 7.6.3 gives uniform bounds for $D^\alpha v_R$ when $|\xi| = 1$, that is, bounds for $|D^\alpha v(\xi)| (1 + |\xi|)^{|\alpha| - m + n/4}$. The argument can be reversed to prove the last statement in the proposition, for the passage from the operators $(x_j - h_j(D)) D_k$ to the general operators in (25.1.1) can be made by the argument preceding Theorem 18.2.7.

A slight modification of the proof gives precise information about the smoothness of elements in I^m. We state the result directly in a global form.

Theorem 25.1.4. *If $U \in I^m(X, \Lambda)$ and $U \in H_{(s_0)}$ at $(x_0, \xi_0) \in \Lambda$, then $U \in I^\mu(X, \Lambda)$ at (x_0, ξ_0) if $\mu + s_0 + n/4 > 0$.*

Proof. Choose $A \in \Psi^0(X)$ properly supported, non-characteristic at (x_0, ξ_0), so that $AU \in H_{(s_0)}$. By Lemma 25.1.2 we have $AU \in I^m$. We can choose A so that $WF(AU)$ is in a small conic neighborhood of (x_0, ξ_0). Writing $u = AU$ we conclude that it is sufficient to prove that $u \in I^\mu$ if $u \in H_{(s_0)}$ and u satisfies the hypotheses in Proposition 25.1.3. With the notation used there we have

$$\int_{\frac{1}{2}<|\xi|<2} |D^\alpha v_R(\xi)|^2 \, d\xi \leqq C_\alpha, \qquad \int_{\frac{1}{2}<|\xi|<2} |v_R(\xi)|^2 \, d\xi \leqq C R^{-2(s_0+m+n/4)}.$$

Let $|\xi| = 1$ and set $V_{R,\xi}(\eta) = v_R(\xi + \eta/R^\delta) R^{-n\delta/2}$ where $\delta > 0$. Then

$$\int_{|\eta|<1} |D^\alpha V_{R,\xi}(\eta)|^2 \, d\eta \leqq C_\alpha R^{-2|\alpha|\delta}, \qquad \int_{|\eta|<1} |V_{R,\xi}(\eta)|^2 \, d\eta \leqq C R^{-2(s_0+m+n/4)}.$$

Now we use the Sobolev inequality

$$|D^\beta V(0)|^2 \leqq C_\beta \int_{|\eta|<1} \Big(\sum_{|\alpha|=s} |D^{\alpha+\beta} V(\eta)|^2 + |V(\eta)|^2 \Big) \, d\eta$$

where $s > n/2$. This is somewhat more general than (7.6.6) but follows from the same proof. Taking s so large that $s\delta > s_0 + m + n/4$ we obtain

$$|D^\beta V_{R,\xi}(0)| \leqq C' R^{-(s_0+m+n/4)},$$

hence

$$|D^\beta v_R(\xi)| \leqq C' R^{\delta(n/2+|\beta|)-(s_0+m+n/4)}, \qquad |\xi|=1,$$
$$|D^\beta v(\xi)| \leqq C' |\xi|^{\delta(n/2+|\beta|)-(s_0+n/2+|\beta|)}, \qquad |\xi|>1.$$

For every β we can choose δ so that the exponent is smaller than $\mu-n/4-|\beta|$, and then we obtain $v\in S^{\mu-n/4}$, hence $u\in I^\mu$.

We shall now prove that elements in $I^m(X,\Lambda)$ can also be represented by means of arbitrary phase functions ϕ parametrizing Λ in the sense of Definition 21.2.15. At first we assume that ϕ is non-degenerate.

Proposition 25.1.5. *Let $\phi(x,\theta)$ be a non-degenerate phase function in an open conic neighborhood of $(x_0,\theta_0)\in\mathbb{R}^n\times(\mathbb{R}^N\setminus 0)$ which parametrizes the Lagrangian manifold Λ in a neighborhood of (x_0,ξ_0); $\xi_0=\phi'_x(x_0,\theta_0)$, $\phi'_\theta(x_0,\theta_0)=0$. If $a\in S^{m+(n-2N)/4}(\mathbb{R}^n\times\mathbb{R}^N)$ has support in the interior of a sufficiently small conic neighborhood Γ of (x_0,θ_0), then the oscillatory integral*

$$u(x)=(2\pi)^{-(n+2N)/4}\int e^{i\phi(x,\theta)}a(x,\theta)\,d\theta \tag{25.1.3}$$

defines a distribution $u\in I^m_{\mathrm{comp}}(\mathbb{R}^n,\Lambda)$. If $\Lambda=\{(H'(\xi),\xi)\}$ as in Proposition 25.1.3 then (for $|\xi|>1$)

$$e^{iH(\xi)}\hat u(\xi)-(2\pi)^{n/4}a(x,\theta)\,e^{\pi i/4\,\mathrm{sgn}\,\Phi}|\det\Phi|^{-\frac12}\in S^{m-n/4-1} \tag{25.1.4}$$

where (x,θ) is determined by $\phi'_\theta(x,\theta)=0$, $\phi'_x(x,\theta)=\xi$, and

$$\Phi=\begin{pmatrix}\phi''_{xx} & \phi''_{x\theta}\\ \phi''_{\theta x} & \phi''_{\theta\theta}\end{pmatrix}.$$

Here $a(x,\theta)$ is interpreted as 0 if there is no such point in Γ. $e^{iH(\xi)}\hat u(\xi)$ is polyhomogeneous if a is. Conversely, every $u\in I^m(X,\Lambda)$ with $WF(u)$ in a small conic neighborhood of (x_0,ξ_0) can, modulo C^∞, be written in the form (25.1.3).

In the proof we shall need an extension of Lemma 18.1.18.

Lemma 25.1.6. *Let $\Gamma_j\subset\mathbb{R}^{n_j}\times(\mathbb{R}^{N_j}\setminus 0)$, $j=1,2$, be open conic sets and let $\psi:\Gamma_1\to\Gamma_2$ be a C^∞ proper map commuting with multiplication by positive scalars in the second variable. If $a\in S^m(\mathbb{R}^{n_2}\times\mathbb{R}^{N_2})$ has support in the interior of a compactly based cone $\subset\Gamma_2$ then $a\circ\psi\in S^m(\mathbb{R}^{n_1}\times\mathbb{R}^{N_1})$ if the composition is defined as 0 outside Γ_1.*

Proof. The support of $a\circ\psi$ belongs to a compactly based cone $\subset\Gamma_1$ where $\psi(x,\xi)=(y,\eta)$ implies $|\xi|/C<|\eta|<C|\xi|$. The hypothesis on a means that

$$|D^\alpha_{y,\eta}a(y,t\eta)|\leqq C_\alpha t^m, \qquad 1/C<|\eta|<C.$$

Since $a\circ\psi(x,t\xi)=a(.,t.)\circ\psi(x,\xi)$ by the homogeneity of ψ, we obtain

$$|D^\alpha_{x,\xi}(a\circ\psi)(x,t\xi)|\leqq C'_\alpha t^m, \quad |\xi|=1,$$

by using Leibniz' rule. This proves the lemma.

Proof of Proposition 25.1.5. By hypothesis $\phi'_x(x_0,\theta_0)=\xi_0\neq 0$, so the oscillatory integral (25.1.3) is well defined. u has compact support if Γ has a compact base. We shall use the method of stationary phase to evaluate

$$e^{iH(\xi)}\hat{u}(\xi)=(2\pi)^{-(n+2N)/4}\iint e^{i(\phi(x,\theta)+H(\xi)-\langle x,\xi\rangle)}a(x,\theta)\,dx\,d\theta. \tag{25.1.5}$$

The exponent has a critical point if

$$\phi'_x(x,\theta)=\xi,\qquad \phi'_\theta=0,$$

which by hypothesis means that $(x,\xi)\in\Lambda$, hence that $x=H'(\xi)$. The critical point is non-degenerate. In fact, the maps

$$C=\{(x,\theta);\ \phi'_\theta=0\}\ni(x,\theta)\mapsto(x,\phi'_x)\in\Lambda \quad\text{and}\quad \Lambda\ni(x,\xi)\mapsto\xi$$

are diffeomorphisms. Hence $C\ni(x,\theta)\mapsto\phi'_x$ is a diffeomorphism, so $d\phi'_x=d\phi'_\theta=0$ implies $dx=d\theta=0$. The matrix Φ is therefore non-singular. If we divide (multiply) the first n (last N) rows (columns) by $|\theta|$ we see that $\det\Phi$ is homogeneous in θ of degree $n-N$. Hence $a(x,\theta)\,|\det\Phi|^{-\frac12}$ is in $S^{m-n/4}$ in a conic neighborhood of C. By Lemma 25.1.6 this remains true for the restriction to C regarded as a function of ξ.

It follows from Theorem 7.7.1 that there is a constant C such that for any N

$$\Big|\int e^{i(\phi(x,\theta)-\langle x,\xi\rangle)}a(x,\theta)\,dx\Big|\leqq C_N(|\xi|+|\theta|)^{-N}, \quad \text{if } |\theta|>C|\xi| \text{ or } |\xi|>C|\theta|. \tag{25.1.6}$$

In fact, $(\phi(x,\theta)-\langle x,\xi\rangle)/(|\xi|+|\theta|)=f(x)$ is homogeneous in (ξ,θ) of degree 0 and bounded in C^∞. If $(x,\theta)\in\operatorname{supp} a$ we have

$$|f'(x)|\geqq(|\xi|-C_1|\theta|)/(|\xi|+|\theta|)\geqq\tfrac12 \qquad \text{if } |\theta|/|\xi| \text{ is small},$$
$$|f'(x)|\geqq(C_2|\theta|-|\xi|)/(|\xi|+|\theta|)>C_2/2 \qquad \text{if } |\xi|/|\theta| \text{ is small}.$$

We can therefore apply Theorem 7.7.1 with $\omega=|\xi|+|\theta|$.

Choose $\chi\in C_0^\infty(\mathbb{R}^N\setminus 0)$ equal to 1 when $1/C<|\theta|<C$. By (25.1.6) the difference between $e^{iH(\xi)}\hat{u}(\xi)$ and

$$U(\xi)=(2\pi)^{-(n+2N)/4}\iint e^{i(\phi(x,\theta)+H(\xi)-\langle x,\xi\rangle)}\chi(\theta/|\xi|)\,a(x,\theta)\,dx\,d\theta$$

is rapidly decreasing as $\xi\to\infty$. Set $|\xi|=t$, $\xi/t=\eta$ and replace θ by $t\theta$. Then

$$U(\xi)=(2\pi)^{-(n+2N)/4}\iint e^{it(\phi(x,\theta)+H(\eta)-\langle x,\eta\rangle)}\chi(\theta)\,a(x,t\theta)t^N\,dx\,d\theta.$$

Here the exponent has only one critical point in the support of the integrand and it is defined by $\phi'_\theta(x,\theta)=0$, $\phi'_x(x,\theta)=\eta$. At that point

$$\phi(x,\theta)=\langle\theta,\phi'_\theta(x,\theta)\rangle=0,\qquad \langle x,\eta\rangle=\langle H'(\eta),\eta\rangle=H(\eta)$$

so the critical value is 0. Using (7.7.13) we obtain an asymptotic expansion of U. Since $\chi=1$ at the critical point, the leading term is

$$(2\pi)^{n/4}a(x,t\theta)t^{(N-n)/2}e^{\pi i/4\operatorname{sgn}\Phi}|\det\Phi|^{-\frac12},$$

that is, the term displayed in (25.1.4) in view of the homogeneity of $\det\Phi$ already pointed out. The k^{th} term will contain another factor t^{-k} and a linear combination of derivatives of $a(x,t\theta)$ with respect to x, θ, so it is in $S^{m-n/4-k}$. In view of Proposition 18.1.4 it follows that we have an asymptotic series in the sense of Proposition 18.1.3, and this completes the proof of the first part of the proposition.

To prove the converse it is by Proposition 25.1.3 sufficient to consider an element $u\in I^m(X,\Lambda)$ with $v=\hat{u}e^{iH}\in S^{m-n/4}$ having support in a small conic neighborhood of ξ_0. Choose a C^∞ map $(x,\theta)\mapsto\psi(x,\theta)\in\mathbb{R}^n\setminus 0$ in a conic neighborhood of (x_0,θ_0) such that ψ is homogeneous of degree 1 and $\psi(x,\theta)=\partial\phi/\partial x$ when $\partial\phi/\partial\theta=0$. Let

$$a_0(x,\theta)=(2\pi)^{-n/4}v\circ\psi(x,\theta)\,e^{-\pi i/4\operatorname{sgn}\Phi}|\det\Phi|^{\frac12}\in S^{m+(n-2N)/4}$$

near C, and define u_0 by (25.1.3) with a replaced by a_0. From the first part of the proposition it follows then that $u-u_0\in I^{m-1}$. Repeating the argument gives a sequence $a_j\in S^{m+(n-2N)/4-j}$ such that $u-u_0-\ldots-u_j\in I^{m-j-1}$ if u_j is defined by (25.1.3) with a replaced by a_j. If a is an asymptotic sum of the series $\sum a_j$ it follows that (25.1.3) is valid modulo C^∞. The proof is complete.

We shall now examine what must be changed in the preceding argument if ϕ is just a clean phase function. We still have (25.1.6) so only $U(\xi)$ is important. However, $\phi(x,\theta)+H(\eta)-\langle x,\eta\rangle$ does not satisfy the hypotheses in Theorem 7.7.6. We do know that (locally)

$$C=\{(x,\theta);\ \partial\phi(x,\theta)/\partial\theta=0\}$$

is a manifold of dimension $e+n$, where e is the excess, and that the composed map $C\to\Lambda\to\mathbb{R}^n$: $(x,\theta)\mapsto(x,\phi'_x)\mapsto\phi'_x$ has surjective differential, hence a fiber C_η of dimension e over η where $x=H'(\eta)$. The critical points of $\phi(x,\theta)+H(\eta)-\langle x,\eta\rangle$ are defined by $\phi'_\theta=0$, $\phi'_x=\eta$, that is, $(x,\theta)\in C_\eta$, and $d\phi'_\theta=0$, $d\phi'_x=0$ precisely along the tangent space of C_η. Note that we have fixed upper and lower bounds for $|\theta|$ on C_η since $|\phi'_x|=1$. We can split the θ variables into two groups θ', θ'' so that the number of θ'' variables is e and the projection $C_\eta\ni(x,\theta)\mapsto\theta''$ has bijective differential. Then $d\phi'_\theta=0$, $d\phi'_x=0$, $d\theta''=0$ implies $dx=d\theta=0$. Thus the Hessian of $\phi(x,\theta)+H(\eta)-\langle x,\eta\rangle$ with respect to (x,θ') is not 0, so the critical point on C_η when θ'' is fixed is non-degenerate. If we change the definition of Φ to

$$\Phi=\begin{pmatrix}\phi''_{xx} & \phi''_{x\theta'}\\ \phi''_{\theta'x} & \phi''_{\theta'\theta'}\end{pmatrix},$$

an application of Theorem 7.7.6 to the integral $U(\xi)$ with respect to the $n+N-e$ variables x, θ' gives, when we integrate with respect to θ'' afterwards,

$$e^{iH(\xi)}\hat{u}(\xi)-(2\pi)^{n/4-e/2}\int_{C_\eta}t^{(N+e-n)/2}a(x,t\theta)\,e^{\pi i/4\operatorname{sgn}\Phi}|\det\Phi|^{-\frac12}\,d\theta''\in S^{m+e/2-n/4-1}.$$

Note that the order has increased by $e/2$ since the stationary phase evaluation is applied to e variables less. For the same reason a factor $(2\pi)^{e/2}$ is lost. If we introduce $t\theta$ as a new variable, noting that $\det\Phi$ is homogeneous of degree $n-N+e$ now, we obtain

Proposition 25.1.5′. *Let $\phi(x,\theta)$ be a clean phase function with excess e in an open conic neighborhood of $(x_0,\theta_0)\in\mathbb{R}^n\times(\mathbb{R}^N\setminus 0)$ which parametrizes the Lagrangian manifold Λ in a neighborhood of (x_0,ξ_0); $\xi_0=\phi'_x(x_0,\theta_0)$, $\phi'_\theta(x_0,\theta_0)=0$. If $a\in S^{m+(n-2N-2e)/4}(\mathbb{R}^n\times\mathbb{R}^N)$ has support in the interior of a sufficiently small conic neighborhood Γ of (x_0,θ_0) then the oscillatory integral*

$$u(x)=(2\pi)^{-(n+2N-2e)/4}\int e^{i\phi(x,\theta)}a(x,\theta)\,d\theta \tag{25.1.3$'$}$$

defines a distribution $u\in I^m_{\mathrm{comp}}(\mathbb{R}^n,\Lambda)$. If $\Lambda=\{(H'(\xi),\xi)\}$ as in Proposition 25.1.3 then

$$e^{iH(\xi)}\hat{u}(\xi)-(2\pi)^{n/4}\int_{C_\xi} a(x,\theta)\,e^{\pi i/4\,\mathrm{sgn}\,\Phi}|\det\Phi|^{-\frac12}\,d\theta''\in S^{m-n/4-1}. \tag{25.1.4$'$}$$

Here $C_\xi=\{(x,\theta);\ \phi'_\theta(x,\theta)=0,\ \phi'_x(x,\theta)=\xi\}$; $\theta=(\theta',\theta'')$ is a splitting of the θ variables in two groups such that $C_\xi\ni(x,\theta)\mapsto\theta''$ has bijective differential; and

$$\Phi=\begin{pmatrix}\phi''_{xx} & \phi''_{x\theta'}\\ \phi''_{\theta' x} & \phi''_{\theta'\theta'}\end{pmatrix}.$$

Conversely, modulo C^∞ every $u\in I^m(X,\Lambda)$ with $WF(u)$ in a small conic neighborhood of (x_0,ξ_0) can be written in the form (25.1.3)′.

Remark. If $f\in C^\infty(Y)$ has a critical point at $y_0\in Y$ then $|\det f''(y_0)|^{\frac12}$ transforms as a density at y_0. This is why in the standard stationary phase formula the density in the integrand is transformed to a scalar in the asymptotic expansion. If on the other hand f is critical on a submanifold $Z\subset Y$ and is non-degenerate in transversal directions, then the square root of the determinant of the Hessian in transversal planes defines a density in the normal bundle. Dividing a density in Y by it gives a density on Z. This confirms the invariant meaning of the integrand in (25.1.4)′.

There is no difficulty in performing a change of local coordinates x in the representation (25.1.3) of an element in $I^m(X,\Lambda)$, so Proposition 25.1.3 contains all that is needed to define a principal symbol isomorphism for I^m extending Theorem 18.2.11. However, it is instructive to establish first a theorem on limits of elements in I^m which connects the definitions in this section with those given in the linear case in Section 21.6.

Proposition 25.1.7. *Let $u\in I^m_{\mathrm{comp}}(\mathbb{R}^n,\Lambda)$, $\Lambda=\{(H'(\xi),\xi),\ \xi\in\mathbb{R}^n\setminus 0\}$, and set $e^{iH}\hat{u}=(2\pi)^{n/4}v$, $v\in S^{m-n/4}$. If $\psi\in C^\infty(\mathbb{R}^n)$ is real valued, $\psi(x_0)=0$, $\psi'(x_0)=\xi_0\neq 0$, $(x_0,\xi_0)\in\Lambda$, then as $t\to+\infty$*

$$t^{-2m-n/2}(ue^{-it^2\psi})(x_0+x/t)-v(t^2\xi_0)t^{-2m+n/2}u^{\psi}_{x_0,\xi_0}(x)\to 0 \quad in\ \mathscr{D}', \tag{25.1.7}$$