Cambridge Tracts in Mathematics

Jean Bertoin

Lévy Processes

$$\mu_B^q(A') = q \int_{\mathbb{R}^d} dx \mathbb{E}_x(\exp\{-qT_B\}, X_{T_B} \in A')$$

$$\leq q \int dx \mathbb{E}_x(\exp\{-qT_{B\cap A'}\})$$

$$\leq q \int_{\mathbb{R}^d} dx \mathbb{E}_x(\exp\{-qT_{A'}\})$$

Lévy过程

$$\leq kq \int_{\mathbb{R}^d} U^q(x, A)dx = km(A) ,$$

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Jean Bertoin
Laboratoire de Probabilités
Université Pierre et Marie Curie

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作　　者： Jean Bertoin

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Preface

Lévy processes can be thought of as random walks in continuous time, that is they are stochastic processes with independent and stationary increments. The state space may be a fairly general topological group, but in this text, we will stick to the Euclidean framework. The best known and most important examples are the Poisson process, Brownian motion, the Cauchy process, and more generally stable processes. Lévy processes concern many aspects of probability theory and its applications. In particular, they are prototypes of Markov processes (actually, they form the class of space-time homogeneous Markov processes) and of semimartingales; they are also used as models in the study of queues, insurance risks, dams, and more recently in mathematical finance. From the viewpoint of functional analysis, they appear in connection with potential theory of convolution semigroups.

Historically, the first researches go back to the late 20's (that is when the foundations of modern probability theory were laid down) with the study of infinitely divisible distributions. Their general structure has been gradually discovered by de Finetti, Kolmogorov, Lévy, Khintchine and Itô; it is described by the celebrated Lévy-Khintchine formula which points out the correspondence between infinitely divisible distributions and processes with independent and stationary increments. After the pioneer contribution of Hunt in the mid-50's, the spreading of the theory of Markov processes and its connection with abstract potential theory has had a considerable impact on Lévy processes; see the works of Doob, Dynkin, Blumenthal and Getoor, Skorohod, Kesten, Bretagnolle, Port and Stone, Berg and Forst, Kanda, Hawkes, $\cdots$. At the same time, the fluctuation theory for random walks developed chiefly by Spitzer, Feller and Borovkov via analytic methods has been extended to continuous time by approximations based on discrete time skeletons; and many important properties of the sample paths of Lévy processes have been noted by Rogozin, Taylor, Fristedt, Pruitt, and others. Path transformations such as reflexion, splitting or time reversal form another set of useful techniques that were applied initially (in continuous time) to Brownian motion. Their importance for Lévy processes was recognized first by Millar, Greenwood and Pitman, who presented a direct approach to fluctuation theory. Further developments in this setting were made quite recently by Bertoin, Doney and others. Local times have received a lot of attention in the last ten years or so; the most impressive result in that field is perhaps the characterization by Barlow and Hawkes of the class of Lévy processes which possess jointly continuous local times; see also the recent works of Marcus and Rosen in the symmetric case. To complete this brief overview, we stress that the so-called general theory of processes also has many important applications to Lévy processes, in particular concerning stochastic calculus and limit theorems.

Several books contain sections or chapters on Lévy processes (e.g. Lévy (1954), Itô (1961), Gihman and Skorohod (1975), Jacod and Shiryaev (1987), Sato (1990, 1995), Skorohod (1991), Rogers and Williams (1994), …); see also the surveys by Taylor (1973), Fristedt (1974) and Bingham (1975). The purpose of this monograph is to present an up-dated and concise account of the theory, which may serve as a reference text. I endeavoured to make it as self-contained as possible; the prerequisite is limited to standard notions in probability and Fourier analysis.

Here is a short description of the content. A chapter of preliminaries introduces the notation and reviews some elementary material on infinitely divisible laws, Poisson processes, martingales, Brownian motion and regularly varying functions. The core of the theory of Lévy processes in connection with the Markov property and the related potential theory is developed in chapters I and II. The theory of general Markov processes is doubtless one of the most fascinating fields of probability, but it is also one of the most demanding. Nonetheless, the special case of Lévy processes is much easier to handle, thanks to techniques of Fourier analysis and the spatial homogeneity. We stress that no prior knowledge of Markov processes is assumed. Chapter III is devoted to the study of subordinators, which form the class of increasing Lévy processes; a special emphasis is given to the properties of their sample paths. Subordinators also have a key part in chapter IV, where we introduce Itô's theory of the excursions of a Markov process away from a point, and in chapter V, where we investigate the local times of Lévy processes. The fluctuation theory is presented in chapter VI, following the Greenwood-Pitman approach based on excursion theory. Chapter VII is devoted to Lévy processes with no positive jumps, for which fluctuation theory becomes remarkably simple. Some path transformations are described, which extend well-known identities for Brownian motion due to Williams and Pitman. Finally, several consequences of the scaling property of stable processes are presented in chapter VIII. Each chapter ends with exercises, which provide additional information on the topic for the interested reader, and with comments, where credits and further references are given. To avoid duplication with the existing literature on semimartingales, we did not include material on stochastic calculus or limit theorems; we refer to Jacod (1979), Protter (1990) and Jacod and Shiryaev (1987) for detailed expositions.

We use the following labels. Roman numbers refer to chapters, arabic numbers to statements, and numbers between parentheses to equations or formulas. For instance, Proposition V.2 designates the proposition with label 2 of chapter V, and (III.10) the equation with label (10) in chapter III. The roman number refering to a chapter is omitted within the same chapter.

This text is partially based on a 'cours de troisième cycle' taught in the Laboratoire de Probabilités de L'Université Pierre-et-Marie-Curie. My work was greatly eased by the position I had at this time in the Centre National de la Recherche Scientifique. I should like to thank warmly my colleagues in the Laboratoire de Probabilités, and to express my deep gratitude to Nick Bingham, Ron Doney, Daniel Revuz and Hrvoje Sikic, who read preliminary versions of the manuscript and corrected uncountable errors, misprints and misuses of the English language.

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Preliminaries

1. Notation

In this section, we set down notation which will be used throughout the text.

We denote by $\mathbf{R}^d$ the d-dimensional Euclidean space, equipped with the standard scalar product $\langle \cdot, \cdot \rangle$ and the Euclidean norm $|\cdot|$. It is endowed with the Borel sigma-field $\mathscr{B}(\mathbf{R}^d)$ and the Lebesgue measure dx. The abbreviation a.e. refers to 'almost everywhere' with respect to the Lebesgue measure. The lower and upper bounds of a subset A of the nonnegative half-line $[0, \infty)$ are denoted by $\inf A$ and $\sup A$, respectively, with the convention that $\inf \emptyset = \infty$ and $\sup \emptyset = 0$. We say that a function $f : [0, \infty) \to [0, \infty]$ is increasing if $f(s) \leq f(t)$ for all $0 \leq s \leq t$. If the preceding condition holds with $\leq$ replaced by $<$, we say that f is strictly increasing. We use Landau's notation $f = o(g)$, $f = O(g)$ and $f \sim g$ for $\lim(f/g) = 0$, $\limsup(f/g) < \infty$ and $\lim(f/g) = 1$, respectively.

Next, we introduce the so-called canonical notation for right-continuous substochastic (i.e. possibly defective) processes having left limits. Specifically, take an isolated point ∂ which will serve as cemetery. Consider

$$\Omega = D([0, \infty), \mathbf{R}^d \cup \{\partial\}),$$

the set of paths $\omega : [0, \infty) \to \mathbf{R}^d \cup \{\partial\}$ with lifetime

$$\zeta(\omega) = \inf\{t \geq 0 : \omega(t) = \partial\}$$

which are right-continuous on $[0, \infty)$, have a left limit denoted by $\omega(s-)$ for any $s \in (0, \infty)$, and stay at the cemetery point ∂ after the lifetime $\zeta(\omega)$. This space is endowed with Skorohod's topology, for which we refer to chapter VI in Jacod and Shiryaev (1987). In particular, Ω is a Polish space, that is it is metric-complete and separable. We shall not use Skorohod's topology directly, but it is crucial to work on a Polish space to apply fundamental theorems of probability theory, such as the existence of conditional laws. The Borel sigma-field of Ω is denoted by $\mathscr{F}$.

We then introduce the coordinate process $X = (X_t, t \geq 0)$, where

$$X_t = X_t(\omega) = \omega(t).$$

We also write $\zeta = \zeta(\omega)$ for the lifetime of X and

$$X_{s-} = X_{s-}(\omega) = \omega(s-) \quad , \quad \Delta X_s = X_s - X_{s-}$$

respectively for the left limit and the jump at time $s \in (0,\zeta)$. The family of mappings $\theta_t : \Omega \to \Omega$ and $\mathrm{k}_t : \Omega \to \Omega$ $(t \geq 0)$, specified by

$$\theta_t\omega(s) = \omega(t+s) \qquad (s \geq 0)$$

and

$$\mathrm{k}_t\omega(s) = \begin{cases} \omega(s) & \text{if } s < t\,, \\ \partial & \text{otherwise} \end{cases}$$

are called the translation and the killing operators, respectively.

Suppose that $\mathbb{P}$ is a probability measure on $(\Omega, \mathscr{F})$, and Y is a random variable, say taking values in $\mathbb{R}^d$. We denote the expectation of Y under $\mathbb{P}$ by $\mathbb{E}(Y)$ whenever it makes sense. We then write $\mathbb{E}(Y, \Lambda_1, \cdots, \Lambda_k)$ for $\mathbb{E}(\mathbf{1}_\Lambda Y)$ with $\Lambda = \Lambda_1 \cap \cdots \cap \Lambda_k$, where $\Lambda_1, \cdots, \Lambda_k \in \mathscr{F}$, and $\mathbb{E}(Y \mid \mathscr{G})$ for the conditional expectation given some subfield $\mathscr{G}$. Finally, we denote either by $\mathbb{P}(Y \in \cdot)$ or by $\mathbb{P}(Y \in dy)$ the distribution of Y under $\mathbb{P}$. We say that a family $(\mathbb{P}(\cdot|Y = y), y \in \mathbb{R}^d)$ of laws on $(\Omega, \mathscr{F})$ is a version of the conditional law $\mathbb{P}$ given Y if the mapping $y \to \mathbb{P}(\cdot|Y = y)$ is measurable, $\mathbb{P}(Y = y|Y = y) = 1$ for all $y \in \mathbb{R}^d$, and

$$\mathbb{P}(\Lambda) = \int_{\mathbb{R}^d} \mathbb{P}(\Lambda \mid Y = y)\mathbb{P}(Y \in dy), \qquad \Lambda \in \mathscr{F}.$$

We refer e.g. to chapter III in Dellacherie and Meyer (1975) for the existence of conditional laws.

2. Infinitely divisible distributions

Consider a probability measure μ on $\mathbb{R}^d$, and its characteristic function

$$\mathscr{F}\mu(\lambda) = \int_{\mathbb{R}^d} \exp\{\mathrm{i}\langle\lambda, x\rangle\}\mu(dx) \quad (\lambda \in \mathbb{R}^d).$$

The law μ is called *infinitely divisible* if for any positive integer n, there exists a probability measure μ_n with characteristic function $\mathscr{F}\mu_n$ such that $\mathscr{F}\mu = (\mathscr{F}\mu_n)^n$. In other words, μ can be expressed as the n-th convolution power of μ_n. The simplest examples of infinitely divisible laws are Dirac point masses, Gaussian and stable distributions, and in dimension $d = 1$, Poisson and Gamma distributions.

Assume now that μ is infinitely divisible. Then its characteristic function never vanishes and can be expressed as follows. There is a unique continuous function $\Psi : \mathbb{R}^d \to \mathbb{C}$, called the *characteristic exponent* of μ, such that $\Psi(0) = 0$ and

$$\mathscr{F}\mu(\lambda) = \exp\{-\Psi(\lambda)\} \quad (\lambda \in \mathbb{R}^d).$$

We see that if μ_1 and μ_2 are two infinitely divisible laws with respective characteristic exponents Ψ_1 and Ψ_2, then the convolution $\mu_1 * \mu_2$ is again infinitely divisible with characteristic exponent $\Psi_1 + \Psi_2$.

The starting point of many studies of infinitely divisible laws is the famous Lévy-Khintchine formula (see for instance section 7.6 in Chung (1968), or chapter XVII in Feller (1971)) which determines the class of characteristic functions corresponding to infinitely divisible laws.

Lévy-Khintchine formula *A function $\Psi : \mathbb{R}^d \to \mathbb{C}$ is the characteristic exponent of an infinitely divisible probability measure on $\mathbb{R}^d$ if and only if there are $a \in \mathbb{R}^d$, a positive semi-definite quadratic form Q on $\mathbb{R}^d$, and a measure Π on $\mathbb{R}^d - \{0\}$ with $\int (1 \wedge |x|^2)\Pi(dx) < \infty$ such that*

$$\Psi(\lambda) = \mathrm{i}\langle a, \lambda\rangle + \frac{1}{2}Q(\lambda) + \int_{\mathbb{R}^d} \left(1 - \mathrm{e}^{\mathrm{i}\langle \lambda, x\rangle} + \mathrm{i}\langle \lambda, x\rangle \mathbf{1}_{\{|x|<1\}}\right)\Pi(dx) \qquad (1)$$

for every $\lambda \in \mathbb{R}^d$.

The parameters a, Q, and Π appearing in (1) are determined by Ψ, and their probabilistic meanings will be clarified in section I.1. The measure Π is called the *Lévy measure* of μ and the quadratic form Q the *Gaussian coefficient* . We mention that some authors use a slightly different expression for the Lévy-Khintchine formula. Specifically, the cut-off function $\mathbf{1}_{\{|x|<1\}}$ is replaced by a bounded smooth function which is equivalent to 1 at the origin, the most common being $(1 + |x|^2)^{-1}$. Such a change in the choice of the cut-off function does not alter the Lévy measure and the Gaussian coefficient, but the parameter a has to be replaced by

$$a' = a + \int_{\mathbb{R}^d} x \left(\frac{1}{1 + |x|^2} - \mathbf{1}_{\{|x|<1\}}\right) \Pi(dx).$$

3. Martingales

Consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ endowed with a filtration $(\mathscr{F}_t)_{t\geq 0}$, i.e. an increasing family of sub-fields, which fulfils the usual conditions. That is each $\mathscr{F}_t$ is $\mathbb{P}$-complete and $\mathscr{F}_t = \bigcap_{s>t} \mathscr{F}_s$ for every t. A real-valued stochastic process $M = (M_t, 0 \leq t < \infty)$ is a *martingale* if

$$\mathbb{E}(M_t \mid \mathscr{F}_s) = M_s, \qquad 0 \leq s \leq t.$$

(It is implicit here that $\mathbb{E}(|M_t|) < \infty$ for all t.) We say that M is right-continuous if its sample paths are right-continuous a.s., and *uniformly integrable* if there exists an increasing function $f : [0, \infty) \to [0, \infty)$ with $x = o(f(x))$ as x goes to ∞, such that $\sup\{\mathbb{E}(f(|M_t|)) : t \geq 0\} < \infty$.

We assume from now on that M is a right-continuous martingale. The following key results are due to Doob, and we refer to chapter VI of Dellacherie and Meyer (1980) for a complete account.

Maximal inequality *For every* $t > 0$, *we have*

$$\mathbb{E}\left(\sup\{|M_s|^2 : 0 \le s \le t\}\right) \le 4\mathbb{E}(|M_t|^2).$$

A nonnegative random variable T is called a *stopping time* if for every $t \ge 0$, $\{T \le t\} \in \mathcal{F}_t$.

Optional sampling theorem *Suppose that T is a stopping time, a.s. finite.*

(i) *The stopped process* $(M_{T\wedge t}, t \ge 0)$ *is again a martingale.*
(ii) *Suppose moreover that M is uniformly integrable. Then* $\mathbb{E}(M_T) = \mathbb{E}(M_0)$.

Convergence theorem *Suppose that M is uniformly integrable. Then* $\lim_{t\to\infty} M_t = M_\infty$ *exists a.s. and in* $L^1(\mathbb{P})$, *and* $M_t = \mathbb{E}(M_\infty \mid \mathcal{F}_t)$ *for all* t.

4. Poisson processes

The proofs of the results stated in this section and the next can be found in section XII.1 in Revuz and Yor (1994).

The Poisson distribution with parameter (or intensity) $c > 0$ is the probability measure on integers which assigns mass $e^{-c}c^k/(k!)$ at point $k \in \mathbb{N}$. Its characteristic function is

$$\sum_{k=0}^{\infty} e^{i\lambda k} e^{-c} \frac{c^k}{k!} = \exp\{-c(1 - e^{i\lambda})\} \quad , \quad \lambda \in \mathbb{R}.$$

The Poisson distribution is infinitely divisible, and the results of section I.1 below guarantee the existence of a unique (in law) increasing right-continuous process N with stationary independent increments, called a Poisson process of parameter (or intensity) c, such that for each $t > 0$, N_t has a Poisson distribution with parameter ct. One can also construct N directly as follows. Consider a probability measure $\mathbb{P}$ and a sequence $\tau_1, \cdots, \tau_n, \cdots$ of independent exponential variables with parameter c, that is $\mathbb{P}(\tau_i > s) = e^{-cs}$ for $s \ge 0$. Introduce the partial sums $S_n = \tau_1 + \cdots + \tau_n$, $n \in \mathbb{N}$, so that S_n has the Gamma(c, n) distribution,

$$\mathbb{P}(S_n \in ds) = \frac{c^n}{(n-1)!} s^{n-1} e^{-cs}\, ds \qquad (s \ge 0)\,.$$

Then consider the right-continuous inverse $N_t = \sup\{n \in \mathbb{N} : S_n \leq t\}$ $(t \geq 0)$, so that for every $t \geq 0$ and $k \in \mathbb{N}$,

$$\mathbb{P}(N_t = k) = \mathbb{P}(S_k \leq t, S_{k+1} > t) = \int_0^t \frac{c^k}{(k-1)!} s^{k-1} \mathrm{e}^{-cs} \mathrm{e}^{-c(t-s)}\, ds$$
$$= \mathrm{e}^{-ct}(ct)^k/(k!)\,.$$

On the other hand, it follows easily from the so-called lack-of-memory property of the exponential law that for every $0 \leq s \leq t$, the increment $N_{t+s} - N_t$ has the Poisson distribution with parameter cs and is independent of the sigma-field generated by $(N_u, u \leq t)$.

Next, let $(\mathcal{G}_t)$ be a filtration which satisfies the usual conditions. We say that N is a $(\mathcal{G}_t)$-Poisson process if N is a Poisson process which is adapted to $(\mathcal{G}_t)$ and for every $s, t \geq 0$, the increment $N_{t+s} - N_t$ is independent of $\mathcal{G}_t$. In particular, N is a $(\mathcal{G}_t)$-Poisson process if $(\mathcal{G}_t)$ is the natural filtration of N.

There are three important families of martingales related to a $(\mathcal{G}_t)$-Poisson process. First, one says that a process $H = (H_t, t \geq 0)$ is *predictable* if it is measurable in the sigma-field generated by the left-continuous adapted processes. If H is a real-valued predictable process with $\mathbb{E}(\int_0^t |H_s| ds) < \infty$ for all $t \geq 0$ and if $N = (N_t, t \geq 0)$ is a $(\mathcal{G}_t)$-Poisson process with parameter $c > 0$, then the *compensated integral*

$$M_t = \int_0^t H_s dN_s \;-\; c\int_0^t H_s ds \quad (t \geq 0)$$

is a $(\mathcal{G}_t)$-martingale. If moreover $\mathbb{E}(\int_0^t H_s^2 ds) < \infty$, then

$$M_t^2 - c\int_0^t H_s^2 ds \quad (t \geq 0)$$

is also a martingale. Finally, if H is predictable and bounded, then the same holds for the exponential process

$$\exp\left\{\int_0^t H_s dN_s + c\int_0^t (1 - \mathrm{e}^{H_s}) ds\right\} \quad (t \geq 0)\,.$$

Here, the various integrals with dN_s as integrator are taken in the sense of Stieltjes.

We conclude this section by recalling a well-known criterion for the independence of Poisson processes.

Proposition 1 *Let $N^{(i)}, i = 1, \cdots, d$, be $(\mathcal{G}_t)$-Poisson processes. They are independent if and only if they never jump simultaneously, that is for* *every i, j with $i \neq j$*

$$N_t^{(i)} - N_{t-}^{(i)} = 0 \text{ or } N_t^{(j)} - N_{t-}^{(j)} = 0 \quad \text{for all } t > 0, \text{ a.s.,}$$

where $N_{t-}^{(k)}$ stands for the left limit of $N^{(k)}$ at time t.

It is crucial in Proposition 1 to assume that the $N^{(i)}$ are Poisson processes in the same filtration. Otherwise, it is easy to construct Poisson processes which never jump simultaneously and which are not independent.

5. Poisson measures and Poisson point processes

Let E be a Polish space and ν a sigma-finite measure on E. We call a random measure φ on E *a Poisson measure with intensity* ν if it satisfies the following. For every Borel subset B of E with $\nu(B) < \infty$, $\varphi(B)$ has a Poisson distribution with parameter $\nu(B)$, and if $B_1, \cdots, B_n$ are disjoint Borel sets, the variables $\varphi(B_1), \cdots, \varphi(B_n)$ are independent. Plainly, φ is then a sum of Dirac point masses.

One can construct Poisson measures as follows. First, assume that the total mass of ν is finite, and put $c = \nu(E)$. Let $\xi_1, \cdots, \xi_n, \cdots$ be a sequence of independent identically distributed random variables with common law $c^{-1}\nu$ and a Poisson variable N with parameter c independent of the ξ_n's. The random measure

$$\varphi = \sum_{j=1}^{N} \delta_{\xi_j},$$

where δ_ϵ stands for the Dirac point mass at $\epsilon \in E$, is a Poisson measure with intensity ν. If ν is merely sigma-finite, there exists a partition $(E_n, n \in \mathbb{N})$ of E into Borel sets such that $\nu(E_n) < \infty$ for every integer n. Then we can construct a sequence φ_n of independent Poisson measures with respective characteristic measures $\mathbf{1}_{E_n}\nu$, and $\varphi = \sum_n \varphi_n$ is a Poisson measure with intensity ν.

We then consider the product space $E \times [0, \infty)$, the measure $\mu = \nu \otimes dx$, and a Poisson measure φ on $E \times [0, \infty)$ with intensity μ. It is easy to check that a.s., $\varphi(E \times \{t\}) = 0$ or 1 for all $t \geq 0$. This enables us to represent φ in terms of a stochastic process taking values in $E \cup \{\Upsilon\}$, where Υ is an isolated additional point. Specifically, if $\varphi(E \times \{t\}) = 0$, then put $e(t) = \Upsilon$. If $\varphi(E \times \{t\}) = 1$, then the restriction of φ to the section $E \times \{t\}$ is a Dirac point mass, say at (ϵ, t), and we put $e(t) = \epsilon$. We can now express the Poisson measure as

$$\varphi = \sum_{t \geq 0} \delta_{(e(t),t)}.$$

The process $e = (e(t), t \geq 0)$ is called a *Poisson point process* with characteristic measure ν. We denote its natural filtration after the usual completion by $(\mathcal{G}_t)$.

For every Borel subset B of E, we call

$$N_t^B = \text{Card}\{s \leq t : e(s) \in B\} = \varphi(B \times [0,t]) \qquad (t \geq 0)$$

the *counting process* of B. It is a $(\mathcal{G}_t)$-Poisson process with parameter $\nu(B)$. Conversely, suppose that $e = (e(t), t \geq 0)$ is a stochastic process taking values in $E \cup \{\Upsilon\}$ such that, for every Borel subset B of E, the counting process $N_t^B = \text{Card}\{s \leq t : e(s) \in B\}$ is a Poisson process with intensity $\nu(B)$ in a given filtration $(\mathcal{G}_t)$. Then observe that counting processes associated to disjoint Borel sets never jump simultaneously and thus are independent according to Proposition 1. One then deduces that the associated random measure $\varphi = \sum_{t\geq 0} \delta_{(e(t),t)}$ is a Poisson measure with intensity μ.

We next present a useful probabilistic interpretation of the characteristic measure ν.

Proposition 2 *Let B be a Borel set with $0 < \nu(B) < \infty$. The first entrance time of e into B, $T_B = \inf\{t \geq 0 : e(t) \in B\}$, is a $(\mathcal{G}_t)$-stopping time and we have*

(i) *T_B has an exponential distribution with parameter $\nu(B)$.*

(ii) *The random variable $e(T_B)$ is independent of T_B and has the law $\nu(\cdot \mid B)$, that is for every Borel set A,*

$$\mathbb{P}(e(T_B) \in A) = \nu(A \cap B)/\nu(B) .$$

(iii) *The process e' given by $e'(t) = \Upsilon$ if $e(t) \in B$ and $e'(t) = e(t)$ otherwise $(t \geq 0)$ is a Poisson point process with characteristic measure $1_{B^c}\nu$, and is independent of $(T_B, e(T_B))$.*

The process $(e_t, 0 \leq t \leq T_B)$ is called stopped at the first point in B, its law is characterized by Proposition 2.

In practice, it is important to calculate certain expressions in terms of the characteristic measure. The following two formulas are the most useful:

Compensation formula *Let $H = (H_t, t \geq 0)$ be a predictable process taking values in the space of nonnegative measurable functions on $E \cup \{\Upsilon\}$, such that $H_t(\Upsilon) = 0$ for all $t \geq 0$. We have*

$$\mathbb{E}\left(\sum_{0\leq t<\infty} H_t(e(t))\right) = \mathbb{E}\left(\int_0^\infty dt \int_E d\nu(\epsilon) H_t(\epsilon)\right) .$$

Exponential formula *Let f be a complex-valued Borel function on $E \cup \{\Upsilon\}$ with $f(\Upsilon)=0$ and*

$$\int_E \nu(d\epsilon)|1-e^{f(\epsilon)}| < \infty .$$

We have for every $t \geq 0$

$$\mathbb{E}\left(\exp\left\{\sum_{0\leq s\leq t} f(e(s))\right\}\right) = \exp\left\{-t\int_E \nu(d\epsilon)(1-e^{f(\epsilon)})\right\} .$$

These two formulas are easy to prove when the space E is finite, using respectively the first and the third special martingale of section 4. The general case then follows from a monotone class theorem.

We conclude this section with a useful inequality which is a consequence of Doob's maximal inequality applied to the first special martingale of section 4.

Maximal inequality for compensated sums *Let f be a Borel function on $E \cup \{\Upsilon\}$ with $f(\Upsilon)=0$. We have for every fixed $T > 0$*

$$\mathbb{E}\left(\sup\left\{\left|\sum_{0\leq s\leq t} f(e(s)) - t\int_E f(\epsilon)d\nu(\epsilon)\right|^2, 0\leq t\leq T\right\}\right) \leq 4T\int_E f(\epsilon)^2 d\nu(\epsilon) .$$

6. Brownian motion

A real-valued stochastic process $B=(B_t, t\geq 0)$ is a (linear) *Brownian motion* if its sample paths are continuous a.s., its law at any fixed time $t > 0$ is the centred Gaussian distribution with variance t,

$$\mathbb{P}(B_t \in dx) = (2\pi t)^{-1/2}\exp\{-x^2/2t\}dx ,$$

and its increments are independent in the sense that for any $s, t > 0$, $B_{t+s} - B_t$ is independent of the σ-field generated by $(B_u, 0\leq u\leq t)$. Note that this implies that $B_{t+s}-B_t$ has the centred Gaussian distribution with variance s, so that B is a Gaussian process with stationary (or, homogeneous) independent increments. Finally, a process $(B^1,\cdots,B^d)$ taking values in the d-dimensional Euclidean space is a Brownian motion if its coordinates $B^1,\cdots,B^d$ are independent linear Brownian motions.

There are several different constructions of Brownian motion; here is one of the simplest (see e.g. section I.1 in Revuz and Yor (1994)). First, a standard result guarantees the existence of a centred Gaussian process $\widetilde{B}=(\widetilde{B}_t, t\geq 0)$ with covariance $\mathbb{E}(\widetilde{B}_t\widetilde{B}_s) = s\wedge t$. Then one applies Kolmogorov's criterion to verify that there is a continuous version B of $\widetilde{B}$, that is $B_t = \widetilde{B}_t$ a.s. for every $t\geq 0$. Actually, Kolmogorov's criterion