J.H. CONWAY
N.J.A. SLOANE

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SPHERE PACKINGS, LATTICES AND GROUPS

THIRD EDITION

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J.H. Conway N.J.A. Sloane

Sphere Packings, Lattices and Groups

Third Edition

With Additional Contributions by
E. Bannai, R.E. Borcherds, J. Leech,
S.P. Norton, A.M. Odlyzko, R.A. Parker,
L. Queen and B.B. Venkov

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Preface to First Edition

The main themes. This book is mainly concerned with the problem of **packing spheres** in Euclidean space of dimensions 1,2,3,4,5,.... Given a large number of equal spheres, what is the most efficient (or densest) way to pack them together? We also study several closely related problems: the **kissing number problem**, which asks how many spheres can be arranged so that they all touch one central sphere of the same size; the **covering problem**, which asks for the least dense way to cover n-dimensional space with equal overlapping spheres; and the **quantizing problem**, important for applications to analog-to-digital conversion (or data compression), which asks how to place points in space so that the average second moment of their Voronoi cells is as small as possible. Attacks on these problems usually arrange the spheres so their centers form a *lattice*. Lattices are described by *quadratic forms*, and we study the **classification of quadratic forms**. Most of the book is devoted to these five problems.

The miraculous enters: the E_8 and Leech lattices. When we investigate those problems, some fantastic things happen! There are two sphere packings, one in eight dimensions, the E_8 *lattice*, and one in twenty-four dimensions, the *Leech lattice* Λ_{24}, which are unexpectedly good and very symmetrical packings, and have a number of remarkable and mysterious properties, not all of which are completely understood even today. In a certain sense we could say that the book is devoted to studying these two lattices and their properties.

At one point while working on this book we even considered adopting a special abbreviation for "It is a remarkable fact that", since this phrase seemed to occur so often. But in fact we have tried to avoid such phrases and to maintain a scholarly decorum of language.

Nevertheless there are a number of astonishing results in the book, and perhaps this is a good place to mention some of the most miraculous. (The technical terms used here are all defined later in the book.)

— The occurrence of the Leech lattice as the unique laminated lattice in 24 dimensions (Fig. 6.1).
— Gleason's theorem describing the weight enumerators of doubly-even self-dual codes, and Hecke's theorem describing the theta series of even unimodular lattices (Theorems 16 and 17 of Chap. 7).
— The one-to-one correspondence between the 23 deep holes in the Leech lattice and the even unimodular 24-dimensional lattices of minimal norm 2 (Chapters 16, 23, 26).
— The construction of the Leech lattice in $II_{25,1}$ as $w^{\perp}/w$, where

$$w = (0,1,2,3,\ldots,24|70)$$

is a vector of zero norm (Theorem 3 of Chap. 26). (We remind the reader that in Lorentzian space $\mathbf{R}^{25,1}$, which is 26-dimensional space equipped with the norm $x \cdot x = x_1^2 + \cdots + x_{25}^2 - x_{26}^2$, there is a unique even unimodular lattice $II_{25,1}$ — see §1 of Chap. 26.)
— The occurrence of the Leech lattice as the Coxeter diagram of the reflections in the automorphism group of $II_{25,1}$ (Chap. 27).

Some further themes. Besides the five problems that we mentioned in the first paragraph, there are some other topics that are always in our minds: *error-correcting codes, Steiner systems* and *t-designs*, and the theory of *finite groups*. The E_8 and Leech lattices are intimately involved with these topics, and we investigate these connections in considerable detail. Many of the *sporadic simple groups* are encountered, and we devote a whole chapter to the *Monster* or *Friendly Giant* simple group, whose construction makes heavy use of the Leech lattice.

The main applications. There are many connections between the geometrical problems mentioned in the first paragraph and other areas of mathematics, chiefly with *number theory* (especially quadratic forms and the geometry of numbers).

The main application outside mathematics is to the *channel coding problem*, the design of signals for data transmission and storage, i.e. the *design of codes for a band-limited channel* with white Gaussian noise. It has been known since the work of Nyquist and Shannon that the design of optimal codes for such a channel is equivalent to the sphere packing problem. *Theoretical* investigations into the information-carrying capacity of these channels requires knowledge of the best sphere packings in large numbers of dimensions. On the other hand *practical* signalling systems (notably the recently developed Trellis Coded Modulation schemes) have been designed using properties of sphere packings in low dimensions, and certain modems now on the market use codes consisting of points taken from the E_8 lattice!

There are beautiful applications of these lattices to the numerical evaluation of n-dimensional integrals (see § 4.2 of Chap. 3).

Of course there are connections with *chemistry*, since crystallographers have studied three-dimensional lattices since the beginning of the subject. We sometimes think of this book as being a kind of higher-dimensional analog of Wells' "Structural Inorganic Chemistry" [Wel4]. Recent work on *quasi-crystals* has made use of six- and eight-dimensional lattices.

Higher-dimensional lattices are of current interest in *physics*. Recent developments in *dual theory* and *superstring theory* have involved the E_8 and Leech lattices, and there are also connections with the Monster group.

Other applications and references for these topics will be found in Chap. 1.

Who should buy this book. Anyone interested in sphere packings, in lattices in n-dimensional space, or in the Leech lattice. Mathematicians interested in finite groups, quadratic forms, the geometry of numbers, or combinatorics. Engineers who wish to construct n-dimensional codes for a band-limited channel, or to design n-dimensional vector quantizers. Chemists and physicists interested in n-dimensional crystallography.

Prerequisites. In the early chapters we have tried to address (perhaps not always successfully) any well-educated undergraduate. The technical level fluctuates from chapter to chapter, and generally increases towards the end of the book.

What's new here. A number of things in this book have not appeared in print before. We mention just a few.
– Tables of the densest sphere packings known in dimensions up to one million (Tables 1.2, 1.3).
– Tables and graphs of the best coverings and quantizers known in up to 24 dimensions (Chap. 2).
– A table of the best coding gain of lattices in up to 128 dimensions (Table 3.2).
– Explicit arrangements of n-dimensional spheres with kissing numbers that grow like $2^{0.003n}$ (Eq. (56) of Chap. 1), the best currently known.
– Explicit constructions of good spherical codes (§2.5 of Chap. 1).
– Explicit constructions of n-dimensional lattice coverings with density that grows like $2^{0.084n}$ (Eq. (13) of Chap. 2), again the best known.
– New formulae for the theta series of many lattices (e.g. for the Leech lattice with respect to an octahedral deep hole, Eq. (144) of Chap. 4).
– Tables giving the numbers of points in successive shells of these lattices (e.g. the first 50 shells of the Leech lattice, Table 4.14).
– A new general construction for sphere packings (Construction A_f, §3 of Chap. 8).
– New descriptions of the lattice packings recently discovered by McKay, Quebbemann, Craig, etc. (Chap. 8).
– Borcherds' classification of the 24-dimensional odd unimodular lattices and his neighborhood graph of the 24-dimensional even unimodular lattices (Chap. 24).
– A list of the 284 types of shallow hole in the Leech lattice (Chap. 25).
– Some easy ways of computing with rational and integral quadradic forms, including elementary systems of invariants for such forms, and a handy notation for the genus of a form (Chap. 15).
– Tables of binary quadratic forms with $-100 \leqslant \det \leqslant 50$, indecomposable ternary forms with $|\det| \leqslant 50$, genera of forms with $|\det| \leqslant 11$, genera of p-elementary forms for all p, positive definite forms

with determinant 2 up to dimension 18 and determinant 3 up to dimension 17 (Chap. 15).
— A simple description of a construction for the Monster simple group (Chap. 29).

Other tables, which up to now could only be found in journal articles or conference proceedings, include:
— Bounds for kissing numbers in dimensions up to 24 (Table 1.5).
— The Minkowski-Siegel mass constants for even and odd unimodular lattices in dimensions up to 32 (Chap. 16).
— The even and odd unimodular lattices in dimensions up to 24 (Table 2.2 and Chaps. 16, 17).
— Vectors in the first eight shells of the E_8 lattice (Table 4.10) and the first three shells of the Leech lattice (Table 4.13).
— Best's codes of length 10 and 11, that produce the densest packing known (P_{10c}) in 10 dimensions and the highest kissing number known (P_{11c}) in 11 dimensions (Chap. 5).
— Improved tables giving the best known codes of length 2^m for $m \leqslant 8$ (Table 5.4) and of all lengths up to 24 (Table 9.1).
— Laminated lattices in dimensions up to 48 (Tables 6.1, 6.3).
— The best integral lattices of minimal norms 2, 3 and 4, in dimensions up to 24 (Table 6.4).
— A description of E_8 lattice vectors in terms of icosians (Table 8.1).
— Minimal vectors in McKay's 40-dimensional lattice M_{40} (Table 8.6).
— The classification of subsets of 24 objects under the action of the Mathieu group M_{24} (Fig. 10.1).
— Groups associated with the Leech lattice (Table 10.4).
— Simple groups that arise from centralizers in the Monster (Chap. 10).
— Second moments of polyhedra in 3 and 4 dimensions (Chap. 21).
— The deep holes in the Leech lattice (Table 23.1).
— An extensive table of Leech roots, in both hyperbolic and Euclidean coordinates (Chap. 28).
— Coxeter-Vinberg diagrams for the automorphism groups of the lattices $\mathrm{I}_{n,1}$ for $n \leqslant 20$ (Chap. 28) and $\mathrm{II}_{n,1}$ for $n \leqslant 24$ (Chap. 27).

The contents of the chapters. Chapters 1-3 form an extended introduction to the whole book. In these chapters we survey what is presently known about the packing, kissing number, covering and quantizing problems. There are sections on quadratic forms and their classification, the connections with number theory, the channel coding problem, spherical codes, error-correcting codes, Steiner systems, t-designs, and the connections with group theory. These chapters also introduce definitions and terminology that will be used throughout the book.

Chapter 4 describes a number of important lattices, including the cubic lattice $\mathbf{Z}^n$, the root lattices A_n, D_n, E_6, E_7, E_8, the Coxeter-Todd lattice K_{12}, the Barnes-Wall lattice Λ_{16}, the Leech lattice Λ_{24}, and their duals. Among other things we give their minimal vectors, densities, covering radii, glue vectors, automorphism groups, expressions for their theta series, and tables of the numbers of points in the first fifty shells. We also include a brief discussion of reflection groups and of the technique of gluing lattices together.

Chapters 5-8 are devoted to techniques for constructing sphere packings. Many of the constructions in Chaps. 5 and 7 are based on error-correcting codes; other constructions in Chapter 5 build up packings by layers. Layered packings are studied in greater detail in Chap. 6, where the formal concept of a *laminated lattice* Λ_n is introduced. Chapter 8 uses a number of more sophisticated algebraic techniques to construct lattices.

Chapter 9 introduces analytical methods for finding bounds on the best codes, sphere packings and related problems. The methods use techniques from harmonic analysis and linear programming. We give a simplified account of Kabatiansky and Levenshtein's recent sphere packing bounds.

Chapters 10 and 11 study the Golay codes of length 12 and 24, the associated Steiner systems $S(5,6,12)$ and $S(5,8,24)$, and their automorphism groups M_{12} and M_{24}. The MINIMOG and MOG (or Miracle Octad Generator) and the Tetracode and Hexacode are computational tools that make it easy to perform calculations with these objects. These two chapters also study a number of related groups, in particular the automorphism group Co_0 (or $\cdot 0$) of the Leech lattice. The Appendix to Chapter 10 describes all the sporadic simple groups.

Chapter 12 gives a short proof that the Leech lattice is the unique even unimodular lattice with no vectors of norm 2. Chapter 13 solves the kissing number problem in 8 and 24 dimensions — the E_8 and Leech lattices have the highest possible kissing numbers in these dimensions. Chapter 14 shows that these arrangements of spheres are essentially unique.

Chapters 15-19 deal with the classification of integral quadratic forms. Chapters 16 and 18 together give three proofs that Niemeier's enumeration of the 24-dimensional even unimodular lattices is correct. In Chap. 19 we find all the extremal odd unimodular lattices in any dimension.

Chapters 20 and 21 are concerned with geometric properties of lattices. In Chap. 20 we discuss algorithms which, given an arbitrary point of the space, find the closest lattice point. These algorithms can be used for vector quantizing or for encoding and decoding lattice codes for a bandlimited channel. Chapter 21 studies the Voronoi cells of lattices and their second moments.

Soon after discovering his lattice, John Leech conjectured that its covering radius was equal to $\sqrt{2}$ times its packing radius, but was unable to find a proof. In 1980 Simon Norton found an ingenious argument which shows that the covering radius is no more than 1.452... times the packing radius (Chap. 22), and shortly afterwards Richard Parker and the authors managed to prove Leech's conjecture (Chap. 23).

Our method of proof involves finding all the "deep holes" in the Leech lattice, i.e. all points of 24-dimensional space that are maximally distant from the lattice. We were astonished to discover that there are precisely 23 distinct types of deep hole, and that they are in one-to-one correspondence with the Niemeier lattices (the 24-dimensional even unimodular lattices of minimal norm 2) — see Theorem 2 of Chap. 23. Chapter 23, or the Deep Holes paper, as it is usually called, has turned out to be extremely fruitful, having stimulated the remaining chapters in the book, also Chap. 6, and several journal articles.

In Chap. 24 we give 23 constructions for the Leech lattice, one for each of the deep holes or Niemeier lattices. Two of these are the familiar constructions based on the Golay codes. In the second half of Chap. 24 we introduce the *hole diagram* of a deep hole, which describes the environs of the hole. Chapter 25 (the Shallow Holes paper) uses the results of Chap. 23 and 24 to classify *all* the holes in the Leech lattice.

Considerable light is thrown on these mysteries by the realization that the Leech lattice and the Niemeier lattices can all be obtained very easily from a single lattice, namely $II_{25,1}$, the unique even unimodular lattice in Lorentzian space $\mathbf{R}^{25,1}$. For any vector $w \in \mathbf{R}^{25,1}$, let

$$w^{\perp} = \{x \in II_{25,1} : x \cdot w = 0\} .$$

Then if w is the special vector

$$w_{25} = (0,1,2,3, \ldots ,23,24 | 70) ,$$

$w^{\perp}/w$ is the Leech lattice, and other choices for w lead to the 23 Niemeier lattices.

The properties of the Leech lattice are closely related to the geometry of the lattice $II_{25,1}$. The automorphism groups of the Lorentzian lattice $I_{n,1}$ for $n \leqslant 19$ and $II_{n,1}$ for $n = 1$, 9 and 17 were found by Vinberg, Kaplinskaja and Meyer. Chapter 27 finds the automorphism group of $II_{25,1}$. This remarkable group has a reflection subgroup with a Coxeter diagram that is, speaking loosely, isomorphic to the Leech lattice. More precisely, a set of fundamental roots for $II_{25,1}$ consists of the vectors $r \in II_{25,1}$ satisfying

$$r \cdot r = 2 , \ r \cdot w_{25} = -1 ,$$

and we call these the *Leech roots*. Chapter 26 shows that there is an isometry between the set of Leech roots and the points of the Leech lattice. Then the Coxeter part of the automorphism group of $II_{25,1}$ is just the Coxeter group generated by the Leech roots (Theorem 1 of Chap. 27).

Since $II_{25,1}$ is a natural quadratic form to study, whose definition certainly does not mention the Leech lattice, it is surprising that the Leech lattice essentially determines the automorphism group of the form.

The Leech roots also provide a better understanding of the automorphism groups of the other lattices $I_{n,1}$ and $II_{n,1}$, as we see in Chap. 28. This chapter also contains an extensive table of Leech roots. Chapter 29 describes a construction for the Monster simple group, and the final chapter describes an infinite-dimensional Lie algebra that is obtained from the Leech roots, and conjectures that it may be related to the Monster.

The book concludes with a bibliography of about 1550 items.

The structure of this book. Our original plan was simply for a collection of reprints. But over the past two years the book has been completely transformed: many new chapters have been added, and the original chapters have been extensively rewritten to bring them up to date, to reduce duplicated material, to adopt a uniform notation and terminology, and to eliminate errors.

We have however allowed a certain amount of duplication to remain, to make for easier reading. Because some chapters were written at different times and by different authors, the reader will occasionally notice differences in style from chapter to chapter.

The arrangement of the chapters presented us with a difficult problem. We feel that readers are best served by grouping them in the present arrangement, even though it means that one or two chapters are not in strict logical order. The worst flaw is that the higher-dimensional part of the laminated lattices chapter (Chap. 6) depends on knowledge of the deep holes in the Leech lattice given in Chap. 23. But the preceding part of Chapter 6, which includes all the best lattice packings known in small dimensions, had to appear as early as possible.

Chapters 1-4, 7, 8, 11, 15, 17, 20, 25 and 29 are new. Chapter 5 is based on J. Leech and N.J.A.S., *Canad. J. Math.* **23** (1971) (see reference [Lee10] in the bibliography for full details); Chap. 6 on J.H.C. and N.J.A.S., *Annals of Math.* **116** (1982) [Con32]; Chap. 9 is based on N.J.A.S., *Contemp. Math.* **9** (1982), published by the American Mathematical Society [Slo13]; Chap. 10 on J.H.C., in *Finite Simple Groups*, edited by M. B. Powell and G. Higman, Academic Press, N.Y. 1971 [Con5]; Chap. 12 on J.H.C., *Invent. Math.* **7** (1969), published by Springer-Verlag, N.Y. [Con4]; Chap. 13 on A. M. Odlyzko and N.J.A.S., *J. Combin. Theory* **A26** (1979), published by Academic Press, N.Y. [Odl5]; Chap. 14 on E. Bannai and N.J.A.S., *Canad. J. Math.* **33** (1981) [Ban13]; Chap. 16 on J.H.C. and N.J.A.S., *J. Number Theory* **15** (1982) [Con27], and *Europ. J. Combin.* **3** (1982) [Con34], both published by Academic Press, N.Y.; Chapter 18 on B. B. Venkov, *Trudy Mat. Inst. Steklov* **148** (1978), English translation in *Proc. Steklov Inst. Math.*, published by the American Mathematical Society [Ven1]; Chap. 19 on J.H.C., A. M. Odlyzko and N.J.A.S., *Mathematika*, **25** (1978) [Con19]; Chap. 21 on J.H.C. and N.J.A.S., *IEEE Trans. Information Theory*, **28** (1982) [Con28]; Chap. 22 on S. P. Norton, *Proc. Royal Society London*, **A380** (1982) [Nor4]; Chap. 23 on J.H.C., R. A. Parker and N.J.A.S., *Proc. Royal Society London*, **A380** (1982) [Con20]; Chap. 24 on J.H.C. and N.J.A.S., *Proc. Royal Society London*, **A381** (1982) [Con30]; Chap. 26 on J.H.C. and N.J.A.S., *Bulletin American Mathematical Society*, **6** (1982) [Con31]; Chap. 27 on J.H.C., *J. Algebra*, **80** (1983), published by Academic Press, N.Y. [Con13]; Chap. 28 on J.H.C. and N.J.A.S., *Proc. Royal Society London*, **A384** (1982) [Con33]; Chap. 30 on R. E. Borcherds, J.H.C., L. Queen and N.J.A.S., *Advances in Math.* **53** (1984), published by Academic Press, N.Y. [Bor5].

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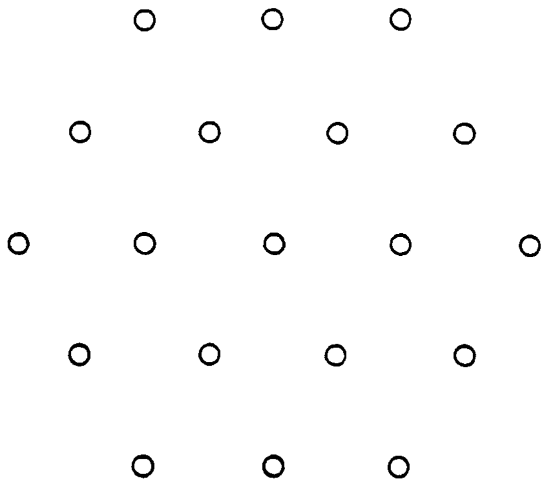
We remark that in two dimensions the familiar hexagonal lattice



solves the packing, kissing, covering and quantizing problems. In a sense this whole book is simply a search for similar nice patterns in higher dimensions.

Preface to Third Edition

Interest in the subject matter of the book continues to grow. The Supplementary Bibliography has been enlarged to cover the period 1988 to 1998 and now contains over 800 items. Other changes from the second edition include a handful of small corrections and improvements to the main text, and this preface (an expanded version of the preface to the Second Edition) which contains a brief report on some of the developments since the appearance of the first edition.

We are grateful to a number of correspondents who have supplied corrections and comments on the first two editions, or who have sent us copies of manuscripts.[1] We thank in particular R. Bacher, R. E. Borcherds, P. Boyvalenkov, H. S. M. Coxeter, Y. Edel, N. D. Elkies, L. J. Gerstein, M. Harada, J. Leech, J. H. Lindsey, II, J. Martinet, J. McKay, G. Nebe, E. Pervin, E. M. Rains, R. Scharlau, F. Sigrist, H. M. Switkay, T. Urabe, A. Vardy, Z.-X. Wan and J. Wills. The new material was expertly typed by Susan K. Pope.

We are planning a sequel, tentatively entitled *The Geometry of Low-Dimensional Groups and Lattices*, which will include two earlier papers [Con36] and [Con37] not included in this book, as well as several recent papers dealing with groups and lattices in low dimensions ([CSLDL1]–[CSLDL8], [CoSl91a], [CoSl95a], etc.).

A Russian version of the first edition, translated by S. N. Litsyn, M. A. Tsfasman and G. B. Shabat, was published by Mir (Moscow) in 1990.

Recent developments, comments, and additional corrections. The following pages attempt to describe recent developments in some of the topics treated in the book. The arrangement roughly follows that of the chapters. Our

[1]We also thank the correspondent who reported hearing the first edition described during a talk as "the bible of the subject, and, like the bible, [it] contains no proofs". This is of course only half true.

coverage is necessarily highly selective, and we apologize if we have failed to mention some important results. In a few places we have also included additional comments on or corrections to the text.

Three books dealing with lattices have recently appeared: in order of publication, these are Ebeling, *Lattices and Codes* [Ebe94], Martinet, *Les Réseaux Parfaits des Espaces Euclidiens* [Mar96] and Conway and Fung, *The Sensual (Quadratic) Form* [CoFu97].

An extensive survey by Lagarias [Laga96] discusses lattices from many points of view not dealt with in our book, as does the Erdős-Gruber-Hammer [ErGH89] collection of **unsolved problems concerning lattices**. The encyclopedic work on **distance-regular graphs** by Brouwer, Cohen and Neumaier [BrCN89] discusses many mathematical structures that are related to topics in our book (see also Tonchev [Ton88]). See also the works by Aschbacher [Asch94] on **sporadic groups**, Engel [Eng86], [Eng93] on **geometric crystallography** [Eng93], Fejes Tóth and Kuperberg [FeK93], and Fejes Tóth [Fej97] on **packing and covering**, Gritzmann and Wills [GriW93b] on **finite packings and coverings**, and Pach and Agarwal [PacA95] on **combinatorial geometry**.

An electronic **data-base of lattices** is now available [NeSl]. This contains information about some 160,000 lattices in dimensions up to 64. The theta-series of the most important lattices can be found in [SloEIS]. The computer languages PARI [BatB91], KANT [Schm90], [Schm91] and especially MAGMA [BosC97], [BosCM94], [BosCP97] have extensive facilities for performing lattice calculations (among many other things).

Notes on Chapter 1: Sphere Packings and Kissing Numbers

Hales [Hal92], [Hal97], [Hal97a], [Hal97b] (see also Ferguson [Ferg97] and Ferguson and Hales [FeHa97]) has described a series of steps that may well succeed in proving the long-standing conjecture (the so-called "**Kepler conjecture**") that no packing of three-dimensional spheres can have a greater density than that of the face-centered cubic lattice. In fact, on August 9, 1998, just as this book was going to press, Hales announced [Hal98] that the final step in the proof has been completed: the Kepler conjecture is now a theorem.

The previous best upper bound known on the density of a three-dimensional packing was due to Muder [Mude93], who showed that the density cannot exceed 0.773055... (compared with $\pi/\sqrt{18} = 0.74048\ldots$ for the f.c.c. lattice).

A paper by W.-Y. Hsiang [Hsi93] (see also [Hsi93a], [Hsi93b]) claiming to prove the Kepler conjecture contains serious flaws. G. Fejes Tóth, reviewing the paper for *Math. Reviews* [Fej95], states: "If I am asked whether the paper fulfills what it promises in its title, namely a proof of Kepler's conjecture, my answer is: no. I hope that Hsiang will