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Jonathan Rosenberg

Algebraic *K*-Theory and Its Applications

代数K理论及其应用

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Preface

Algebraic K-theory is the branch of algebra dealing with linear algebra (especially in the limiting case of large matrices) over a general ring R instead of over a field. It associates to any ring R a sequence of abelian groups $K_i(R)$. The first two of these, K_0 and K_1, are easy to describe in concrete terms; the others are rather mysterious. For instance, a finitely generated projective R-module defines an element of $K_0(R)$, and an invertible matrix over R has a "determinant" in $K_1(R)$. The entire sequence of groups $K_i(R)$ behaves something like a homology theory for rings.

Algebraic K-theory plays an important role in many areas, especially number theory, algebraic topology, and algebraic geometry. For instance, the class group of a number field is essentially $K_0(R)$, where R is the ring of integers, and "Whitehead torsion" in topology is essentially an element of $K_1(\mathbb{Z}\pi)$, where π is the fundamental group of the space being studied. K-theory in algebraic geometry is basic to Grothendieck's approach to the Riemann-Roch problem. Some formulas in operator theory, involving determinants and determinant pairings, are best understood in terms of algebraic K-theory. There is also substantial evidence that the higher K-groups of fields and of rings of integers are related to special values of L-functions and encode deep arithmetic information.

This book is based on a one-semester course I gave at the University of Maryland in the fall of 1990. Most of those attending were second- or third-year graduate students interested in algebra or topology, though there were also a number of analysis students and faculty colleagues from other areas. I tried to make the course (and this book) fairly self-contained, and to assume as a prerequisite only the standard one-year graduate algebra course, based on a text such as [Hungerford], [Jacobson], or [Lang], and the standard introductory graduate course on algebraic and geometric topology, covering the fundamental group, homology, the notions of simplicial and CW-complex, and the definition and basic properties of manifolds. As taught at Maryland, the graduate algebra course includes the most basic definitions and concepts of category theory; a student who hasn't yet seen these ideas could consult any of the above algebra texts or an introduction to category theory such as [Mac Lane]. Since many graduate algebra courses do not include much in the way of algebraic number theory, I have included many topics such as the basic theory of Dedekind rings and the Dirichlet unit theorem, which may be familiar to some readers but not to all. I've tried in this book to presuppose as little topology as possible beyond a typical introductory course, and to develop what is needed as I go along, but to give the reader a flavor of some of the important applications of the subject. A reader with almost no topology background should still be able to follow most of the book except for parts of Sections 1.6, 1.7, 2.4, 4.4, and 6.3, and most of Chapter 5 (though I would hope this book might encourage him or her to take a more systematic course in topology). A problem one always has in writing a book such as this is to decide what to do about spectral sequences. They are usually not mentioned in first-year graduate courses, and yet they are indispensable for serious work in homological algebra and K-theory. To avoid having to give an introduction to spectral sequences which might scare off many readers, I have avoided using spectral sequences directly anywhere in the text. On the other hand, I have made indirect reference to them in many places, so that the reader who has heard of them will often see why they are relevant to the subject and how they could be used to simplify some of the proofs.

For the most part, this book tends to follow the notes of the original course, with a few additions here and there. The major exceptions are that Chapters 3 and 5 have been greatly expanded, and Chapter 6 on cyclic homology has been added even though there was no time for it in the original course. Cyclic homology is a homology theory for rings which may be viewed as the "linearized version" of algebraic K-theory, and it's becoming increasingly clear that it is both a useful computational tool and a subject of independent interest with its own applications.

Each chapter of this book is divided into sections, and I have used a single numbering system for all theorems, lemmas, exercises, definitions, and formulas, to make them easier to locate. Thus a reference such as 1.4.6 means the 6th numbered item in Section 4 of Chapter 1, whether that item is a theorem, a corollary, an exercise, or a displayed formula. The exercises are an integral part of the book, and I have tried to put at least one interesting exercise at the end of every section. The reader should not be discouraged if he finds some of the exercises too difficult, since the exercises vary from the routine to the very challenging.

I have used a number of more-or-less standard notations without special reference, but the reader who is puzzled by them will be able to find most of them listed in the Notational Index in the back of the book.

Why This Book?

The reader might logically ask how this book differs from its "competition." [Bass] remains an important reference, but it is too comprehensive to use as a text for an elementary course, and also it predates the definition of K_2, let alone of higher K-theory or of cyclic homology. My original course was based on the notes by Milnor [Milnor], which are highly recommended. However, I found that [Milnor] is hard to use as a textbook, for the following three reasons:

(1) Milnor writes for a working mathematician, and sometimes leaves out details that graduate students might not be able to provide for themselves.
(2) There are no exercises, at least in the formal sense.
(3) The subject has changed quite a bit since Milnor's book was written.

For the working algebraist already familiar with the contents of [Milnor] who wants to learn about Quillen K-theory and its applications in algebraic geometry, [Srinivas] is an excellent text, but it would have been far beyond the reach of my audience. The notes of Berrick [Berrick] give a more elementary introduction to Quillen K-theory than [Srinivas], but are rather sketchy and do not say much about applications, and thus again are not too suitable for a graduate text. And [LluisP] is very good for an up-to-date survey, but is, as the title says, an overview rather than a textbook. For cyclic homology, the recent book by Loday [LodayCH] is excellent, but to be most useful requires the reader already to know something about K-theory. Also, I do not believe that there is any book available that discusses the applications of algebraic K-theory in functional analysis (which are discussed here in 2.2.10–2.2.11, 4.4.19–4.4.24, 4.4.30, 6.3.8–6.3.17, and 6.3.29-30). Thus for all these reasons it seemed to me that another book on K-theory is needed. I hope this book helps at least in part to fulfill that need.

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Added August, 1995. I am quite grateful for the opportunity to correct a few mistakes in the second printing. I thank several colleagues for their comments on this book, and especially thank Bruce Blackadar, Andreas Rosenschon, and Dan Voiculescu for their corrections.

Contents

1

K_0 of Rings

1. Defining K_0

K-theory as an independent discipline is a fairly new subject, only about 35 years old. (See [Bak] for a brief history, including an explanation of the choice of the letter K to stand for the German word *Klasse*.) However, special cases of K-groups occur in almost all areas of mathematics, and particular examples of what we now call K_0 were among the earliest studied examples of abelian groups. More sophisticated examples of the idea of the definition of K_0 underlie the Euler-Poincaré characteristic in topology and the Riemann-Roch theorem in algebraic geometry. (The latter, which motivated Grothendieck's first work on K-theory, will be briefly described below in §3.1.) The Euler characteristic of a space X is the alternating sum of the Betti numbers; in other words, the alternating sum of the dimensions of certain vector spaces or free R-modules $H_i(X;\, R)$ (the homology groups with coefficients in a ring R). Similarly, when expressed in modern language, the Riemann-Roch theorem gives a formula for the difference of the dimensions of two vector spaces (cohomology spaces) attached to an algebraic line bundle over a non-singular projective curve. Thus both involve a **formal difference** of two free modules (over a ring R which can be taken to be $\mathbb{C}$). The group $K_0(R)$ makes it possible to define a similar formal difference of two finitely generated **projective modules** over any ring R.

We begin by recalling the definition and a few basic properties of projective modules. **Unless we say otherwise, we shall assume all rings have a unit, we shall require all ring homomorphisms to be unit-preserving, and we shall always use the word module to mean "left module."**

1.1.1. Definition. Let R be a ring. A **projective module** over R means an R-module P with the property that any surjective R-module homomorphism $\alpha : M \to P$ has a right inverse $\beta : P \to M$. An equivalent way of phrasing this is that whenever one has a diagram of R-modules and

R-module maps

$$\begin{array}{ccc} & & P \\ & & \downarrow{\scriptstyle \varphi} \\ M & \xrightarrow{\psi} & N \end{array}$$

with $M \xrightarrow{\psi} N$ surjective, one can fill this in to a commutative diagram

$$\begin{array}{ccc} & & P \\ & {\scriptstyle \theta}\swarrow & \downarrow{\scriptstyle \varphi} \\ M & \xrightarrow{\psi} & N. \end{array}$$

Indeed, given the diagram-completion property and a surjective R-module homomorphism $\alpha : M \to P$, one can take $N = P$, $\varphi = id_P$, and $\psi = \alpha$, and the resulting $\theta : P \to M$ is a right inverse for α, *i.e.*, satisfies $\alpha \circ \theta = id_P$.

In the other direction, suppose any surjective R-module homomorphism $\alpha : M \to P$ has a right inverse $\beta : P \to M$, and suppose one is given a diagram of R-modules and R-module maps

$$\begin{array}{ccc} & & P \\ & & \downarrow{\scriptstyle \varphi} \\ M & \xrightarrow{\psi} & N \end{array}$$

with $M \xrightarrow{\psi} N$ surjective. Replacing $M \xrightarrow{\psi} N$ by $M \oplus P \xrightarrow{\psi \oplus id_P} N \oplus P$ and $\varphi : P \to N$ by $(\varphi, id_P) : P \to N \oplus P$, we may suppose φ is one-to-one, and then replacing N by the image of φ and M by $\psi^{-1}(\operatorname{im}\varphi)$, we may assume it's an isomorphism. Then take $\alpha = \varphi^{-1} \circ \psi$ and the right inverse $\beta : P \to M$ enables us to complete the diagram.

When $\alpha : M \to P$ is surjective and $\beta : P \to M$ is a right inverse for α, then $p = \beta \circ \alpha$ is an idempotent endomorphism of M, since

$$\begin{aligned} (\beta \circ \alpha)^2 &= (\beta \circ \alpha) \circ (\beta \circ \alpha) \\ &= \beta \circ (\alpha \circ \beta) \circ \alpha \\ &= \beta \circ id_P \circ \alpha = \beta \circ \alpha, \end{aligned}$$

and then $x \mapsto (\alpha(x), (1-p)(x))$ gives an isomorphism $M \cong P \oplus (1-p)(M)$.

Using this observation, we can now prove the fundamental characterization of projective modules.

1.1.2. Theorem. *Let R be a ring. An R-module is projective if and only if it is isomorphic to a direct summand in a free R-module. It is finitely generated and projective if and only if it is isomorphic to a direct summand in R^n for some n.*

Proof. If P is projective, choose a free module F and a surjective R-module homomorphism $\alpha : F \to P$ by taking F to be the free module on some generating set for P, and α to be the obvious map sending a generator of F to the corresponding generator of P. We are using the universal property of a free module: To define an R-module homomorphism out of a free module, it is necessary and sufficient to specify where the generators should go. If P is finitely generated, then F will be isomorphic to R^n for some n. The observation above then shows P is isomorphic to a direct summand in a free R-module, which we can take to be R^n for some n if P is finitely generated.

For the converse, observe first that free modules F are projective, since given a surjective R-module homomorphism $\alpha : M \to F$ with F free, one can for each generator x_i of F choose some $y_i \in M$ with $\alpha(y_i) = x_i$, and then one can define a right inverse to α by using the universal property of a free module to define an R-module homomorphism $\beta : F \to M$ with $\beta(x_i) = y_i$. Next, suppose $F = P \oplus Q$ and F is a free module. Given a surjective R-module homomorphism $\alpha : M \to P$, $\alpha \oplus id_Q$ is a surjective R-module homomorphism $(M \oplus Q) \to (P \oplus Q) = F$, so it has a right inverse. Now restrict this right inverse to P and project into M to get a right inverse for α. Finally, if $F = R^n$ with standard generators $x_1, \ldots, x_n$, then P is generated by $p(x_i)$, where p is the identity on P and 0 on Q. Thus a direct summand in R^n is finitely generated and projective. □

We're now almost ready to define K_0 of a ring R. First of all, note that the isomorphism classes of finitely generated projective modules over R form an abelian semigroup $\operatorname{Proj} R$, in fact a monoid, with $\oplus$ as the addition operation and with the 0-module as the identity element. To see that this makes sense, there are a few easy things to check. First of all, $\operatorname{Proj} R$ is a set! (This wouldn't be true if we didn't take isomorphism classes, but in fact we have a very concrete model for $\operatorname{Proj} R$ as the set of split submodules of the R^n, $n \in \mathbb{N}$, divided out by the equivalence relation of isomorphism.) Secondly, direct sum is well defined on isomorphism classes, *i.e.*, if $P \cong P'$ and $Q \cong Q'$, then $P \oplus Q \cong P' \oplus Q'$. And thirdly, direct sum is commutative ($P \oplus Q \cong Q \oplus P$) and associative ($(P \oplus Q) \oplus V \cong P \oplus (Q \oplus V)$) once we pass to isomorphism classes.

In general, though, $\operatorname{Proj} R$ is not a group, and may not even have the cancellation property

$$a + b = c + b \Rightarrow a = c.$$

It's therefore convenient to **force it** into being a group, even though this may result in the loss of some information. The idea of how to do this is very simple and depends on the following, which is just a generalization of the way $\mathbb{Z}$ is constructed from the additive semigroup of positive integers, or $\mathbb{Q}^\times$ is constructed from the multiplicative semigroup of non-zero integers, or a ring is "localized" by the introduction of formal inverses for certain elements.

1.1.3. Theorem. *Let S be a commutative semigroup (not necessarily having a unit). There is an abelian group G (called the* **Grothendieck group or group completion** *of S), together with a semigroup homo-*

morphism $\varphi : S \to G$, such that for any group H and homomorphism $\psi : S \to H$, there is a unique homomorphism $\theta : G \to H$ with $\psi = \theta \circ \varphi$.

Uniqueness holds in the following strong sense: if $\varphi' : S \to G'$ is any other pair with the same property, then there is an isomorphism $\alpha : G \to G'$ with $\varphi' = \alpha \circ \varphi$.

Proof. We will outline two constructions. The simplest is to define G to be the set of equivalence classes of pairs (x, y) with $x, y \in S$, where $(x, y) \sim (u, v)$ if and only if there is some $t \in S$ such that

$$x + v + t = u + y + t \quad \text{in } S. \tag{1.1.4}$$

Denote by $[(x, y)]$ the equivalence class of (x, y). Then addition is defined by the rule

$$[(x, y)] + [(x', y')] = [(x + x', y + y')].$$

(It is easy to see that this is consistent with the equivalence relation, and that the associative rule holds.)

Note that for any x and y in S,

$$[(x, x)] = [(y, y)]$$

since $x + y = y + x$. Let 0 be this distinguished element $[(x, x)]$. This is an identity element for G, *i.e.*, G is a monoid, since for any x, y, and t in S,

$$(x + t, y + t) \sim (x, y).$$

Also, G is a group since

$$[(x, y)] + [(y, x)] = [(x + y, x + y)] = 0.$$

We define $\varphi : S \to G$ by

$$\varphi(x) = [(x + x, x)],$$

and it is easy to see that this is a homomorphism. Note that the image of φ generates G as a group, since

$$[(x, y)] = \varphi(x) - \varphi(y)$$

in G. Given a group H and homomorphism $\psi : S \to H$, the homomorphism $\theta : G \to H$ with $\psi = \theta \circ \varphi$ is defined by

$$\theta([(x, y)]) = \psi(x) - \psi(y).$$

Alternatively, one may define G to be the free abelian group on generators $[x]$, $x \in S$, divided out by the relations that if $x + y = z$ in S, then the elements $[x] + [y] = [z]$ in G. Note that $[(x, y)]$ in the previous construction corresponds to $[x] - [y]$ in this second construction. The map

φ is $x \mapsto [x]$, and of course any homomorphism from S into a group H must factor through G by construction.

To prove the uniqueness, suppose $\varphi' : S \to G'$ has the same universal property. First of all, $\varphi'(S)$ must generate G', since otherwise, if G'' is the subgroup generated by the image of φ', then there are two homomorphisms $\theta : G' \to G' \oplus G'/G''$ with

$$(\varphi', 0) = \theta \circ \varphi',$$

namely, $\theta = (id, 0)$ and $\theta = (id, q)$, q the quotient map. By the universal properties for G and G', there must be maps $\alpha : G \to G'$ with $\varphi' = \alpha \circ \varphi$ and $\beta : G' \to G$ with $\varphi = \beta \circ \varphi'$. But then $\alpha \circ \beta = id$ on the image of φ', hence on all of G', so α is a left inverse to β. Similarly $\beta \circ \alpha = id$ on the image of φ; hence α is also a right inverse to β, as required. ☐

Remarks. The assignment $S \rightsquigarrow G = G(S)$ is in fact a functor from the category of abelian semigroups to the category of abelian groups, since if $\gamma : S \to S'$ is a homomorphism of semigroups, it induces a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\gamma} & S' \\ \varphi \downarrow & & \varphi' \downarrow \\ G(S) & \longrightarrow & G(S'), \end{array}$$

where the arrow at the bottom is uniquely determined by the universal property of $G(S)$.

In fancier language, Theorem 1.1.3 just asserts that the forgetful functor F from the category of abelian groups to the category of abelian semigroups has a left adjoint, since

$$\mathrm{Hom}_{\mathbf{Semigroups}}(S, FH) \cong \mathrm{Hom}_{\mathbf{Groups}}(G, H).$$

This could also have been deduced from the adjoint functor theorem (see [Freyd] or [Mac Lane]).

It is convenient that we do not have to assume that cancellation ($x + z = y + z \Rightarrow x = y$) holds in S. Indeed, the map $\varphi : S \to G$ is injective if and only if cancellation holds in S. One of the reasons for introducing Grothendieck groups is that semigroups without cancellation are usually very hard to handle; yet in many cases their Grothendieck groups are fairly tractable.

1.1.5. Definition. Let R be a ring (with unit). Then $K_0(R)$ is the Grothendieck group (in the sense of Theorem 1.1.3) of the semigroup $\mathrm{Proj}\, R$ of isomorphism classes of finitely generated projective modules over R.

Note that K_0 is a **functor**; in other words, if $\varphi : R \to R'$ is an R-module homomorphism, there is an induced homomorphism $K_0(\varphi) = \varphi_* : K_0(R) \to K_0(R')$ satisfying the usual conditions $id_* = id$, $(\varphi \circ \psi)_* = \varphi_* \circ \psi_*$. To see this, observe first that φ induces a homomorphism $\mathrm{Proj}\, R \to \mathrm{Proj}\, R'$

via $[P] \mapsto [R' \otimes_\varphi P]$, for P a finitely generated projective module over R. As required, $R' \otimes_\varphi P$ is finitely generated and projective over R', since if $P \oplus Q \cong R^n$, then

$$(R' \otimes_\varphi P) \oplus (R' \otimes_\varphi Q) \cong R' \otimes_\varphi (P \oplus Q) \cong (R' \otimes_\varphi R^n) = R'^n.$$

And of course, the tensor product commutes with direct sums so we get a homomorphism. Functoriality of K_0 now follows from functoriality of the Grothendieck group construction.

1.1.6. Example. If R is a field, or more generally a division ring (*i.e.*, a skew-field), then any finitely generated R-module is a finitely generated R-vector space and so has a basis and a well-defined dimension. This dimension is the only isomorphism invariant of the module, so we see that $\operatorname{Proj} R \cong \mathbb{N}$, the additive monoid of natural numbers. Since the group completion of $\mathbb{N}$ is $\mathbb{Z}$, $K_0(R) \cong \mathbb{Z}$, with the isomorphism induced by the dimension isomorphism $\operatorname{Proj} R \to \mathbb{N}$. The inclusion of a field F into an extension field F' induces the identity map from $\mathbb{Z}$ to itself, since $\dim_{F'}(F' \otimes_F P) = \dim_F P$ for any F-vector space P.

This same example also shows why we only use **finitely generated** projective modules in defining K_0. If R is a field, the same arguments show that the monoid of isomorphism classes of **countably generated** modules is isomorphic to the extended natural numbers $\mathbb{N} \cup \{\infty\}$, with the usual rule of transfinite arithmetic, $n + \infty = \infty$ for any n. This is no longer a monoid with cancellation; in fact, any two elements become isomorphic after adding ∞ to each one. Thus the Grothendieck group of this monoid is trivial. A similar phenomenon happens with infinitely generated modules over an arbitrary ring; see Exercise 1.1.8.

1.1.7. Exercise. Let S be the abelian monoid with elements $a_{n,m}$, where $n \in \mathbb{N}$, and

$$\begin{cases} m = 0 \text{ if } n = 0 \text{ or } 1, \\ m \in \mathbb{Z} \text{ if } n = 2, \\ m \in \mathbb{Z}/2 \text{ if } n \geq 3. \end{cases}$$

The semigroup operation is given by the formula

$$a_{n,m} + a_{n',m'} = a_{n+n',m+m'},$$

where $m + m'$ is to be computed in $\mathbb{Z}$ if $n + n' \leq 2$ and in $\mathbb{Z}/2$ if $n + n' \geq 3$. (If for instance $n = 2$ and $n' \geq 1$, then m is to be interpreted mod 2.) We shall see in §1.6 that S is isomorphic to $\operatorname{Proj} R$ with $R = C^{\mathbb{R}}(S^2)$, the continuous real-valued functions on the 2-sphere. Compute $G(S)$ and the map $\varphi : S \to G(S)$. Determine the image of S in G, and show that while $\varphi^{-1}(0) = 0$, φ is not injective.

1.1.8. Exercise (the "Eilenberg swindle"). Show that for any ring R, the Grothendieck group of the semigroup of isomorphism classes of **countably** generated projective R-modules vanishes.

1.1.9. Exercise. Recall that if a ring R is commutative, then every left R-module is automatically a right R-module as well, so that the tensor product of two left R-modules makes sense.

(1) Show that the tensor product of two finitely generated projective modules is again finitely generated and projective.
(2) Show that the tensor product makes $K_0(R)$ into a commutative ring with unit. (The class of the free R-module R is the unit element.)

2. K_0 from idempotents

There is another approach to K_0 which is a little more concrete and therefore often convenient. If P is a finitely generated projective R-module, we may assume (replacing P by an isomorphic module) that $P \oplus Q = R^n$ for some n, and we can consider the R-module homomorphism p from R^n to itself which is the identity on P and 0 on Q. Clearly p is idempotent, *i.e.*, $p^2 = p$. Since any R-module homomorphism $R^n \to R^n$ is determined by the n coordinates of the images of each of the standard basis vectors, it corresponds to multiplication on the **right** (since R is acting on the left) by an $n \times n$ matrix. In other words, P is given by an idempotent $n \times n$ matrix p which determines P up to isomorphism.

On the other hand, different idempotent matrices can give rise to the same isomorphism class of projective modules. (When R is a field, the only invariant of a projective module P is its dimension, which corresponds to the rank of the matrix p. When the characteristic of the field is zero, the rank of an idempotent matrix is just its trace.) So to compute $K_0(R)$ from idempotent matrices, we need to describe the equivalence relation on the idempotent matrices that corresponds to isomorphism of the corresponding modules.

1.2.1. Lemma. *If p and q are idempotent matrices over a ring R (of possibly different sizes), the corresponding finitely generated projective R-modules are isomorphic if and only if it is possible to enlarge the sizes of p and q (by adding zeroes in the lower right-hand corner) so that they have the same size $N \times N$ and are conjugate under the group of invertible $N \times N$ matrices over R, $GL(N, R)$.*

Proof. The condition is sufficient since if $u \in GL(N, R)$ and $upu^{-1} = q$, then right multiplication by u induces an isomorphism from $R^N q$ to $R^N p$. So the problem is to prove necessity of the condition. Suppose p is $n \times n$ and q is $m \times m$, and $R^n p \cong R^m q$. We can extend an isomorphism $\alpha : R^n p \to R^m q$ to an R-module homomorphism $R^n \to R^m$ by taking $\alpha = 0$ on the complementary module $R^n(1-p)$, and by viewing the image $R^m q$

as embedded in R^m. Similarly extend α^{-1} to an R-module homomorphism $\beta : R^m \to R^n$ which is 0 on $R^m(1-q)$. Once we've done this, α is given by right multiplication by an $n \times m$ matrix a, and β is given by right multiplication by an $m \times n$ matrix b. We also have the relations $ab = p$, $ba = q$, $a = pa = aq$, $b = qb = bp$. The trick is now to take $N = n + m$ and to observe that

$$\begin{pmatrix} 1-p & a \\ b & 1-q \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(with usual block matrix notation) and that

$$\begin{pmatrix} 1-p & a \\ b & 1-q \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1-p & a \\ b & 1-q \end{pmatrix}$$
$$= \begin{pmatrix} 1-p & a \\ b & 1-q \end{pmatrix}\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}.$$

Thus $\begin{pmatrix} 1-p & a \\ b & 1-q \end{pmatrix}$ is invertible and conjugates $p \oplus 0$ to $0 \oplus q$. The latter is of course conjugate to $q \oplus 0$ by a permutation matrix. □

Now we can give a simple description of Proj R.

1.2.2. Definition. Let R be a ring. Denote by $M(n, R)$ the collection of $n \times n$ matrices over R and by $GL(n, R)$ the group of $n \times n$ matrices over R. We embed $M(n, R)$ in $M(n+1, R)$ by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ (this is a **non-unital** ring homomorphism) and $GL(n, R)$ in $GL(n+1, R)$ by the group homomorphism $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$. Denote by $M(R)$ and $GL(R)$ the infinite unions of the the $M(n, R)$, resp. $GL(n, R)$. Note that $M(R)$ is a ring **without unit** and $GL(R)$ is a group. It is important to remember that each matrix in $M(R)$ has finite size. Let Idem(R) be the set of idempotent matrices in $M(R)$, and note that $GL(R)$ acts on Idem(R) by conjugation.

Now we can restate Lemma 1.2.1.

1.2.3. Theorem. *For any ring R,* Proj R *may be identified with the set of conjugation orbits of $GL(R)$ on* Idem(R). *The semigroup operation is induced by $(p, q) \mapsto \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$. (One only has commutativity and associativity after passage to conjugacy classes.) $K_0(R)$ is the Grothendieck group of this semigroup.*

Using this fact we can now show that K_0 is invariant under passage from R to $M_n(R)$ and commutes with direct limits. We will also construct an example of a ring for which K_0 vanishes.

1.2.4. Theorem ("Morita invariance"). *For any ring R and any positive integer n, there is a natural isomorphism $K_0(R) \xrightarrow{\cong} K_0(M_n(R))$.*

Proof. Via the usual identification of $M_k(M_n(R))$ with $M_{kn}(R)$,

$$\text{Idem}(M_n(R)) = \text{Idem}(R) \quad \text{and} \quad GL(M_n(R)) = GL(R).$$