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Albert C.J. Luo
Valentin Afraimovich
Editors

Hamiltonian Chaos Beyond the KAM Theory

Dedicated to George M. Zaslavsky (1935–2008)

超越 KAM 理论的哈密顿混沌

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Preface

George M. Zaslavsky was born in Odessa, Ukraine in 1935 in a family of an artillery officer. He received education at the University of Odessa and moved in 1957 to Novosibirsk, Russia. In 1965, George joined the Institute of Nuclear Physics where he became interested in nonlinear problems of accelerator and plasma physics. Roald Sagdeev and Boris Chirikov were those persons who formed his interest in the theory of dynamical chaos. In 1968 George introduced a separatrix map that became one of the major tools in theoretical study of Hamiltonian chaos. The work "Stochastical instability of nonlinear oscillations" by G. Zaslavsky and B. Chirikov, published in Physics Uspekhi in 1971, was the first review paper "opened the eyes" of many physicists to power of the theory of dynamical systems and modern ergodic theory. It was realized that very complicated behavior is possible in dynamical systems with only a few degrees of freedom. This complexity cannot be adequately described in terms of individual trajectories and requires statistical methods. Typical Hamiltonian systems are not integrable but chaotic, and this chaos is not homogeneous. At the same values of the control parameters, there coexist regions in the phase space with regular and chaotic motion. The results obtained in the 1960s were summarized in the book "Statistical Irreversibility in Nonlinear Systems" (Nauka, Moscow, 1970).

The end of the 1960s was a hard time for George. He was forced to leave the Institute of Nuclear Physics in Novosibirsk for signing letters in defense of some Soviet dissidents. George got a position at the Institute of Physics in Krasnoyarsk, not far away from Novosibirsk. There he founded a laboratory of the theory of nonlinear processes which exists up to now. In Krasnoyarsk George became interested in the theory of quantum chaos. The first rigorous theory of quantum resonance was developed in 1977 in collaboration with his co-workers. They introduced the important notion of quantum break time (the Ehrenfest time) after which quantum evolution began to deviate from a semiclassical one. The results obtained in Krasnoyarsk were summarized in the book "Chaos in Dynamical Systems" (Nauka, Moscow and Harwood, Amsterdam, 1985).

In 1984, R. Sagdeev invited George to the Institute of Space Research in Moscow. There he has worked on the theory of degenerate and almost degenerate Hamiltonian systems, anomalous chaotic transport, plasma physics, and theory of chaos in waveguides. The book "Nonlinear Physics: from the Pendulum to Turbulence and Chaos" (Nauka, Moscow and Harwood, New York, 1988), written with R. Sagdeev, has been a classical textbook for everybody who studies chaos theory. When studying interaction of a charged particle with a wave packet, George with colleagues from the Institute discovered that stochastic layers of different separatrices in degenerated Hamiltonian systems may merge producing a stochastic web. Unlike the famous Arnold diffusion in non-degenerated Hamiltonian systems, that appears only if the number of degrees of freedom exceeds 2, diffusion in the Zaslavsky webs is possible at one and half degrees of freedom. This diffusion is rather universal phenomenon and its speed is much greater than that of Arnold diffusion. Beautiful symmetries of the Zaslavsky webs and their properties in different branches of physics have been described in the book "Weak chaos and Quasi-Regular Structures" (Nauka, Moscow, 1991 and Cambridge University Press, Cambridge, 1991) coauthored with R. Sagdeev, D. Usikov and A. Chernikov.

In 1991, George emigrated to the USA and became a Professor of Physics and Mathematics at Physical Department of the New York University and at the Courant Institute of Mathematical Sciences. The last 17 years of his life he devoted to principal problems of Hamiltonian chaos connected with anomalous kinetics and fractional dynamics, foundations of statistical mechanics, chaotic advection, quantum chaos, and long-range propagation of acoustic waves in the ocean. In his New York period George published two important books on the Hamiltonian chaos: "Physics of Chaos in Hamiltonian Systems" (Imperial College Press, London, 1998) and "Hamiltonian chaos and Fractional Dynamics" (Oxford University Press, New York, 2005). His last book "Ray and wave chaos in ocean acoustics: chaos in waveguides" (World Scientific Press, Singapore, 2010), written with D. Makarov, S. Prants, and A. Virovlynsky, reviews original results on chaos with acoustic waves in the underwater sound channel.

George was a very creative scientist and a very good teacher whose former students and collaborators are working now in America, Europe and Asia. He authored and co-authored 9 books and more than 300 papers in journals. Many of his works are widely cited. George worked hard all his life. He loved music, theater, literature and was an expert in good vines and food. Only a few people knew that he loved to paint. In the last years he has spent every summer in Provence, France working, writing books and papers and painting in water-colors. The album with his water-colors was issued in 2009 in Moscow.

George Zaslavsky was one of the key persons in the theory of dynamical chaos and made many important contributions to a variety of other subjects. His books and papers influenced very much in advancing modern nonlinear science.

Sergey Prants
Albert C.J. Luo
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Contents

Chapter 1
Stochastic and Resonant Layers in Nonlinear Hamiltonian Systems

Albert C.J. Luo

Abstract In this chapter, stochastic and resonant layers in 2-dimensional nonlinear Hamiltonian systems are presented. The chaos in the stochastic layer is formed by the primary resonance interaction in nonlinear Hamiltonians systems. However, the chaos in the resonant layer is formed by the sub-resonance interaction. The chapter presented herein is to memorize Professor George M. Zaslavsky for his contributions in stochastic layers.

1.1 Introduction

The modern theory of dynamics originates from Poincaré's qualitative analysis. Poincaré (1892) discovered that the motion of nonlinear a coupled oscillator is sensitive to the initial condition, and qualitatively presented that the inherent characteristics of the motion in the vicinity of unstable fixed points of nonlinear oscillation systems may be *stochastic* under regular applied forces. In addition, Poincaré developed the perturbation theory for periodic motions in planar dynamical systems. Birkhoff (1913) continued Poincaré's work, and provided a proof of Poincaré's geometric theorem. Birkhoff (1927) showed that both stable and unstable fixed points of nonlinear oscillation systems with 2-degrees of freedom must exist whenever their frequency ratio (or called *resonance*) is rational. The sub-resonances in periodic motions of such systems change the topological structures of phase trajectories, and the island chains are obtained when the dynamical systems renormalized with fine scales are used. The work of Poincaré and Birkhoff implies that the complexity of topological structures in phase space exists for nonlinear dynamic systems. The question is whether the complicated trajectory can fill the entire phase space or not.

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The formal and normal forms in the vicinity of equilibrium are developed through the Taylor series to investigate the complexity of trajectory in the neighborhood of the equilibrium. Since the trajectory complexity exists in the vicinity of hyperbolic points, one focused on investigating the dynamics in such vicinity of hyperbolic points.

From a topological point of view, Smale's horseshoe was presented in Smale (1967). Further, a differentiable dynamical system theory was developed. Such a theory has been extensively used to interpret the homoclinic tangle phenomenon in nonlinear dynamics. Smale found the infinite, many periodic motions, and a perfect minimal Cantor set near a homoclinic motion can be formed. However, Smale's results cannot apply to Hamiltonian systems with more than 2-degrees of freedom. Because the differentiable dynamical system theory is based on the linearization of dynamical systems at hyperbolic points, it may not be adequate to explain the complexity of chaotic motions in nonlinear dynamical systems. To continue Birkhoff's formal stability, Glimm (1963) investigated the formal stability of an equilibrium (or a periodic solution) of Hamiltonian systems through the rational functions instead of the power series expansion. Such an investigation just gave another kind of approximation. Though those theories are extensively applied in nonlinear dynamical systems, such analyses based on the formal and normal forms are still the local analyses in the vicinity of equilibrium. Those theories cannot be applied for the global behaviors of nonlinear dynamical systems.

To understand the complexity of motion in nonlinear Hamiltonian systems, based on the non-rigorous theory of perturbation, Kolmogorov (1954) postulated the KAM theorem. In the KAM theorem, Kolmogorov suggested a procedure which ultimately led to the stability proof of the periodic solutions of the Hamiltonian systems with 2-degrees of freedom. This problem is intimately connected with the difficulty of small divisors. The aforementioned theorem was proved under different restrictions (e.g., Arnold, 1963; Moser, 1962). Further, Arnold (1964) investigated the instability of dynamical systems with several degrees of freedom, and the diffusion of motion along the generic separatrix was discussed. The results of Arnold (1964) extended Kolmogorov's results to the Hamiltonian system with several degrees of freedom system. The stability in the sense of Lyapunov cannot be inferred. The KAM theory is based on the separable oscillators with weak interactions. In fact, once the perturbation exists, the dynamics of the perturbed Hamiltonian systems may not be well-behaved to the separable dynamical systems. In physical systems, the interaction between two oscillators in a nonlinear dynamical system cannot be very small. The KAM theorem may provide an acceptable prediction only when the interaction perturbation is very weak. The KAM theory is based on separable, integrable Hamiltonian systems. In fact, the complexity of motions in non-integrable, nonlinear Hamiltonian systems is much beyond what the KAM theory stated.

The instability zone (or *stochastic layer*) of Hamiltonian systems, as investigated in Arnold (1964), is a domain of chaotic motion in the vicinity of the generic separatrix. Even if the width of the separatrix splitting was estimated, the dynamics of the separatrix splitting was not developed. Henon and Heiles (1964) gave a numerical investigation on the nonlinear Hamiltonian systems with 2-degrees of freedom in order to determine whether or not a well-behaved constant of the motion exists for such Hamiltonian Systems. Izrailev and Chirikov (1966) first pointed out that the periodically forced, nonlinear Hamiltonian system with 1-degree of freedom exhibits a KAM instability leading to the stochastic behavior (or stochastic and resonance layers). Walker and Ford (1969) investigated the amplitude instability and ergodic behavior for nonlinear Hamiltonian systems with 2-degrees of freedom to develop the verifiable scheme for prediction of the onset of the amplitude instability. Isolated resonance and double resonance were investigated and the resonance was determined through the transformed coordinates. Such ergodic behavior in nonlinear Hamiltonian system originates from Birkhoff (1927). In other words, to investigate the enormous complexity of non-special motions in dynamical systems from geodesic flows, Birkhoff (1927) presented that the set of non-special motions (or chaotic motions) is measurable in the sense of Lebesgue, and the set of the special motions (or regular motion) is of zero measure. Furthermore, the ergodic theory had been developed in the 20^{th} century and it is as a fundamental base for fractal theory. The thorough study of the geodesic flows in the ergodic theory can be found in Hopf (1937). Those ideas were generalized by Anosov (1962) to study a class of differential equations, which can be also referenced to (Sinai,1976). Even though the ergodic theory is a foundation for fractality of chaotic motions in nonlinear dynamical systems, such a theory still cannot provide enough hopes to understand the complexity of chaotic motions in nonlinear dynamics.

For a nonlinear Hamiltonian system with n-degrees of freedom, it is very difficult to understand the mechanism of chaotic motions. To date, such a problem is unsolved. Around (1960) considered extremely simple, nonlinear Hamiltonian systems to investigate such a mechanism. Melnikov (1962) used the concept in Poincaré (1892) to investigate the behavior of trajectories of perturbed systems near autonomous Hamiltonian systems. Melnikov (1963) further investigated the behavior of trajectories of perturbed Hamiltonian systems and the width of the separatrix splitting were approximately estimated. The width gives the domain of the chaotic motion in the vicinity of the generic separatrix. Even if the width of the separatix splitting was approximately estimated, the dynamics of the separatrix splitting was not developed. From a physical point of view, Chirikov (1960) investigated the resonance processes in magnetic traps, and the resonance overlap was presented initially. Zaslavsky and Chirikov (1964) discussed the mechanism of 1-dimensional Fermi acceleration and determined the stochastic property of such a system. Rosenblut et al. (1966) investigated the appearance of a stochastic instability (or chaotic motion) of trapped particles in the magnetic field of a traveling wave under a perturbation. Filonenko et al. (1967) further discussed the destruction of magnetic surface generated by the resonance harmonics of perturbation. The destruction of such a magnetic surface demonstrates the formation and destruction of the resonant surface. Zaslavsky and Filonenko (1968) gave a systematic investigation of the stochastic instability of trapped particles through the separatrix map (or whisker motion in Arnold (1964)), and the fractional equation for diffusion was developed. Zaslavsky and Chirikov (1972) further presented the stochastic instability of nonlinear oscillations. Chirikov (1979) refined the resonance overlap criterion to predict the onset of chaos in stochastic layers. In addition, the most important achievements for prediction of the appearance of chaotic motions were summarized. Escande and Doveil (1981) used the resonance overlap concept and gave a criterion through a renormalization group method (also see, Escande, 1985). The details for the resonance overlap theory and renormalization group scheme can be referred to references (e.g., Lichtenberg and Lieberman, 1992; Reichl, 1992). Though the resonant overlap criterion can provide a rough prediction of the onset of chaotic motion in the stochastic layers, the mechanism of the chaotic motion in the stochastic layers still cannot be fully understood until now.

Luo (1995) proposed the resonance theory for chaotic motions in the vicinity of generic separatrix in nonlinear Hamiltonian systems (also see, Luo and Han, 2001), and it was asserted that chaotic motions in nonlinear Hamiltonian systems are caused by the resonant interaction. Furthermore, the mechanism for the formation, growth and destruction of stochastic layers in nonlinear Hamiltonian systems was discussed in Luo and Han (2001). In Luo et al. (1999), the resonant webs formed in the stochastic layer were presented, and it was observed that the webs are similar to the stochastic layer of the parametrically forced pendulum system. The recent investigations (e.g., Han and Luo, 1998; Luo, 2001b, c, 2002) discovered that the resonance interaction generates the resonant separatrix, and the chaotic motion forms in vicinity of such a resonant separatrix. The corresponding criteria were presented for analytical predictions of chaotic motions in 1-DOF nonlinear Hamiltonian systems with periodic perturbations. The maximum and minimum energy spectrum methods were developed for numerical predictions of chaotic motions in nonlinear Hamiltonian systems (also see, Luo et al., 1999; and Luo, 2002). The energy spectrum approach is applicable not only for small perturbations but for the large perturbation. The recent achievements for stochastic layers in periodically forced Hamiltonians with 1-degree of freedom were summarized in Luo (2004). Luo (2006a) investigated quasi-periodic and chaotic motions in n-dimensional nonlinear Hamiltonian systems. The energy spectrum method was systematically presented for arbitrary interactions of the integrable nonlinear Hamiltonian systems. The internal resonance was discussed analytically for weak interactions, and the chaotic and quasi-periodic motions can be predicted. From a theory for discontinuous dynamical system in Luo (2006b), Luo (2007a) presented a general theory for n-dimensional nonlinear dynamical systems. The global tangency and transversality to the separatrix were discussed from the first integral quantity. The first integral quantity increment was introduced to investigate the periodic and chaotic flows. In this chapter, only the stochastic and resonant layers in nolinear Hamilton systems will be presented. For more materials, readers can refer to Luo (2008).

1.2 Stochastic layers

In this section, the stochastic layers in nonlinear Hamiltonian systems will be described geometrically, and the approximate criterions for onset and destruction of the stochastic layers will be presented.

1.2.1 Geometrical description

Consider a 2-dimensional Hamiltonian system with a time periodically perturbed vector field, i.e.,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\boldsymbol{\mu}) + \mu\mathbf{g}(\mathbf{x},t,\boldsymbol{\pi}); \quad \mathbf{x} = (x,y)^{\mathrm{T}} \in \mathbb{R}^2, \tag{1.1}$$

where $\mathbf{f}(\mathbf{x},\boldsymbol{\mu})$ is an unperturbed Hamiltonian vector field on $\mathbb{R}^2$ and $\mathbf{g}(\mathbf{x},t)$ is a periodically perturbed vector field with period $T = 2\pi/\Omega$, and

$$\mathbf{f}(\mathbf{x},\boldsymbol{\mu}) = (f_1(\mathbf{x},\boldsymbol{\mu}), f_2(\mathbf{x},\boldsymbol{\mu}))^{\mathrm{T}} \text{ and } \mathbf{g}(\mathbf{x},t,\boldsymbol{\pi}) = (g_1(\mathbf{x},t,\boldsymbol{\pi}), g_2(\mathbf{x},t,\boldsymbol{\pi}))^{\mathrm{T}} \tag{1.2}$$

are sufficiently smooth $(C^r, r \geq 2)$ and bounded on a bounded set $D \subset \mathbb{R}^2$ in phase space. $f_1 = \partial H_0(x,y)/\partial y,\ f_2 = -\partial H_0(x,y)/\partial x; g_1 = \partial H_1(x,y,\Omega t)/\partial y, g_2 = -\partial H_1(x,y,\Omega t)/\partial x$. If the perturbation (or forcing term) $\mathbf{g}(\mathbf{x},t)$ vanishes, Equation (1.1) reduces to a 2-dimensional autonomous system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\boldsymbol{\mu})$ corresponding to a 1-degree of freedom system in nonlinear Hamiltonian systems. Therefore the total Hamiltonian of Eq. (1.1) can be expressed by

$$H(x,y,t,\mathbf{p}) = H_0(x,y,\boldsymbol{\mu}) + \mu H_1(x,y,\Omega t,\boldsymbol{\pi}), \tag{1.3}$$

with excitation frequency Ω and strength μ of the perturbed Hamiltonian $H_1(x,y,t,\boldsymbol{\pi})$ as well. For comparison with the other approximate analysis, such a perturbation parameter is introduced herein. The Hamiltonian of the integrable system in Eq. (1.1) is $H_0(x,y,\boldsymbol{\mu})$. Once the initial condition is given, the Hamiltonian $H_0(x,y)$ is invariant (i.e., $H_0(x,y,\boldsymbol{\mu}) = E$), which is the first integral manifold.

To restrict this investigation to the 2-dimensional stochastic layer, four assumptions for Eq.(1.1) are introduced as follows:

(H1) The unperturbed system of Eq.(1.1) possesses a bounded, closed separatrix $q_0(t)$ with at least one hyperbolic point $p_0 : (x_h, y_h)$.
(H2) The neighborhood of $q_0(t)$ for the point $p_0 : (x_h, y_h)$ is filled with at least three families of periodic orbits $q_\sigma(t)$ $(\sigma = \alpha, \beta, \gamma)$ with $\alpha, \beta, \gamma \in (0,1]$.
(H3) For the Hamiltonian energy E_σ of $q_\sigma(t)$, its period T_σ is a differentiable function of E_σ.
(H4) The perturbed system of Eq.(1.1) possesses a *perturbed* orbit $q(t)$ in the neighborhood of the *unperturbed* separatrix $q_0(t)$.

From the foregoing hypothesis, the phase portrait of the unperturbed Hamiltonian system in the vicinity of the separatrix is sketched in Fig.1.1. The following point sets and the corresponding Hamiltonian energy are introduced, i.e.,

$$\Gamma_0 \equiv \{(x,y) \,|\, (x,y) \in q_0(t),\ t \in \mathbb{R}\} \cup \{p_0\} \text{ and } E_0 = H_0(q_0(t)) \tag{1.4}$$

for the separatrix,

$$\Gamma_\sigma \equiv \{(x,y)\,|\,(x,y) \in q_\sigma(t),\ t \in \mathbb{R}\} \text{ and } E_\sigma = H_0(q_\sigma(t)) \tag{1.5}$$

for the unperturbed, σ-periodic orbit and

$$\Gamma = \{(x,y)\,|\,(x,y) \in q(t),\ t \in \mathbb{R}\} \text{ and } E = H_0(q(t)) \tag{1.6}$$

for the perturbed orbit $q(t)$.

The Hamiltonian energies in Eqs. (1.4) and (1.5) are constant for any periodic orbit of the unperturbed system but the Hamiltonian energy in Eq. (1.6) varies with $(x,y) \in q(t)$ of the perturbed system. Note that the unperturbed Hamiltonian $H_0(q_\sigma(t))$ $(\sigma = \alpha, \beta, \gamma)$ and $H_0(q_0(t))$ are C^r $(r \geq 2)$ smooth in Luo and Han (2001). The hypotheses (H2)–(H3) imply that $T_\sigma \to \infty$ monotonically as $\sigma \to 0$ (i.e., the periodic orbit $q_\sigma(t)$ approaches to $q_0(t)$ as $\sigma \to 0$).

The δ-sets of the first integral quantity (or the energy) of the unperturbed Eq. (1.1) in Γ_σ $(\sigma = \alpha, \beta, \gamma)$, are defined as

$$N_\sigma^\delta(E_0) = \left\{E_\sigma \,\middle|\, |E_\sigma - E_0| < \delta_\sigma, \text{ for small } \delta_\sigma > 0\right\} \tag{1.7}$$

and the union of the three δ-sets with E_0 is

$$N^\delta(E_0) = \bigcup\nolimits_\sigma N_\sigma^\delta(E_0) \cup \{E_0\}. \tag{1.8}$$

For some time t, there is a point $\mathbf{x}_\sigma = (x_\sigma(t), y_\sigma(t))^{\mathrm{T}}$ on the orbit $q_\sigma(t)$ and this point is also on the normal $\mathbf{f}^\perp(\mathbf{x}_0) = (-f_2(\mathbf{x}_0), f_1(\mathbf{x}_0))^{\mathrm{T}}$ of the tangential vector of the separatrix $q_0(t)$ at a point $\mathbf{x}_0 = (x_0(t), y_0(t))^{\mathrm{T}}$, as shown in Fig. 1.2. Therefore, the distance is defined as

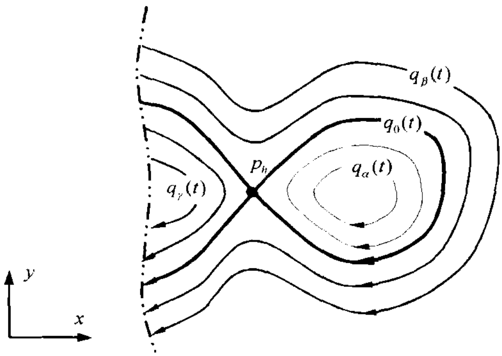

Fig. 1.1 The phase portrait of the unperturbed system of Eq. (1.1) near a hyperbolic point p_h. $q_0(t)$ is a separatrix going through the hyperbolic point and splitting the phase into three parts near the hyperbolic point, and the corresponding orbits $q_\sigma(t)$ are termed the σ-orbit $(\sigma = \{\alpha, \beta, \gamma\})$.