Avinash Khare

Fractional Statistics and Quantum Theory

分数统计与量子理论

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Avinash Khare
Institute of Physics,
Bhubaneswar 751005, India

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作　　者：A. Khare

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Fractional Statistics and Quantum Theory

Dedicated to My Parents

Tai and Late Bhayasaheb

Preface

Sometime in early 1994, I received a letter from World Scientific, Singapore, inviting me to write a book on anyons. That made me think about the whole thing. Initially I was not sure if I should spend about two years of my research career in such a venture. At that time I went through the literature on this field and I had a careful look at the only monograph on this subject by Lerda as well as the book on anyon superconductivity edited by Wilczek. While I liked parts of these books (and their influence is apparent in parts of the book), I also felt that our perceptions are some what different. Having worked on both the non-relativistic as well as the relativistic anyon models, I was finally convinced that perhaps the time had come to accept this opportunity and put forward my point of view and hence this book.

As I started the planning of the book, I realized that a lot of work has been done in the last two decades in this field and it was not possible to cover everything in this monograph. I then decided that instead of pretending to be objective, it was better to include those topics which I consider important. However, I felt that I should at least give a brief description of the omitted topics and also give a couple of references for each of these topics so that the interested reader can follow the developments in these fields. I am somewhat lucky in that many of these topics have been adequately covered in the literature.

Even though so much work has been done in this field in the last two decades, I feel that this area is still in its early developmental stage and it is not completely clear as to what direction it will take in the future. This is because even the basic problem of the statistical mechanics of a non-interacting anyon gas is still an unsolved problem. I strongly believe that unless one can solve this basic problem, no qualitative progress is

possible in this field, since in the absence of this bench-mark study, any calculation including interactions will always be unreliable.

I am grateful to R.K. Patra who typed the first draft of the book, and to Umesh Salian, Sandip Joshi, and Koushik Ray for help with drawing figures and putting everything together properly. I also thank my daughter Anupama for a careful proof reading. Some of the chapters were written while I was visiting Los Alamos National Laboratory, Univ. of Oslo and Univ. Of Trondehim. It is my pleasure to thank Fred Cooper, Jon Magne Leinaas and Jan Myrheim for the kind invitations and for a warm hospitality.

It is a great pleasuie to thank all my colleagues and collaborators who, over the years, have enhanced my knowledge of theoretical physics in general and anyons in particular. I would specially like to thank Surjyo Behera, Rajat Bhaduri, Alan Comtet, Fred Cooper, Trilochan Pradhan and Uday Sukhatme.

Further, I am extremely grateful to all the members of Institute of Physics, Bhubaneswar, and specially the three Directors, Trilochan Pradhan, V.S. Ramamurthy and Surjyo Behera who in the last twenty two years have given all the help and encouragement and most importantly provided a very stimulating atmosphere to carry out research.

I am grateful and indebted to my parents, Tai and late Bhayasaheb who in spite of difficult economic condition always encouraged us to look for intellectual pursuits and emphasized the need for hard work with single minded devotion. It is only because of their sacrifices that I am today what I am. Finally, I am grateful to my wife Pushpa who always set very high standards and encouraged and inspired me to try to achieve these standards.

Avinash Khare
Bhubaneswar, July, 1997

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Chapter 1

Introduction

O brave new worlds, that have such people in them!
— E.A. Abbott in *Flatland*

Many of us have wondered some time or the other if one can have non-trivial science and technology in two space dimensions; but the usual feeling is that two space dimensions do not offer enough scope for it. This question, to the best of my knowledge, was first addressed in 1884 by E.A. Abbot in his satirical novel *Flatland* [1]. The first serious book on this topic appeared in 1907 entitled *An episode of Flatland* [2]. In this book C.H. Hinton offered glimpses of the possible science and technology in the flatland. A nice summary of these two books appeared as a chapter entitled *Flatland* in a book in 1969 edited by Martin Gardner [3]. Inspired by this summary, in 1979 A.K. Dewdney [4] published a book which contains several laws of physics, chemistry, astronomy and biology in the flatland. However, all these people missed one important case where physical laws are much more complex, nontrivial and hence interesting in the flatland than in our three dimensional world. I am referring here to the case of quantum statistics. In last two decades it has been realized that whereas in three and higher space dimensions all particles must either be bosons or fermions (i.e. they must have spin of $n\hbar$ or $(2n+1)\hbar/2$ with n=0,1,2,... and must obey Bose-Einstein or Fermi-Dirac statistics respectively), in two space dimensions the particles can have any fractional spin and can satisfy *any* fractional statistics which is interpolating between the two. The particles obeying such statistics are generically called as *anyons*. In other words, if one takes one anyon slowly around the other then in general the phase acquired is $exp(\pm i\theta)$. If θ =0 or π (modulo 2π) then the particles are bosons or fermions respectively while if $0 < \theta < \pi$ then the particles are termed as anyons.

In this book I plan to explore in detail the various facets of anyons. Before I go into the details, one might wonder if our discussion is merely of academic interest? The answer to the question is no. In fact it is a surprising fact that two, one and even zero dimensional experimental physics is possible in our three-dimensional world. A few lines of digression are called for here to explain how this is possible in our three dimensional world. The point is that because of the third law of thermodynamics, which states that all the degrees of freedom freeze out in the limit of zero temperature, it is possible to strictly confine the electrons to surfaces, or even to lines or points. Thus it may happen that in a strongly confining potential, or at sufficiently low temperatures, the excitation energy in one or more directions may be much higher than the average thermal energy of the particles, so that those dimensions are effectively frozen out. An illustration might be worth while here. Consider a two dimensional electron gas on which the first experiment was in fact done in 1966 [5]. The electrons are confined to the surface of a semiconductor by a strong electric field, and they move more or less freely along the surface. On the other hand, the energy E required to excite motion in the direction perpendicular to the surface is of the order of several milli-electron-Volt (meV). Now at a temperature of say $T = 1K$, the thermal energy is kT, where k is the Boltzmann constant. Thus if the transverse excitation energy is say 10 meV, then the fraction of electrons in the lowest excited transverse energy level is

$$e^{-\frac{E}{kT}} = e^{-100} \approx 10^{-44}. \tag{1.1}$$

which is zero for all practical purposes. Thus the electron gas is truly a two-dimensional gas. Few examples where planar experimental physics is possible are electron gas, surface layer studies and copper-oxide materials. Of course, even there, at the most basic level, the fundamental particles are certainly fermions or bosons. However, the most direct and appropriate discussion of the low energy behavior of a material is usually in terms of the quasi-particles. The hope is that at least in some of these cases the quasi-particles could be anyons. This hope has in fact been realized in the case of the fractionally quantized Hall effect where the quasi-particles are believed to be charged vortices i.e. charged anyons [6, 7, 8]. Three rather different experiments [9] seem to confirm the existence of fractionally charged excitations and hence indirectly of anyons.

There is another place where, at one stage many believed that anyons could play a major role. I have in mind here the high-T_c superconductors. To date the mechanism of superconductivity in these high-T_c materials is not known. Few years ago, several people were excited by the suggestion that anyons could provide the mechanism for superconductivity in these materials [10]. It soon turned out that these models provide a unique test of these ideas. In particular, they predicted the violation of the discrete symmetries of parity and time reversal invariance in these materials. Unfortunately the experiments performed in several laboratories [11, 12] failed to observe the parity and the time reversal violation in the high-T_c superconductors. While these experiments have certainly dampened the interest of the physics community in anyons, they also showed that the anyon ideas are not merely esoteric and have experimental consequences which could be tested in the laboratories.

Another reason why I believe that the anyons would have relevance to the real world is because of the unwritten first law of physics which states that 'anything that is not forbidden is compulsory!'. Finally, anyons represent a challenge to all those people who think that they *know* quantum mechanics and statistical mechanics and that they could have contributed significantly to the development of these fields if only they had been born 60-70 years earlier!

At this stage it might be worthwhile to give a short historical review of the field. By its very nature, such a review is bound to be subjective and I apologize in advance to those authors who may feel that their contribution has not been given its due credit.

1.1 Historical Review

The concept of the indistinguishability of the identical particles has a deep meaning in quantum mechanics. Actually, this concept was introduced by John Willard Gibbs in classical statistical mechanics, much before the advent of quantum mechanics in order to resolve the famous Gibbs paradox. However, its ramifications are far deeper in quantum mechanics vis a vis the classical mechanics. For example, it was realized quite early that in quantum mechanics, the identical particles always interact simply because they are identical. As a result, the physical behavior of a collection of identical particles is influenced not only by the conventional interactions but also by the statistics they obey. In particular, it was realized that there are two kinds of quantum statistics in nature. It was shown that all particles have either half-integral or integral spin (in units of the Planck constant $\hbar$) and accordingly they satisfy Bose-Einstein [13, 14] or Fermi-Dirac [15, 16] statistics respectively. It was also soon realized that there is an effective attraction between the bosons and an effective repulsion between the fermions, both of which are purely quantum mechanical in nature, and are refereed to as statistical interaction. It may be noted that it is this statistical attraction which gives rise to the accumulation of the Bose particles in the ground state which is at the heart of the phenomenon of Bose condensation. Similarly, it is the statistical repulsion between the fermions which gives rise to the famous Pauli exclusion principle. In fact, the stability of matter very much depends on the fermionic nature of the matter. Recall that according to our current picture, all matter in nature consists of quarks and leptons which are fermions.

The question that one wants to ask is whether Bose-Einstein and Fermi-Dirac are the only possible forms of quantum statistics in nature? For almost fifty years, it was believed that the answer to this question is yes. To understand why, let us go back in history a little bit. Immediately after quantum mechanics was formulated by Schrödinger and Heisenberg as it is known today, Heisenberg and Dirac extended the theory to systems of identical particles [17, 16, 18]. Their key observation was that all operators representing observables in such systems are necessarily symmetric under the interchange of the particle labels, if the particles are really indistinguishable. This statement has profound consequences since the symmetry operators preserve the symmetry of the wave functions. Clearly, if the operator O and the wave function ψ are both totally symmetric, then $O\psi$ is also totally symmetric while if O is symmetric but ψ is anti-symmetric, then $O\psi$ is totally anti-symmetric. This immediately explained as to why one has quantum theories of identical particles using only symmetric or only anti-symmetric wave functions. Since several consequences of both the Bose-Einstein and the Fermi-Dirac statistics were soon experimentally verified, hence no one really bothered to construct a more satisfactory theory.

The Heisenberg-Dirac theory, even though experimentally so successful, could however be questioned on philosophical ground. Consider for example the case when two particles are so far apart that they cannot be physically interchanged. Then, clearly, it does not matter if we symmetrize or anti-symmetrize the wave functions or do neither! So the question arises whether there is some postulate which is more fundamental than the symmetrization or anti-symmetrization postulate. Strictly speaking, the interchange of particle labels is a slightly misleading concept. If the particles are strictly identical, then an interchange of the identical particles is obviously an identity transformation. Now in quantum mechanics it is not uncommon that a physical identity transformation may be represented mathematically by a phase factor. As is well known, any permutation of bosons gives the trivial phase factor of $+1$ while even and odd permutation of fermions gives the phase factor of $+1$ or -1 respectively. An obvious question is, can we also have a more general complex phase factor instead of just $+1$ or -1?

This question was partly answered by Laidlaw and DeWitt [19] in 1971. They applied the Feynman path integral formalism to systems of identical particles. Note that in the path integral formalism, the interchange of identical particles has a clear physical meaning as a continuous process in which each particle moves along a continuous path. The path dependence of the interchange is all important since it relates the quantum mechanical concept of particle statistics to the topology of the configuration space. The phase factors associated with different interchange paths must define a representation of the first homotopy group of the configuration space. Unfortunately, Laidlaw and DeWitt confined their attention to only three dimensions and hence concluded that only bosons and fermions can exist thereby missing the more exotic possibilities in two space dimensions.

In 1977, using a more traditional approach to quantization, Leinaas and Myrheim [20] derived the same relationship between the particle statistics and topology but were bold enough to enquire about the possible quantum statistics in two dimensions. Their approach was based on the geometrical interpretation of wave functions which is the basis of the modern day gauge theories. They showed that in two dimensions, the space is multiply connected which results in the possibility of what they termed as the *intermediate statistics*. In particular, they showed that there exists a continuously variable parameter, which one can choose to be the phase angle θ, that characterizes different statistics ; θ equal to 0 or π correspond to bosons or fermions respectively while $0 < \theta < \pi$ corresponds to more exotic possibilities. It may be noted that θ can be a rational or irrational fraction of π. As an illustration of their ideas, they explicitly solved the spectrum of two such exotic particles in a two-dimensional harmonic oscillator potential, and showed that as θ goes from zero to π, there is a continuous interpolation between the bosonic and the fermionic spectra. This simple calculation also showed that even the 2-anyon spectrum is not related to the corresponding single particle energy levels thereby suggesting that even the simplest problem of a noninteracting anyon gas may not be easy to solve.

Few years later, Goldin, Menikoff and Sharp obtained the same results by an entirely different approach [21]. They studied the representations of the commutator algebra of of particle density and current operators. This algebra has commutation relations that are independent of the particle statistics, but the inequivalent representations correspond to the different statistics.

It is fair to say that the idea of the intermediate statistics did not receive enough attention in the physics community till the papers of Wilczek [22]. It is he who coined the name *anyons* for the two-dimensional identical particles obeying the intermediate statistics and since then it is being called as the anyonic or more generally as the *fractional statistics*. Apart from proposing the name, wilczek's main contribution was the flux-tube model for anyons in which anyons turn out to be point particles having both the electric charge and the magnetic flux i.e. they are point charged vortices. Wilczek also clearly spelled out the concept of statistical transmutation i.e. the fact that one can treat the noninteracting anyons as interacting bosons or interacting fermions. This idea has proved extremely useful in trying to work out the statistical mechanics of an ideal gas of anyons by treating it as an interacting Bose (or Fermi) gas. Now it is well known that the interacting Bose or Fermi gas problems are notoriously difficult and so this analogy is extremely useful in appreciating the difficulties involved in understanding even the seemingly simple problem of the noninteracting anyon gas. The flux-tube