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Vasily E. Tarasov

Fractional Dynamics

Applications of Fractional Calculus to
Dynamics of Particles, Fields and Media

分数维动力学

分数阶积分在粒子、场及介质
动力学中的应用



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Preface

Fractional calculus is a theory of integrals and derivatives of any arbitrary real (or complex) order. It has a long history from 30 September 1695, when the derivative of order $\alpha = 1/2$ was mentioned by Leibniz. The fractional differentiation and fractional integration go back to many great mathematicians such as Leibniz, Liouville, Grünwald, Letnikov, Riemann, Abel, Riesz and Weyl. The integrals and derivatives of non-integer order, and the fractional integro-differential equations have found many applications in recent studies in theoretical physics, mechanics and applied mathematics.

New possibilities in mathematics and theoretical physics appear, when the order α of the differential operator D_x^α or the integral operator I_x^α becomes an arbitrary parameter. The fractional calculus is a powerful tool to describe physical systems that have long-term memory and long-range spatial interactions. In general, many usual properties of the ordinary (first-order) derivative D_x are not realized for fractional derivative operators D_x^α . For example, a product rule, chain rule and semi-group property have strongly complicated analogs for the operators D_x^α .

Most of the processes associated with complex systems have nonlocal dynamics and it can be characterized by long-term memory in time. The fractional integration and fractional differentiation operators allow one to consider some of those characteristics. Using fractional calculus, it is possible to obtain useful dynamical models, where fractional integro-differential operators in the time and space variables describe the long-term memory and nonlocal spatial properties of the complex media and processes. We should note that close connections exist between fractional differential and integral equations, and the dynamics of many complex systems, anomalous processes and fractal media.

There are many interesting books about fractional calculus, fractional differential equations, and their physical applications. The first book dedicated specifically to the theory of fractional integrals and derivatives, is the one by Oldham and Spanier published in 1974. There exists the remarkably comprehensive encyclopedic-type monograph by Samko, Kilbas and Marichev, which was published in Russian in 1987 and in English in 1993. The works devoted substantially to fractional differential equations are the book by Miller and Ross (1993), and the book by Podlubny

(1999). In 2006, Kilbas, Srivastava and Trujillo published a very important and remarkable book, in which one can find a modern encyclopedic, detailed and rigorous theory of fractional differential equations. This book can be recommended as a main modern mathematical handbook for graduate students and researchers, who are interested in this subject. There exist some mathematical monographs devoted to special questions of fractional calculus, including the book by McBride (1979), the work by Kiryakova (1993), the monograph by Rubin (1996) and the volume edited by Srivastava and Owa (1989). The physical applications of fractional calculus to describe complex media and processes were considered in the very interesting volumes edited by Carpintery and Mainardi in 1997 and by Hilfer in 2000. The book by West, Bologna, and Grigolini published in 2003 is devoted to physical application of fractional calculus to fractal processes. The first book devoted exclusively to the fractional dynamics and application of fractional calculus to chaos is the one by Zaslavsky published in 2005. One of the most recent books on the subject of fractional calculus is the edited volume of Sabatier, Agrawal and Tenreiro Machado published in 2007. In 2010, the book by Mainardi will be published to devote to applications of fractional calculus in dynamics of viscoelastic materials. Note that there are international journals such as "Journal of Fractional Calculus" and "Fractional Calculus and Applied Analysis", which are dedicated entirely to the fractional calculus.

The content of the noted books and edited volumes about applications of fractional calculus in physics, mechanics and applied mathematics does not include all of modern fractional theoretical models, methods and approaches. A lot of new results, obtained recently in the fractional dynamics, are not reflected in the books. In this monograph, some modern applications of fractional calculus to complex physical systems and new results of last years are described. Therefore the book is supposed to be useful for physicists and mathematicians, who are interested in the modern theories of complex processes and media. Some of interesting subjects in the theoretical physics are not described in this monograph, since it is not possible to realize in one book a complete description of all fractional dynamics. For example, the applications of fractional calculus to the viscoelastic media and the continuous time random walk processes are not considered in the book, since there are interesting monographs and reviews, where these subjects are discussed.

The text is self-contained and can be used without previous courses in fractional calculus and theory of fractals. The necessary information, which is beyond to undergraduate courses of the mathematics, is suggested in the book. Therefore this book can be used in the courses for graduate students. In the book the modern approaches and new fundamental results of last years are described. Therefore the manuscript is supposed to be useful for physicists and mathematicians who are interested in the electrodynamics, statistical and condensed matter physics, quantum dynamics, complex media theories and kinetics, discrete maps and chain models, nonlinear dynamics and chaos.

The book consists of five parts. The first part is devoted the fractional continuous models of fractal distributions of particles. The fractional integral equations are used to describe fractal distributions of mass, charge and probability. In the second part, we consider the fractional dynamics that describes the media with long-range

interaction of particles. The close connection of discrete models with long-range interactions and continuous medium equations with fractional derivatives is proved. The fractional coordinate derivatives are used to describe nonlocal properties of the complex media. In the third part, we suggest the fractional vector calculus, fractional exterior calculus, and fractional variation calculus to describe generalized dynamical systems, fractional statistical mechanics and kinetics, fractional electrodynamics of complex media. The suggested generalizations of vector operations and variations are considered with respect to coordinate variables. In the fourth part, we describe the fractional temporal dynamics, where derivatives with respect to time variable have non-integer orders. The nonholonomic systems with generalized constraints to describe a long-term memory are considered. The electrodynamics of dielectric media is described as a fractional temporal electrodynamics. The discrete maps with memory are obtained from the fractional differential equations of kicked dynamical systems. In the fifth part, we consider an application of fractional derivatives in quantum dynamics. These derivatives are defined as fractional powers of self-adjoint derivatives. A fractional generalization of quantum Markovian dynamics is suggested. The quantization of different fractional derivatives and fractal functions are suggested. The numerous recent publications cited are listed in the references at the end of each chapter.

The author is greatly indebted to professor George M. Zaslavsky for his invaluable discussions and comments. The author wishes to express his gratitude to professor Albert C.J. Luo for his support of the edition of this book. The author would like to express his appreciation for the kind hospitality by the Courant Institute of Mathematical Studies of New York University during his visits in 2005, 2007, 2008 and 2009.

Vasily E. Tarasov

12 October, 2009, Moscow

Contents

Part I Fractional Continuous Models of Fractal Distributions

1	Fractional Integration and Fractals	3
1.1	Riemann-Liouville fractional integrals	4
1.2	Liouville fractional integrals	6
1.3	Riesz fractional integrals	7
1.4	Metric and measure spaces	9
1.5	Hausdorff measure	10
1.6	Hausdorff dimension and fractals	14
1.7	Box-counting dimension	16
1.8	Mass dimension of fractal systems	19
1.9	Elementary models of fractal distributions	20
1.10	Functions and integrals on fractals	22
1.11	Properties of integrals on fractals	25
1.12	Integration over non-integer-dimensional space	26
1.13	Multi-variable integration on fractals	28
1.14	Mass distribution on fractals	29
1.15	Density of states in Euclidean space	31
1.16	Fractional integral and measure on the real axis	32
1.17	Fractional integral and mass on the real axis	34
1.18	Mass of fractal media	36
1.19	Electric charge of fractal distribution	38
1.20	Probability on fractals	39
1.21	Fractal distribution of particles	41
	References	44
2	Hydrodynamics of Fractal Media	49
2.1	Introduction	49
2.2	Equation of balance of mass	50

2.3	Total time derivative of fractional integral	51
2.4	Equation of continuity for fractal media	54
2.5	Fractional integral equation of balance of momentum	55
2.6	Differential equations of balance of momentum	56
2.7	Fractional integral equation of balance of energy	57
2.8	Differential equation of balance of energy	58
2.9	Euler's equations for fractal media	60
2.10	Navier-Stokes equations for fractal media	62
2.11	Equilibrium equation for fractal media	63
2.12	Bernoulli integral for fractal media	64
2.13	Sound waves in fractal media	66
2.14	One-dimensional wave equation in fractal media	67
2.15	Conclusion	69
	References	69
3	Fractal Rigid Body Dynamics	73
3.1	Introduction	73
3.2	Fractional equation for moment of inertia	74
3.3	Moment of inertia of fractal rigid body ball	76
3.4	Moment of inertia for fractal rigid body cylinder	78
3.5	Equations of motion for fractal rigid body	81
3.6	Pendulum with fractal rigid body	82
3.7	Fractal rigid body rolling down an inclined plane	84
3.8	Conclusion	85
	References	86
4	Electrodynamics of	
	Fractal Distributions of Charges and Fields	89
4.1	Introduction	89
4.2	Electric charge of fractal distribution	90
4.3	Electric current for fractal distribution	92
4.4	Gauss' theorem for fractal distribution	93
4.5	Stokes' theorem for fractal distribution	93
4.6	Charge conservation for fractal distribution	94
4.7	Coulomb's and Biot-Savart laws for fractal distribution	95
4.8	Gauss' law for fractal distribution	96
4.9	Ampere's law for fractal distribution	97
4.10	Integral Maxwell equations for fractal distribution	98
4.11	Fractal distribution as an effective medium	100
4.12	Electric multipole expansion for fractal distribution	101
4.13	Electric dipole moment of fractal distribution	103
4.14	Electric quadrupole moment of fractal distribution	104
4.15	Magnetohydrodynamics of fractal distribution	107
4.16	Stationary states in magnetohydrodynamics of fractal distributions	110

4.17	Conclusion	111
	References	112
5	Ginzburg-Landau Equation for Fractal Media	115
5.1	Introduction	115
5.2	Fractional generalization of free energy functional	116
5.3	Ginzburg-Landau equation from free energy functional	117
5.4	Fractional equations from variational equation	118
5.5	Conclusion	121
	References	121
6	Fokker-Planck Equation for Fractal Distributions of Probability	123
6.1	Introduction	123
6.2	Fractional equation for average values	124
6.3	Fractional Chapman-Kolmogorov equation	125
6.4	Fokker-Planck equation for fractal distribution	127
6.5	Stationary solutions of generalized Fokker-Planck equation	130
6.6	Conclusion	132
	References	132
7	Statistical Mechanics of Fractal Phase Space Distributions	135
7.1	Introduction	135
7.2	Fractal distribution in phase space	136
7.3	Fractional phase volume for configuration space	136
7.4	Fractional phase volume for phase space	139
7.5	Fractional generalization of normalization condition	139
7.6	Continuity equation for fractal distribution in configuration space ..	141
7.7	Continuity equation for fractal distribution in phase space	142
7.8	Fractional average values for configuration space	144
7.9	Fractional average values for phase space	145
7.10	Generalized Liouville equation	146
7.11	Reduced distribution functions	147
7.12	Conclusion	148
	References	150
 Part II Fractional Dynamics and Long-Range Interactions		
8	Fractional Dynamics of Media with Long-Range Interaction	153
8.1	Introduction	153
8.2	Equations of lattice vibrations and dispersion law	155
8.3	Equations of motion for interacting particles	160
8.4	Transform operation for discrete models	162

8.5	Fourier series transform of equations of motion	163
8.6	Alpha-interaction of particles	166
8.7	Fractional spatial derivatives	170
8.8	Riesz fractional derivatives and integrals	174
8.9	Continuous limits of discrete equations	177
8.10	Linear nearest-neighbor interaction	180
8.11	Linear integer long-range alpha-interaction	181
8.12	Linear fractional long-range alpha-interaction	184
8.13	Fractional reaction-diffusion equation	187
8.14	Nonlinear long-range alpha-interaction	190
8.15	Fractional 3-dimensional lattice equation	194
8.16	Fractional derivatives from dispersion law	195
8.17	Fractal long-range interaction	198
8.18	Fractal dispersion law	203
8.19	Grünwald-Letnikov-Riesz long-range interaction	206
8.20	Conclusion	208
	References	209
9	Fractional Ginzburg-Landau Equation	215
9.1	Introduction	215
9.2	Particular solution of fractional Ginzburg-Landau equation	216
9.3	Stability of plane-wave solution	220
9.4	Forced fractional equation	221
9.5	Conclusion	222
	References	223
10	Psi-Series Approach to Fractional Equations	227
10.1	Introduction	227
10.2	Singular behavior of fractional equation	228
10.3	Resonance terms of fractional equation	229
10.4	Psi-series for fractional equation of rational order	230
10.5	Next to singular behavior	233
10.6	Conclusion	235
	References	236
Part III Fractional Spatial Dynamics		
11	Fractional Vector Calculus	241
11.1	Introduction	241
11.2	Generalization of vector calculus	242
11.3	Fundamental theorem of fractional calculus	247
11.4	Fractional differential vector operators	250
11.5	Fractional integral vector operations	253
11.6	Fractional Green's formula	254
11.7	Fractional Stokes' formula	257
11.8	Fractional Gauss' formula	259

11.9 Conclusion	261
References	262
12 Fractional Exterior Calculus and Fractional Differential Forms	265
12.1 Introduction	265
12.2 Differential forms of integer order	266
12.3 Fractional exterior derivative	269
12.4 Fractional differential forms	274
12.5 Hodge star operator	279
12.6 Vector operations by differential forms	281
12.7 Fractional Maxwell's equations in terms of fractional forms	282
12.8 Caputo derivative in electrodynamics	284
12.9 Fractional nonlocal Maxwell's equations	285
12.10 Fractional waves	287
12.11 Conclusion	288
References	289
13 Fractional Dynamical Systems	293
13.1 Introduction	293
13.2 Fractional generalization of gradient systems	294
13.3 Examples of fractional gradient systems	301
13.4 Hamiltonian dynamical systems	305
13.5 Fractional generalization of Hamiltonian systems	307
13.6 Conclusion	311
References	312
14 Fractional Calculus of Variations in Dynamics	315
14.1 Introduction	315
14.2 Hamilton's equations and variations of integer order	315
14.3 Fractional variations and Hamilton's equations	317
14.4 Lagrange's equations and variations of integer order	319
14.5 Fractional variations and Lagrange's equations	321
14.6 Helmholtz conditions and non-Lagrangian equations	323
14.7 Fractional variations and non-Hamiltonian systems	326
14.8 Fractional stability	328
14.9 Conclusion	330
References	331
15 Fractional Statistical Mechanics	335
15.1 Introduction	335
15.2 Liouville equation with fractional derivatives	336
15.3 Bogolyubov equation with fractional derivatives	340
15.4 Vlasov equation with fractional derivatives	343
15.5 Fokker-Planck equation with fractional derivatives	345
15.6 Conclusion	349

References	350
Part IV Fractional Temporal Dynamics	
16 Fractional Temporal Electrodynamics	357
16.1 Introduction	357
16.2 Universal response laws	358
16.3 Linear electrodynamics of medium	360
16.4 Fractional equations for laws of universal response	362
16.5 Fractional equations of the Curie-von Schweidler law	364
16.6 Fractional Gauss' laws for electric field	366
16.7 Universal fractional equation for electric field	369
16.8 Universal fractional equation for magnetic field	370
16.9 Fractional damping of magnetic field	372
16.10 Conclusion	373
References	374
17 Fractional Nonholonomic Dynamics	377
17.1 Introduction	377
17.2 Nonholonomic dynamics	378
17.3 Fractional temporal derivatives	385
17.4 Fractional dynamics with nonholonomic constraints	388
17.5 Constraints with fractional derivatives	394
17.6 Equations of motion with fractional nonholonomic constraints	396
17.7 Example of fractional nonholonomic constraints	398
17.8 Fractional conditional extremum	401
17.9 Hamilton's approach to fractional nonholonomic constraints	403
17.10 Conclusion	405
References	406
18 Fractional Dynamics and Discrete Maps with Memory	409
18.1 Introduction	409
18.2 Discrete maps without memory	410
18.3 Caputo and Riemann-Liouville fractional derivatives	415
18.4 Fractional derivative in the kicked term and discrete maps	418
18.5 Fractional derivative in the kicked term and dissipative discrete maps	422
18.6 Fractional equation with higher order derivatives and discrete map	425
18.7 Fractional generalization of universal map for $1 < \alpha \leq 2$	429
18.8 Fractional universal map for $\alpha > 2$	434
18.9 Riemann-Liouville derivative and universal map with memory	436
18.10 Caputo fractional derivative and universal map with memory	441
18.11 Fractional kicked damped rotator map	445
18.12 Fractional dissipative standard map	447

18.13 Fractional Hénon map	449
18.14 Conclusion	450
References	451

Part V Fractional Quantum Dynamics

19 Fractional Dynamics of Hamiltonian Quantum Systems	457
19.1 Introduction	457
19.2 Fractional power of derivative and Heisenberg equation	458
19.3 Properties of fractional dynamics	460
19.4 Fractional quantum dynamics of free particle	462
19.5 Fractional quantum dynamics of harmonic oscillator	463
19.6 Conclusion	464
References	465
20 Fractional Dynamics of Open Quantum Systems	467
20.1 Introduction	467
20.2 Fractional power of superoperator	468
20.3 Fractional equation for quantum observables	471
20.4 Fractional dynamical semigroup	473
20.5 Fractional equation for quantum states	475
20.6 Fractional non-Markovian quantum dynamics	477
20.7 Fractional equations for quantum oscillator with friction	478
20.8 Quantum self-reproducing and self-cloning	482
20.9 Conclusion	486
References	487
21 Quantum Analogs of Fractional Derivatives	491
21.1 Introduction	491
21.2 Weyl quantization of differential operators	492
21.3 Quantization of Riemann-Liouville fractional derivatives	494
21.4 Quantization of Liouville fractional derivative	496
21.5 Quantization of nondifferentiable functions	497
21.6 Conclusion	500
References	501
Index	503

Part I
Fractional Continuous Models of
Fractal Distributions

Chapter 1

Fractional Integration and Fractals

The theory of integration and differentiation of non-integer order has a long history from 30 September 1695, when the derivatives of order $\alpha = 1/2$ was described by Leibniz in a letter to L'Hospital (Oldham and Spanier, 1974; Samko et al., 1993; Ross, 1975). The earliest theory of integrals and derivatives of non-integer order goes back to Liouville and Riemann (Ross, 1975). There are many interesting books about fractional calculus and fractional differential equations (Oldham and Spanier, 1974; Samko et al., 1993; Miller and Ross, 1993; Podlubny, 1999; Kilbas et al., 2006; Nahushev, 2003; Pshu, 2005); see also (McBride, 1976, 1986; Srivastava and Owa, 1989; Nishimoto, 1989; Kiryakova, 1994; Rubin, 1996). Derivatives and integrals of non-integer order, and fractional integro-differential equations have found many applications in recent studies in physics (for example, books (West et al., 2003; Zaslavsky, 2005; Uchaikin, 2008; Mainardi, 2010), edited volumes (Carpiniteri and Mainardi, 1997; Hilfer, 2000; Sabatier et al., 2007), and reviews (Metzler and Klafter, 2000; Zaslavsky, 2002; Montroll and Shlesinger, 1984; Metzler and Klafter, 2004)). Now we can state that fractional dynamics form a new paradigm in science.

The fractional integrals have different definitions (Samko et al., 1993; Kilbas et al., 2006), and exploiting any of them depends on the boundary conditions and the specifics of the considered physical systems and processes. We note that some of probability, geometric and physical interpretations of fractional integrals and derivatives were discussed in (Nigmatullin, 1992; Rutman, 1994, 1995; Ren et al., 1996; Yu et al., 1997; Ren et al., 1997; Machado, 2003; Tatom, 1995; Stanislavsky, 2004; Stanislavsky and Weron, 2002; Tarasov, 2004) and (Heymans and Podlubny, 2006; Ben Adda, 1997; Mainardi, 1998; Gorenflo, 1998; Kempfle and Schafer, 2000; Monsrefi-Torbati and Hammond, 1998; Podlubny, 2002).

In this chapter, we consider some basic concepts of fractional continuous models of fractal distribution of particles and fields. The fractional integral equations are used to describe fractal distributions of mass, charge and probability. Equations of motion for the fractal distributions are considered in next chapters.

1.1 Riemann-Liouville fractional integrals

Two of the most known definitions of fractional integrals are the so-called Riemann-Liouville and Liouville integrals. First, let us consider the definition of Riemann-Liouville fractional integration on a finite interval of the real line. More detailed information can be found in (Samko et al., 1993; Kilbas et al., 2006).

If $f(x)$ is a continuous function on the real line, then we can form the definite integral from a to x :

$$(I^1 f)(x) = \int_a^x f(x_1) dx_1.$$

Repeating this process gives

$$(I^2 f)(x) = \int_a^x (I^1 f)(x_1) dx_1 = \int_a^x dx_1 \int_a^{x_1} dx_2 f(x_2), \quad (1.1)$$

and this can be extended arbitrarily

$$(I^n f)(x) = \int_a^x dx_1 \int_a^{x_1} dx_2 \cdots \int_a^{x_{n-1}} dx_n f(x_n).$$

The Cauchy formula for repeated integration is

$$(I^n f)(x) = \frac{1}{(n-1)!} \int_a^x (x-z)^{n-1} f(z) dz. \quad (1.2)$$

A proof is given by induction. Equation (1.2) can be generalized for non-integer n . Using the Gamma function $\Gamma(n) = (n-1)!$, to remove the discrete nature of the factorial, we obtain a fractional generalization of the integration

$$(I^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-z)^{\alpha-1} f(z) dz \quad (\alpha > 0).$$

This equation can be considered as a definition of fractional integral.

Let $[a, b]$ be a finite or infinite interval of the real axis \mathbb{R} . We denote by $L_p(a, b)$, where $p \geq 1$, the set of those Lebesgue real-valued measurable functions $f = f(x)$ on $[a, b]$ for which

$$\left(\int_a^b |f(x)|^p dx \right)^{1/p} < \infty.$$

If $f(x)$ is a function on a finite interval $[a, b]$ such that $f \in L_1(a, b)$, then the left- and right-sided Riemann-Liouville fractional integrals are

$$({}_a I_x^\alpha f)(x) = {}_a I_x^\alpha [f](z) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-z)^{\alpha-1} f(z) dz, \quad (1.3)$$

$$({}_x I_b^\alpha f)(x) = {}_x I_b^\alpha [f](z) = \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} f(z) dz. \quad (1.4)$$

The fractional integration operators ${}_aI_x^\alpha$ and ${}_xI_b^\alpha$ with $\alpha > 0$ are bounded (Samko et al., 1993) in $L_p(a, b)$ for all $p \geq 1$. Let us note the connection between the operators ${}_aI_x^\alpha$ and ${}_xI_b^\alpha$ of the form

$$Q{}_aI_x^\alpha = {}_xI_b^\alpha Q,$$

$$Q{}_xI_b^\alpha = {}_aI_x^\alpha Q,$$

where Q is defined by

$$(Qf)(x) = f(a + b - x).$$

An important property of the fractional integration operators is given by the following statement (see Theorem 2.6 in (Samko et al., 1993)). The fractional integration operators ${}_aI_x^\alpha$ and ${}_xI_b^\alpha$ form a semigroup. If $\alpha > 0$ and $\beta > 0$, then the equations

$$({}_aI_x^\alpha {}_aI_x^\beta f)(x) = ({}_aI_x^{\alpha+\beta} f)(x), \quad ({}_xI_b^\alpha {}_xI_b^\beta f)(x) = ({}_xI_b^{\alpha+\beta} f)(x) \quad (1.5)$$

are satisfied at almost every point $x \in [a, b]$ for $f(x) \in L_p(a, b)$ with $p \geq 1$. If $\alpha + \beta > 1$, then relations (1.5) holds at any point of $[a, b]$. These equations with $\alpha > 0$ and $\beta > 0$ are satisfied at any point of $[a, b]$ for $f \in C(a, b)$, i.e., f is continuous on $[a, b]$. This is the semigroup property of fractional integration.

The fractional integral can be given in terms of elementary functions for a small number of functions. For example, the Riemann-Liouville integrations give

$${}_aI_x^\alpha (x-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} (x-a)^{\alpha+\beta}, \quad (1.6)$$

$${}_xI_b^\alpha (b-x)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} (b-x)^{\alpha+\beta}, \quad (1.7)$$

for $\beta > -1$ and $\alpha \geq 0$. For the constant C , we have

$${}_aI_x^\alpha C = \frac{C}{\Gamma(\alpha+1)} (x-a)^\alpha, \quad (1.8)$$

$${}_xI_b^\alpha C = \frac{C}{\Gamma(\alpha+1)} (b-x)^\alpha. \quad (1.9)$$

For $\alpha = 1$, Equation (1.6) leads us to the usual relation

$${}_aI_x^1 (x-a)^n = \frac{\Gamma(n+1)}{\Gamma(n+2)} (x-a)^{n+1} = \frac{1}{n+1} (x-a)^{n+1}, \quad (1.10)$$

where we use $\Gamma(z+1) = z\Gamma(z)$. Other relations can be found in Table 9.1 of (Samko et al., 1993).