NONLINEAR
PHYSICAL
SCIENCE

Vasily E. Tarasov

Fractional Dynamics

Applications of Fractional Calculus to Dynamics of Particles, Fields and Media

分数维动力学

分数阶积分在粒子、场及介质动力学中的应用

高等教育出版社
HIGHER EDUCATION PRESS

Author
Vasily E. Tarasov
Skobeltsyn Institute of Nuclear Physics
Moscow State University
119992 Moscow, Russia
Email: tarasov@theory.sinp.msu.ru

图书在版编目(CIP)数据

分数维动力学:分数阶积分在粒子、场及介质动力学中的应用=Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media:英文/(俄罗斯)塔拉索夫(Tarasov,V.E.)著.—北京:高等教育出版社,2010.8
(非线性物理科学/罗朝俊,(瑞典)伊布拉基莫夫主编)
ISBN 978-7-04-029473-6

Ⅰ.①分… Ⅱ.①塔… Ⅲ.①维-动力学-英文 Ⅳ.①O313

中国版本图书馆 CIP 数据核字(2010)第 097428 号

策划编辑 王丽萍　**责任编辑** 李　鹏　**封面设计** 杨立新
责任校对 王　超　**责任印制** 陈伟光

出版发行	高等教育出版社	**购书热线**	010-58581118
社　　址	北京市西城区德外大街 4 号	**咨询电话**	400-810-0598
邮政编码	100120	**网　　址**	http://www.hep.edu.cn
			http://www.hep.com.cn
		网上订购	http://www.landraco.com
经　　销	蓝色畅想图书发行有限公司		http://www.landraco.com.cn
印　　刷	涿州市星河印刷有限公司	**畅想教育**	http://www.widedu.com
开　　本	787 × 1092　1/16	**版　　次**	2010 年 8 月第 1 版
印　　张	32.75	**印　　次**	2010 年 8 月第 1 次印刷
字　　数	780 000	**定　　价**	89.00 元

本书如有缺页、倒页、脱页等质量问题,请到所购图书销售部门联系调换。

物料号　29473-00

Preface

Fractional calculus is a theory of integrals and derivatives of any arbitrary real (or complex) order. It has a long history from 30 September 1695, when the derivative of order $\alpha = 1/2$ was mentioned by Leibniz. The fractional differentiation and fractional integration go back to many great mathematicians such as Leibniz, Liouville, Grünwald, Letnikov, Riemann, Abel, Riesz and Weyl. The integrals and derivatives of non-integer order, and the fractional integro-differential equations have found many applications in recent studies in theoretical physics, mechanics and applied mathematics.

New possibilities in mathematics and theoretical physics appear, when the order α of the differential operator D_x^α or the integral operator I_x^α becomes an arbitrary parameter. The fractional calculus is a powerful tool to describe physical systems that have long-term memory and long-range spatial interactions. In general, many usual properties of the ordinary (first-order) derivative D_x are not realized for fractional derivative operators D_x^α. For example, a product rule, chain rule and semi-group property have strongly complicated analogs for the operators D_x^α.

Most of the processes associated with complex systems have nonlocal dynamics and it can be characterized by long-term memory in time. The fractional integration and fractional differentiation operators allow one to consider some of those characteristics. Using fractional calculus, it is possible to obtain useful dynamical models, where fractional integro-differential operators in the time and space variables describe the long-term memory and nonlocal spatial properties of the complex media and processes. We should note that close connections exist between fractional differential and integral equations, and the dynamics of many complex systems, anomalous processes and fractal media.

There are many interesting books about fractional calculus, fractional differential equations, and their physical applications. The first book dedicated specifically to the theory of fractional integrals and derivatives, is the one by Oldham and Spanier published in 1974. There exists the remarkably comprehensive encyclopedic-type monograph by Samko, Kilbus and Marichev, which was published in Russian in 1987 and in English in 1993. The works devoted substantially to fractional differential equations are the book by Miller and Ross (1993), and the book by Podlubny

(1999). In 2006, Kilbas, Srivastava and Trujillo published a very important and remarkable book, in which one can find a modern encyclopedic, detailed and rigorous theory of fractional differential equations. This book can be recommended as a main modern mathematical handbook for graduate students and researchers, who are interested in this subject. There exist some mathematical monographs devoted to special questions of fractional calculus, including the book by McBride (1979), the work by Kiryakova (1993), the monograph by Rubin (1996) and the volume edited by Srivastava and Owa (1989). The physical applications of fractional calculus to describe complex media and processes were considered in the very interesting volumes edited by Carpintery and Mainardi in 1997 and by Hilfer in 2000. The book by West, Bologna, and Grigolini published in 2003 is devoted to physical application of fractional calculus to fractal processes. The first book devoted exclusively to the fractional dynamics and application of fractional calculus to chaos is the one by Zaslavsky published in 2005. One of the most recent books on the subject of fractional calculus is the edited volume of Sabatier, Agrawal and Tenreiro Machado published in 2007. In 2010, the book by Mainardi will be published to devote to applications of fractional calculus in dynamics of viscoelastic materials. Note that there are international journals such as "Journal of Fractional Calculus" and "Fractional Calculus and Applied Analysis", which are dedicated entirely to the fractional calculus.

The content of the noted books and edited volumes about applications of fractional calculus in physics, mechanics and applied mathematics does not include all of modern fractional theoretical models, methods and approaches. A lot of new results, obtained recently in the fractional dynamics, are not reflected in the books. In this monograph, some modern applications of fractional calculus to complex physical systems and new results of last years are described. Therefore the book is supposed to be useful for physicists and mathematicians, who are interested in the modern theories of complex processes and media. Some of interesting subjects in the theoretical physics are not described in this monograph, since it is not possible to realize in one book a complete description of all fractional dynamics. For example, the applications of fractional calculus to the viscoelastic media and the continuous time random walk processes are not considered in the book, since there are interesting monographs and reviews, where these subjects are discussed.

The text is self-contained and can be used without previous courses in fractional calculus and theory of fractals. The necessary information, which is beyond to undergraduate courses of the mathematics, is suggested in the book. Therefore this book can be used in the courses for graduate students. In the book the modern approaches and new fundamental results of last years are described. Therefore the manuscript is supposed to be useful for physicists and mathematicians who are interested in the electrodynamics, statistical and condensed matter physics, quantum dynamics, complex media theories and kinetics, discrete maps and chain models, nonlinear dynamics and chaos.

The book consists of five parts. The first part is devoted the fractional continuous models of fractal distributions of particles. The fractional integral equations are used to describe fractal distributions of mass, charge and probability. In the second part, we consider the fractional dynamics that describes the media with long-range interaction of particles. The close connection of discrete models with long-range interactions and continuous medium equations with fractional derivatives is proved. The fractional coordinate derivatives are used to describe nonlocal properties of the complex media. In the third part, we suggest the fractional vector calculus, fractional exterior calculus, and fractional variation calculus to describe generalized dynamical systems, fractional statistical mechanics and kinetics, fractional electrodynamics of complex media. The suggested generalizations of vector operations and variations are considered with respect to coordinate variables. In the fourth part, we describe the fractional temporal dynamics, where derivatives with respect to time variable have non-integer orders. The nonholonomic systems with generalized constraints to describe a long-term memory are considered. The electrodynamics of dielectric media is described as a fractional temporal electrodynamics. The discrete maps with memory are obtained from the fractional differential equations of kicked dynamical systems. In the fifth part, we consider an application of fractional derivatives in quantum dynamics. These derivatives are defined as fractional powers of self-adjoint derivatives. A fractional generalization of quantum Markovian dynamics is suggested. The quantization of different fractional derivatives and fractal functions are suggested. The numerous recent publications cited are listed in the references at the end of each chapter.

The author is greatly indebted to professor George M. Zaslavsky for his invaluable discussions and comments. The author wishes to express his gratitude to professor Albert C.J. Luo for his support of the edition of this book. The author would like to express his appreciation for the kind hospitality by the Courant Institute of Mathematical Studies of New York University during his visits in 2005, 2007, 2008 and 2009.

Vasily E. Tarasov

12 October, 2009, Moscow

Contents

Part I
Fractional Continuous Models of Fractal Distributions

Chapter 1
Fractional Integration and Fractals

The theory of integration and differentiation of non-integer order has a long history from 30 September 1695, when the derivatives of order $\alpha = 1/2$ was described by Leibniz in a letter to L'Hospital (Oldham and Spanier, 1974; Samko et al., 1993; Ross, 1975). The earliest theory of integrals and derivatives of non-integer order goes back to Liouville and Riemann (Ross, 1975). There are many interesting books about fractional calculus and fractional differential equations (Oldham and Spanier, 1974; Samko et al., 1993; Miller and Ross, 1993; Podlubny, 1999; Kilbas et al., 2006; Nahushev, 2003; Pshu, 2005); see also (McBride, 1976, 1986; Srivastava and Owa, 1989; Nishimoto, 1989; Kiryakova, 1994; Rubin, 1996). Derivatives and integrals of non-integer order, and fractional integro-differential equations have found many applications in recent studies in physics (for example, books (West et al., 2003; Zaslavsky, 2005; Uchaikin, 2008; Mainardi, 2010), edited volumes (Carpinteri and Mainardi, 1997; Hilfer, 2000; Sabatier et al., 2007), and reviews (Metzler and Klafter, 2000; Zaslavsky, 2002; Montroll and Shlesinger, 1984; Metzler and Klafter, 2004)). Now we can state that fractional dynamics form a new paradigm in science.

The fractional integrals have different definitions (Samko et al., 1993; Kilbas et al., 2006), and exploiting any of them depends on the boundary conditions and the specifics of the considered physical systems and processes. We note that some of probability, geometric and physical interpretations of fractional integrals and derivatives were discussed in (Nigmatullin, 1992; Rutman, 1994, 1995; Ren et al., 1996; Yu et al., 1997; Ren et al., 1997; Machado, 2003; Tatom, 1995; Stanislavsky, 2004; Stanislavsky and Weron, 2002; Tarasov, 2004) and (Heymans and Podlubny, 2006; Ben Adda, 1997; Mainardi, 1998; Gorenflo, 1998; Kempfle and Schafer, 2000; Monsrefi-Torbati and Hammond, 1998; Podlubny, 2002).

In this chapter, we consider some basic concepts of fractional continuous models of fractal distribution of particles and fields. The fractional integral equations are used to describe fractal distributions of mass, charge and probability. Equations of motion for the fractal distributions are considered in next chapters.

1.1 Riemann-Liouville fractional integrals

Two of the most known definitions of fractional integrals are the so-called Riemann-Liouville and Liouville integrals. First, let us consider the definition of Riemann-Liouville fractional integration on a finite interval of the real line. More detailed information can be found in (Samko et al., 1993; Kilbas et al., 2006).

If $f(x)$ is a continuous function on the real line, then we can form the definite integral from a to x:

$$(I^1 f)(x) = \int_a^x f(x_1)\, dx_1.$$

Repeating this process gives

$$(I^2 f)(x) = \int_a^x (I^1 f)(x_1) dx_1 = \int_a^x dx_1 \int_a^{x_1} dx_2\, f(x_2), \tag{1.1}$$

and this can be extended arbitrarily

$$(I^n f)(x) = \int_a^x dx_1 \int_a^{x_1} dx_2 \cdots \int_a^{x_{n-1}} dx_n\, f(x_n).$$

The Cauchy formula for repeated integration is

$$(I^n f)(x) = \frac{1}{(n-1)!} \int_a^x (x-z)^{n-1} f(z)\, dz. \tag{1.2}$$

A proof is given by induction. Equation (1.2) can be generalized for non-integer n. Using the Gamma function $\Gamma(n) = (n-1)!$, to remove the discrete nature of the factorial, we obtain a fractional generalization of the integration

$$(I^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-z)^{\alpha-1} f(z)\, dz \quad (\alpha > 0).$$

This equation can be considered as a definition of fractional integral.

Let $[a,b]$ be a finite or infinite interval of the real axis $\mathbb{R}$. We denote by $L_p(a,b)$, where $p \geqslant 1$, the set of those Lebesgue real-valued measurable functions $f = f(x)$ on $[a,b]$ for which

$$\left(\int_a^b |f(x)|^p dx \right)^{1/p} < \infty.$$

If $f(x)$ is a function on a finite interval $[a,b]$ such that $f \in L_1(a,b)$, then the left- and right-sided Riemann-Liouville fractional integrals are

$$({}_aI_x^\alpha f)(x) = {}_aI_x^\alpha[z] f(z) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-z)^{\alpha-1} f(z) dz, \tag{1.3}$$

$$({}_xI_b^\alpha f)(x) = {}_xI_b^\alpha[z] f(z) = \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} f(z) dz. \tag{1.4}$$

The fractional integration operators ${}_aI_x^\alpha$ and ${}_xI_b^\alpha$ with $\alpha > 0$ are bounded (Samko et al., 1993) in $L_p(a,b)$ for all $p \geqslant 1$. Let us note the connection between the operators ${}_aI_x^\alpha$ and ${}_xI_b^\alpha$ of the form

$$Q\,{}_aI_x^\alpha = {}_xI_b^\alpha Q,$$

$$Q\,{}_xI_b^\alpha = {}_aI_x^\alpha Q,$$

where Q is defined by

$$(Qf)(x) = f(a+b-x).$$

An important property of the fractional integration operators is given by the following statement (see Theorem 2.6 in (Samko et al., 1993)). The fractional integration operators ${}_aI_x^\alpha$ and ${}_xI_b^\alpha$ form a semigroup. If $\alpha > 0$ and $\beta > 0$, then the equations

$$({}_aI_x^\alpha\, {}_aI_x^\beta\, f)(x) = ({}_aI_x^{\alpha+\beta}\, f)(x), \quad ({}_xI_b^\alpha\, {}_xI_b^\beta\, f)(x) = ({}_xI_b^{\alpha+\beta}\, f)(x) \tag{1.5}$$

are satisfied at almost every point $x \in [a,b]$ for $f(x) \in L_p(a,b)$ with $p \geqslant 1$. If $\alpha+\beta > 1$, then relations (1.5) holds at any point of $[a,b]$. These equations with $\alpha > 0$ and $\beta > 0$ are satisfied at any point of $[a,b]$ for $f \in C(a,b)$, i.e., f is continuous on $[a,b]$. This is the semigroup property of fractional integration.

The fractional integral can be given in terms of elementary functions for a small number of functions. For example, the Riemann-Liouville integrations give

$${}_aI_x^\alpha (x-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(x-a)^{\alpha+\beta}, \tag{1.6}$$

$${}_xI_b^\alpha (b-x)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(b-x)^{\alpha+\beta}, \tag{1.7}$$

for $\beta > -1$ and $\alpha \geqslant 0$. For the constant C, we have

$${}_aI_x^\alpha C = \frac{C}{\Gamma(\alpha+1)}(x-a)^\alpha, \tag{1.8}$$

$${}_xI_b^\alpha C = \frac{C}{\Gamma(\alpha+1)}(b-x)^\alpha. \tag{1.9}$$

For $\alpha = 1$, Equation (1.6) leads us to the usual relation

$${}_aI_x^1 (x-a)^n = \frac{\Gamma(n+1)}{\Gamma(n+2)}(x-a)^{n+1} = \frac{1}{n+1}(x-a)^{n+1}, \tag{1.10}$$

where we use $\Gamma(z+1) = z\Gamma(z)$. Other relations can be found in Table 9.1 of (Samko et al., 1993).