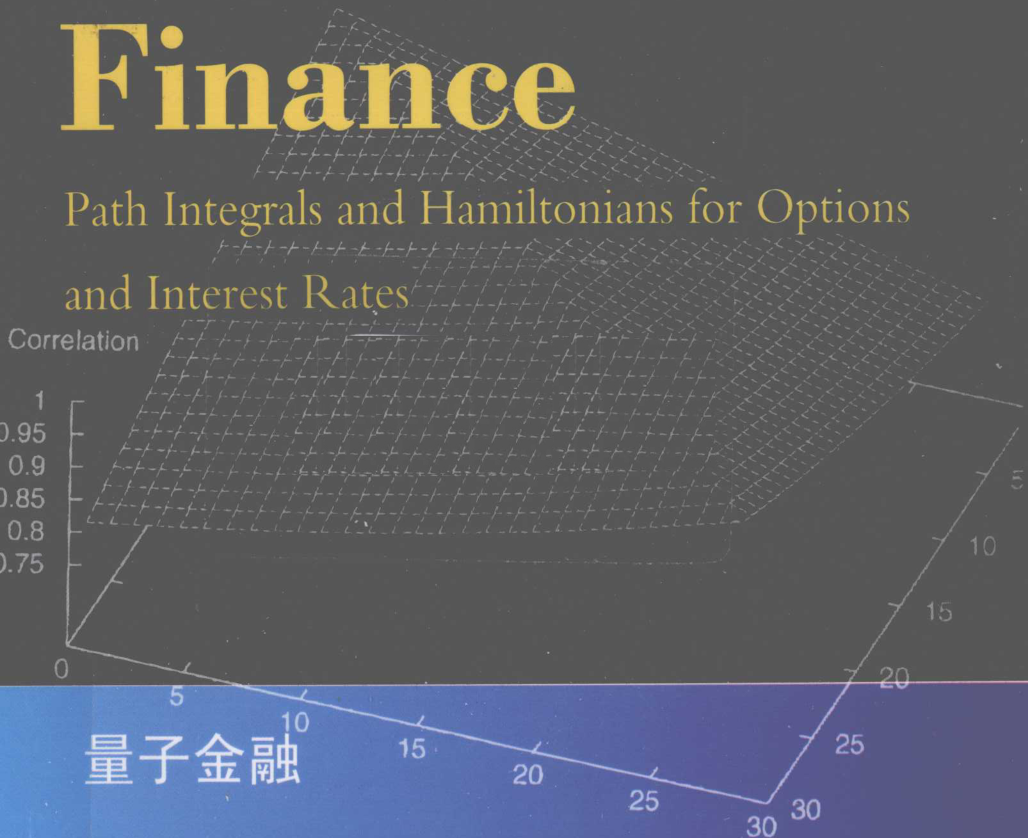
Belal E. Baaquie

Quantum Finance

Path Integrals and Hamiltonians for Options and Interest Rates



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BELAL E. BAAQUIE
National University of Singapore

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I dedicate this book to my father Mohammad Abdul Baaquie and to the memory of my mother Begum Ajmeri Roanaq Ara Baaquie, for their precious lifelong support and encouragement.

Foreword

After a few early isolated cases in the 1980s, since the mid-1990s hundreds of papers dealing with economics and finance have invaded the physics preprint server xxx.lanl.gov/cond-mat, initially devoted to condensed matter physics, and now covering subjects as different as computer science, biology or probability theory. The flow of paper posted on this server is still increasing – roughly one per day – addressing a range of problems, from financial data analysis to analytical option-pricing methods, agent-based models of financial markets and statistical models of wealth distribution or company growth. Some papers are genuinely beautiful, others are rediscoveries of results known by economists, and unfortunately some are simply crazy.

A natural temptation is to apply the tools one masters to other fields. In the case of physics and finance, this temptation is extremely strong. The sophisticated tools developed in the last 50 years to deal with statistical mechanics and quantum mechanics problems are often of immediate interest in finance and in economics. Perturbation theory, path integral (Feynman–Kac) methods, random matrix and spin-glass theory are useful for option pricing, portfolio optimization and game theoretical situations, and many new insights have followed from such transfers of knowledge.

Within theoretical physics, quantum field theory has a special status and is regarded by many as the queen of disciplines, that has allowed one to unravel the most intimate intricacies of nature, from quantum electrodynamics to critical phenomena. In the present book, Belal Baaquie tells us how these methods can be applied to finance problems, and in particular to the modelling of interest rates. The interest rate curve can be seen as a string of numbers, one for each maturity, fluctuating in time. The 'one-dimensional' nature of these randomly fluctuating rates imposes subtle correlations between different maturities, that are most naturally described using quantum field theory, which was indeed created to deal with nontrivial correlations between fluctuating fields. The level of complexity of the bond market (reflecting the structure of the interest rate curve) and its derivatives (swaps, caps, floors, swaptions) requires a set of efficient and adapted techniques. My feeling is that the methods of quantum field theory, which naturally grasp complex structures, are particularly well suited for this type of problems. Belal Baaquie's book, based on his original work on the subject, is particularly useful for those who want to learn techniques which will become, in a few years, unavoidable. Many new ideas and results improving our understanding of interest rate markets will undoubtedly follow from an in-depth exploration of the paths suggested in this fascinating (albeit sometimes demanding) opus.

Jean-Philippe Bouchaud
Capital Fund Management and CEA-Saclay

Preface

Financial markets have undergone tremendous growth and dramatic changes in the past two decades, with the volume of daily trading in currency markets hitting over a trillion US dollars and hundreds of billions of dollars in bond and stock markets. Deregulation and globalization have led to large-scale capital flows; this has raised new problems for finance as well as has further spurred competition among banks and financial institutions.

The resulting booms, bubbles and busts of the global financial markets now directly affect the lives of hundreds of millions of people, as was witnessed during the 1998 East Asian financial crisis.

The principles of banking and finance are fairly well established [16,76,87] and the challenge is to apply these principles in an increasingly complicated environment. The immense growth of financial markets, the existence of vast quantities of financial data and the growing complexity of the market, both in volume and sophistication, has made the use of powerful mathematical and computational tools in finance a necessity. In order to meet the needs of customers, complex financial instruments have been created; these instruments demand advanced valuation and risk assessment models and systems that quantify the returns and risks for investors and financial institutions [63, 100].

The widespread use in finance of stochastic calculus and of partial differential equations reflects the traditional presence of probabilists and applied mathematicians in this field. The last few years has seen an increasing interest of theoretical physicists in the problems of applied and theoretical finance. In addition to the vast corpus of literature on the application of stochastic calculus to finance, concepts from theoretical physics have been finding increasing application in both theoretical and applied finance. The influx of ideas from theoretical physics, as expressed for example in [18] and [69], has added a whole collection of new mathematical and computational techniques to finance, from the methods of classical and quantum physics to the use of path integration, statistical mechanics and so on. This book is part of the on-going process of applying ideas from physics to finance.

The long-term goal of this book is to contribute to a quantum theory of finance; towards this end the theoretical tools of quantum physics are applied to problems in finance. The larger question of applying the formalism and insights of (quantum) physics to economics, and which forms a part of the larger subject of econophysics [88, 89], is left for future research.

The mathematical background required of the readership is the following:

- A good grasp of calculus
- Familiarity with linear algebra
- Working knowledge of probability theory

The material covered in this book is primarily meant for physicists and mathematicians conducting research in the field of finance, as well as professional theorists working in the finance industry. Specialists working in the field of derivative instruments, corporate and Treasury Bonds and foreign currencies will hopefully find that the theoretical tools and mathematical ideas introduced in this book broadens their repertoire of quantitative approaches to finance.

This book could also be of interest to researchers from the theoretical sciences who are thinking of pursuing research in the field of finance as well as graduates students with the required mathematical training. An earlier draft of this book was taught as an advanced graduate course to a group of students from financial engineering, physics and mathematics.

Given the diverse nature of the potential audience, fundamental concepts of finance have been reviewed to motivate readers new to the field. The chapters on 'Introduction to finance' and on 'Derivative securities' are meant for physicists and mathematicians unfamiliar with concepts of finance. On the other hand, discussions on quantum mechanics and quantum field theory are meant to introduce specialists working in finance and in mathematics to concepts from quantum theory.

Acknowledgments

I am deeply grateful to Lawrence Ma for introducing me to the subject of theoretical finance; most of my initial interest in mathematical finance is a result of the patient explanations of Lawrence.

I thank Jean-Philippe Bouchaud for instructive and enjoyable discussions, and for making valuable suggestions that have shaped my thinking on finance; the insights that Jean-Philippe brings to the interface of physics and finance have been particularly enlightening.

I would like to thank Toh Choon Peng, Sanjiv Das, George Chacko, Mitch Warachka, Omar Foda, Srikant Marakani, Claudio Coriano, Michael Spalinski, Bertrand Roehner, Bertrand Delamotte, Cui Liang and Frederick Willeboordse for many helpful and stimulating interactions.

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I thank Science and Finance for kindly providing data on Eurodollar futures, and the Laboratoire de Physique Théorique et Hautes Energies, Universités Paris 6 et 7, and in particular François Martin, for their kind hospitality during the completion of this book.

Contents

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Synopsis

Two underlying themes run through this book: first, defining and analyzing the subject of quantitative finance in the conceptual and mathematical framework of quantum theory, with special emphasis on its path-integral formulation, and, second, the introduction of the techniques and methodology of quantum field theory in the study of interest rates.

No attempt is made to apply quantum theory in re-working the fundamental principles of finance. Instead, the term 'quantum' refers to the abstract mathematical constructs of quantum theory that include probability theory, state space, operators, Hamiltonians, commutation equations, Lagrangians, path integrals, quantized fields, bosons, fermions and so on. All these theoretical structures find natural and useful applications in finance.

The path integral and Hamiltonian formulations of (random) quantum processes have been given special emphasis since they are equivalent to, as well as independent of, the formalism of stochastic calculus – which currently is one of the cornerstones of mathematical finance. The starting point for the application of path integrals and Hamiltonians in finance is in stock option pricing. Path integrals are subsequently applied to the modelling of linear and nonlinear theories of interest rates as a two-dimensional quantum field, something that is beyond the scope of stochastic calculus. Path integrals have the additional advantage of providing a framework for efficiently implementing the mathematical procedure of renormalization which is necessary in the study of nonlinear quantum field theories.

The term 'Quantum Finance' represents the synthesis of the concepts, methods and mathematics of quantum theory, with the field of theoretical and applied finance.

To ease the reader's transition to the mathematics of quantum theory, and of path integration in particular, the presentation of new material starts in a few cases with well-established models of finance. New ideas are introduced by first carrying out the relatively easier exercise of recasting well-known results in the formalism of quantum theory, and then going on to derive new results. One unexpected advantage of this approach is that theorists, working in the field of finance – presently focussed on notions drawn from stochastic calculus and partial differential equations – obtain a formalism that completely parallels and mirrors stochastic calculus, and prepares the ground for a (smooth) transition to the mathematics of quantum field theory.

All important equations are derived from first principles of finance and, as far as possible, a complete and self-contained mathematical treatment of the main results is given. A few of the exactly soluble models that appear in finance are closely studied since these serve as exemplars for demonstrating the general principles of quantum finance. In particular, the workings of the path-integral and Hamiltonian formulations are demonstrated by concretely working out, in complete mathematical detail, some of the more instructive models. The models themselves are interesting in their own right, thus providing a realistic context for developing the applications of path integrals to finance.

The book consists of the following three major components:[1]

Fundamental concepts of finance

The standard concepts of finance and equations of option theory are reviewed in this component.

Chapter 2 is an 'Introduction to finance' that is meant for readers who are unfamiliar with the essential ideas of finance. Fundamental concepts and terminology of finance, necessary for understanding the particular set of equations that arise in finance, are introduced. In particular, the concepts of risk and return, time value of money, arbitrage, hedging and, finally, Treasury Bonds and fixed-income securities are briefly discussed.

Chapter 3 on 'Derivative securities' introduces the concept of financial derivatives and discusses the pricing of derivatives. The classic analysis of Black and Scholes is discussed, the mathematics of stochastic calculus briefly reviewed and the connection of stochastic processes with the Langevin equation is elaborated. A derivation from first principles is given of the price of a stock option with stochastic volatility. The material covered in these two chapters is standard, and defines the framework and context for the next two chapters.

Systems with finite number of degrees of freedom

In this part Hamiltonians and path integrals are applied to the study of stock options and stochastic interest rates models. These models are characterized by having

[1] The path-integral formulation of problems in finance opens the way for applying powerful computational algorithms; these numerical algorithms are a specialized subject, and are not addressed except for a passing reference in Section 5.16.

finite number of **degrees of freedom**, which is defined to be the **number of independent random variables at each instant of time** t. Examples of such systems are a randomly evolving equity $S(t)$ or the spot interest rate $r(t)$, each of which have one degree of freedom. All quantities computed for quantum systems with a finite number of degrees of freedom are completely finite, and do not need the procedure of renormalization to have a well-defined value.

In Chapter 4 on 'Hamiltonians and stock options', the problem of the pricing of derivative securities is recast as a problem of quantum mechanics, and the Hamiltonians driving the prices of options are derived for both stock prices with constant and stochastic volatility. The martingale condition required for risk-neutral evolution is re-expressed in terms of the Hamiltonian. Potential terms in the Hamiltonian are shown to represent a class of path-dependent options. Barrier options are solved exactly using the appropriate Hamiltonian.

In Chapter 5 on 'Path integrals and stock options', the problem of option pricing is expressed as a Feynman path integral. The Hamiltonians derived in the previous chapter provide a link between the partial differential equations of option pricing and its path-integral realization. A few path integrals are explicitly evaluated to illustrate the mathematics of path integration. The case of stock price with stochastic volatility is solved exactly, as this is a nontrivial problem which is a good exemplar for illustrating the subtleties of path integration.

Certain exact simplifications emerge due to the path-integral representation of stochastic volatility and lead to an efficient Monte Carlo algorithm that is discussed to illustrate the numerical aspects of the path integral.

In Chapter 6 on 'Stochastic interest rates' Hamiltonians and path integrals', some of the important existing stochastic models for the spot and forward interest rates are reviewed. The Fokker–Planck Hamiltonian and path integral are obtained for the spot interest rate, and a path-integral solution of the Vasicek model is presented.

The Heath–Jarrow–Morton (HJM) model for the forward interest rates is recast as a problem of path integration, and well-known results of the HJM model are re-derived using the path integral.

Chapter 6 is a preparation for the main thrust of this book, namely the application of quantum field theory to the modelling of the interest rates.

Quantum field theory of interest rates models

Quantum field theory is a mathematical structure for studying systems that have infinitely many degrees of freedom; there are many new features that emerge for such systems that are beyond the formalism of stochastic calculus, the most important being the concept of renormalization for nonlinear field theories. All the chapters in this part treat the forward interest rates as a quantum field.

In Chapter 7 on 'Quantum field theory of forward interest rates', the formalism of path integration is applied to a randomly evolving curve: the forward interest rates are modelled as a randomly fluctuating curve that is naturally described by quantum field theory. Various linear (Gaussian) models are studied to illustrate the theoretical flexibility of the field theory approach. The concept of psychological future time is shown to provide a natural extension of (Gaussian) field theory models. The martingale condition is solved for Gaussian models, and a field theory derivation is given for the change of numeraire. Nonlinear field theories are shown to arise naturally in modelling positive-valued forward interest rates as well as forward rates with stochastic volatility.

In Chapter 8 on 'Empirical forward interest rates and field theory models', the empirical aspects of the forward rates are discussed in some detail, and it is shown how to calibrate and test field theory models using market data on Eurodollar futures. The most important result of this chapter is that a so-called 'stiff' Gaussian field theory model provides an almost exact fit for the market behaviour of the forward rates. The empirical study provides convincing evidence on the efficacy of the field theory in modelling the behaviour of the forward interest rates.

In Chapter 9 on 'Field theory of Treasury Bonds' derivatives and hedging', the pricing of Treasury Bond futures, bond options and interest caps are studied. The hedging of Treasury Bonds in field theory models of interest rates is discussed, and is shown to be a generalization of the more standard approaches. Exact results for both instantaneous and finite time hedging are derived, and a semi-empirical analysis of the results is carried out.

In Chapter 10 on 'Field theory Hamiltonian of the forward interest rates' the state space and Hamiltonian is derived for linear and nonlinear theories. The Hamiltonian formulation yields an exact solution of the martingale condition for the nonlinear forward rates, as well as for forward rates with stochastic volatility. A Hamiltonian derivation is given of the change of numeraire for nonlinear theories, of bond option price, and of the pricing kernel for the forward interest rates.

All chapters focus on the conceptual and theoretical aspects of the quantum formalism as applied to finance, with material of a more mathematical nature being placed in the Appendices that follow each chapter. In a few instances where the reader might benefit from greater detail the derivations are included in the main text, but in a smaller font size. The Appendix at the end of the book contains mathematical results that are auxiliary to the material covered in the book. The reason for including the Appendices is to present a complete and comprehensive treatment of all the topics discussed, and a reader who intends to carry out some computations would find this material useful. In principle, the Appendices and the derivations in smaller type can be skipped without any loss of continuity.