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交互作用流的超过程

SUPERPROCESSES ARISING FROM INTERACTING STOCHASTIC FLOWS

◎ Zhao Qiaoling

China Agricultural Science and Technology Press

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Preface

This paper is divided into 7 chapters.

The central theme of these lectures is the construction and study of a new class of superprocesses named as "superprocesses arising from interacting stochastic flows" (abbreviated to SAISF).

In Chapter 1 ~ Chapter 2, we will introduction some elementary theories about measure-valued processes.

In Chapter 3, we will construct a new class of superprocesses named as "superprocesses arising from interacting stochastic flows" (abbreviated to SAISF). These superprocesses are characterized by their generators as:

$$\mathscr{L}F_{m,f}(\mu) = F_{m,G^m f} + \mathscr{B}F_{m,f}(\mu); \quad \text{for any } m \in \mathbb{N}, f \in C_b^2((\mathbb{R}^d)^m),$$

where $F_{m,f}(\mu)$ denotes the integral $\int_{(\mathbb{R}^d)^m} f d\mu^m$ and

$$\mathscr{B}F(\mu) = \int_{\mathbb{R}^d} \beta(x) \frac{\delta F(\mu)}{\delta\mu(x)} \mu(dx) + \frac{1}{2} \int_{\mathbb{R}^d} \sigma(x) \frac{\delta^2 F(\mu)}{\delta\mu(x)^2} \mu(dx),$$

$$G^m f = \frac{1}{2} \sum_{i,j=1, i \neq j}^{m} \sum_{p,q=1}^{d} a^{p,q}(x_i, x_j) \frac{\partial^2 f}{\partial x_i^p \partial x_j^q} + \frac{1}{2} \sum_{i=1}^{m} \sum_{p,q=1}^{d} c^{p,q}(x_i) \frac{\partial^2 f}{\partial x_i^p \partial x_i^q} + \sum_{i=1}^{m} \sum_{p=1}^{d} b^p(x_i) \frac{\partial f}{\partial x_i^p}$$

This class of superprocesses is the unified setting of some new born classes of superprocesses considered by many authors in their papers. Here we use the duality method developed by Dawson, Li and Wang to prove their strong Markov property and the technique of branching particle system approximation to prove their existence. In the end of this chapter, we shall give some variance of this class of superprocesses.

In Chapter 4, we shall investigate its probabilistic properties. Firstly, we shall prove the atomic property of the SAISF if its parameters satisfies the condition that $a^{p,q}(x,x) = c^{p,q}(x)$ for any $x \in \mathbb{R}^d$ in Section 4.1. Secondly, we will deduce the stochastic partial differential equation associated with 1-dimensional SAISF in Section 4.2. Thirdly, we will consider some rescaled limit for the SAISF under some conditions.

In Chapter 5, we will use "piecing" technique to investigate the SAISF with branching mechanism depending on population size and general superprocesses with branching mechanism depending on population size. The limit duality method and "piecing" technique are main methods in this chapter.

In Chapter 6, the stochastic flow of mappings generated by a Feller convolution semigroup on a compact metric space is studied. This kind of flow is the generalization of superprocesses of stochastic flows and stochastic diffeomorphism induced by the strong solutions of stochastic differential equations.

In Chapter 7, we reconstruct the superprocesses of stochastic flows by martingale method, and prove that if and only if the infinitesimal particles never hit each other, then atomic part and diffuse part of this kind of superprocesses will be also superprocesses of stochastic flows.

Zhao Qiaoling

February, 2008

Table of Contents

Chapter 1

From particle systems to measure-valued processes

In this chapter, we shall introduce two classes of measure-valued processes using only rather elementary methods and in particular the method of moment measures. Our staring point is a class of exchangeable finite particle systems and we have studied the limits of their normalized empirical measures as the number of initial particles tends to infinity. The resulting limit process is the Fleming-Viot probability-measure-valued process. Before this, we introduced the sitting of measure-valued Feller processes. The last two sections of this chapter provide a preliminary introduction to weak convergence of processes, martingale problems and branching particle systems, all of which will be developed in some generality in subsequent chapters.

1.1 Measure-valued feller processes

Let (E, d) be a compact metric space, $C(E)$ the space of continuous functions, $\varepsilon = B(E)$ *the* σ - *algebra* of Borel subsets of E, and $M_1(E)$ the space of probability measures on E. We denote by $b\varepsilon$ (resp. $pb\varepsilon$) the bounded (resp. nonnegative bounded) ε - measurable functions on E. If $\mu \in M_1(E)$ and $f \in b\varepsilon$, we define $\langle \mu, f\rangle = \int_E f d\mu$. Note that $M_1(E)$ furnished with the topology of weak convergence is a compact metric space[where $\mu_n \xrightarrow{w} \mu$ if and only if $\langle \mu_n, f\rangle \to \langle \mu, f\rangle$, $\forall f \in bC(E)$].

Let $D = D([0, \infty), M_1(E))$ be furnished with the usual Skorohod topology and $X_t: D \to M_1(E)$, $X_t(\omega) \doteq \omega(t)$ for $\omega \in D$. Let $\mathscr{D}_t^\circ = \sigma\{X_s: 0 \leqslant s \leqslant t\}$, $\mathscr{D} = \bigvee \mathscr{D}_t^\circ = \mathscr{B}(D)$, $\mathscr{D}_t = \mathscr{D}_{t+}^\circ \doteq \bigcap_{\varepsilon>0} \mathscr{D}_{t+\varepsilon}^\circ$. For any $\mathscr{D}_t$ - stopping time τ, $\mathscr{D}_\tau \doteq \{A \in \mathscr{D}: A \cap \{\tau \leqslant t\} \in \mathscr{D}_t, \forall t\}$. Then $(D, (\mathscr{D}_t)_{t\geqslant 0}, \mathscr{D}, (X_t)_{t\geqslant 0})$ defines the canonical probability - measure-valued process.

Recall that D and $M_1(D)$ are both Polish spaces. If $P \in M_1(D)$, and $F \in b\mathscr{D}$, we let $P(F) \doteq \int_D F dP$ (we sometimes also use the notation $E(X)$ to denote the expectation of a random variable X).

For $t \geqslant 0$, define $\Pi_t: M_1(D) \to M_1(M_1(E))$ by $\Pi_t P \doteq P o X_t^{-1}$. Then for fixed $P \in M_1(D)$, the mapping $t \to \Pi_t P \in D([0, \infty), M_1(M_1(E)))$. (N. B. However the mapping $P \to \Pi. P$ is not continuous from $M_1(D)$ to $D([0, \infty), M_1(M_1(E)))$.)

By an $M_1(E)$ - valued stochastic process we mean a family of probability measures $\{P_\mu: \mu \in M_1(E)\}$ on $(D, \mathscr{D}, (\mathscr{D}_t)_{t\geqslant 0})$ such that

(i) $P_\mu(X(0) = \mu) = 1$, that is $\Pi_0 P_\mu = \delta_\mu$,

(ii) the mapping $\mu \to P_\mu$ from $M_1(E)$ to $M_1(D)$ is measurable.

It is said to be time homogeneous strong Markov if for every $\{\mathscr{D}_t\}$ – stopping time τ, $\mu \in M_1(E)$, with $P_\mu(\tau < \infty) = 1$.

(iii) $P_\mu[F(X(\tau + t)) | \mathscr{D}_\tau] = T_\tau F(X(t)), P_\mu - a.s.$

for all $F \in b\mathscr{B}(M_1(E))$, $t \geq 0$, where

$$T_t F(\mu) \doteq P_\mu F(X(t)) = \int_{M_1(E)} F(v) p(t, \mu, dv).$$

The transition function is defined by $p(t, \mu, \cdot) \doteq \Pi_t P_\mu(\cdot)$. Let $(C(M_1(E)), \| \cdot \|)$ denote the Banach space of continuous functions on $M(E)$ with $\| F \| \doteq \sup_\mu |F(\mu)|$. The process is a *Feller process* if in addition $T_t : C(M_1(E)) \to C(M_1(E))$, $\forall t > 0$ and $\| T_t F - F \| \to 0$ as $t \to 0$. Then $\{T_t : t \geq 0\}$ forms a strongly continuous semigroup of positive contraction operators on $(C(M_1(E))$, that is, $T_t F \geq 0$ if $F \geq 0$ and $\| T_t F \| \geq \| F \|$ for $F \in C(M_1(E))$. Give a Feller semigroup the strong infinitesimal generator is defined by

$$\mathscr{B} F \doteq \lim_{t \downarrow 0} \frac{T_t F - F}{t} \text{ (where the limit is taken in the norm topology).}$$

The domain $\mathscr{D}(\mathscr{B})$ of $\mathscr{B}$ is the subspace of $C(M_1(E))$ for which this limit exists. Since $\int_0^\infty e^{-\lambda t} T_t F\, dt \in \mathscr{D}(\mathscr{B})$ if $\lambda > 0$ and $F \in C(M_1(E))$, it follows that $\mathscr{D}(\mathscr{B})$ is dense in $C(M_1(E))$. A subspace $\mathscr{D}_0 \subset \mathscr{D}(\mathscr{B})$ is a core for $\mathscr{B}$ if the closure of the restriction of $\mathscr{B}$ to $\mathscr{D}_0$ is equal to $\mathscr{B}$.

Lemma 1.1.1 *Let $\mathscr{B}$ be the generator of a strongly continuous contraction semigroup T_t on $C(M_1(E))$. Let $\mathscr{D}_0$ be a dense subspace of $C(M_1(E))$ and $\mathscr{D}_0 \subset \mathscr{D}(\mathscr{B})$. If $T_t : \mathscr{D}_0 \to \mathscr{D}_0$, then it is a core for $\mathscr{B}$.*

(A similar statement is true for a semigroup S_t defined on $C(E)$, with generator A and domain $D(A)$.)

Proof [EK1, Ch. 1, Prop. 3.3]

In order to formulate an $M_1(E)$ – valued Feller process we must first introduce some appropriate subspace of C $(M_1(E)$ which can serve as a core for the generator.

The algebra of polynomials, $C_P(M_1(E))$, is defined to be the linear span of monomials of the form

$$\begin{aligned} F_{f,n}(\mu) &= \int f(x) \mu^n(dx) \\ &= \int_E \cdots \int_E f(x_1, \cdots, x_n) \mu(dx_1) \cdots \mu(dx_n) \end{aligned}$$

where $f \in C(E^n)$.

The function $F \in C(M_1(E))$ is said to be differentiable if the limit

$$F^{(1)}(\mu; x) \doteq \frac{\delta F(\mu)}{\delta \mu(x)} \doteq \lim_{\varepsilon \downarrow 0} (F(\mu + \varepsilon \delta_x) - F(\mu)) / \varepsilon = \frac{\partial}{\partial \varepsilon} F(\mu + \varepsilon \delta_x) |_{\varepsilon = 0}$$

exists for each $x \in E$ and belongs to $C(E)$ $\forall \mu \in M_1(E)$. The set of functions for which

$F^{(1)}(\mu;x)$ is jointly continuous in μ and x is denoted by $C^{(1)}(M_1(E))$.
The second derivative is defined by

$$F^{(2)}(\mu;\ x,\ y) \doteq \frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)} = \frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_2}F(\mu + \varepsilon_1\delta_x + \varepsilon_2\delta_y)\mid_{\varepsilon_1=\varepsilon_2=0}$$

if it exists for each x and y and belongs to $C(E\times E)\ \forall\mu\in M_1(E)$.
Let $C^{(k)}(M_1(E))$ denote the set of functions for which $F^{(k)}(\mu;x_1,\cdots,x_k)$ exists and is continuous on $M_1(E)\times E^k$

Lemma 1.1.2 (i) *$C_p(M_1(E))$ is dence in $C(M_1(E))$ and convergence determining in $M_1(M_1(E))$.*

(ii) *Function in $C_p(M_1(E))$ are infinitely differentiable, and the first and second derivatives are given by*

$$\frac{\delta F_{f,n}(\mu)}{\delta\mu(x)} = \sum_{j=1}^{n}\int_E\cdots\int_E f(x_1\cdots x_{j-1},\ x,\ x_{j+1}\cdots x_n)\Pi_{i\neq j}\mu(dx_i)$$

$$\frac{\delta^2 F_{f,n}(\mu)}{\delta\mu(x)\delta\mu(y)} = \sum_{j=1,j\neq k}^{n}\sum_{k=1}^{n}\int_E\cdots\int_E f(x_1,\cdots,\ x_{j-1},\ x,\ x_{j+1},\ \cdots,\ x_{k-1},\ y,\ x_{k+1},\ \cdots x_n)\Pi_{i\neq j,k}\mu(dx_i)$$

Proof (i) The linear span of the space in question is an algebra of function on the compact metric space $M_1(E)$. In order to verify that $C_p(M_1(E))$ separates points it suffices to note that $\mu\in M_1(E)$ is uniquely determined by $\{\langle\mu,\phi\rangle:\phi\in C(E)\}$. The first part of the result is then an immediate consequence of the Stone – Welerstrass theorem.

If $\int F(\mu)p_n(d\mu)\to\int F(\mu)p(d\mu)$ as $n\to\infty$ for all F belonging to a dense subset of $C(M_1(E))$, then it is true for all $F\in C(M_1(E))$. This proves that $C_p(M_1(E))$ is convergence determining in $M_1(M_1(E))$.

(ii) Follows by a simple calculation.

1.2 Independent particle systems: dynamical law of large numbers

Let S_t: $t\geq 0$ be a Feller semigroup on the Banach space $(C(E),\|\cdot\|)$ where $\|\cdot\|$ is the supremum norm, with E compact. Then the domain $D(A)$ of the infinitesimal generator A is a dense subspace of $C(E)$. We assume that there exists a separating algebra of function

$$D_0\subset D(A),\ S_t D_0\subset D_0.$$

Consequently D_0 is a core for A (cf. Lemma 1.1.1).

Let $P(t,\ x,\ dy)$ denote the transition function of $\{S_t\}$, that is,

$$S_t f(x) = \int_E f(y)P(t,\ x,\ dy),\ f\in C(E).$$

It will be convenient to work with a canonical version of the Feller process which will be described in the following result. Let $D_E = D([0,\infty),\ E)$ denote the space of *càdlàg* functions from $[0,\infty)$ into E. Then D_E is a Polish space if it is furnished with the Skorhod topology.

Proposition 1.2.1 *Let $\{S_t\}$ be a Feller semigroup on $C(E)$ with E compact. Then*

(*a*) *For each $x\in E$ there exists a probability measure P_x on $\mathscr{B}(D_E)$ satisfying*

$$P_x(\omega(0) = x) = 1, \text{ and for } s \leqslant t, \tag{1.2.1}$$

$$P_x(f(\omega(t)) \mid \sigma(\omega(u) : u \leqslant s)) = (S_{t-s}f)(\omega(s)), \ P_x - a.s.\ \forall f \in C(E). \tag{1.2.2}$$

(b) There exists a standard probability space $(\Omega, \mathscr{F}, Q_A)$ *and a measurable mapping* $\zeta : (E \times \Omega, \varepsilon \times \mathscr{F}) \to (D_E, \mathscr{B}(D_E))$*, such that for each* $x \in E$

$$Q_A(\{\omega : \zeta(x, \omega) \in B\}) = P_x(B), \forall B \in \mathscr{B}(D_E). \tag{1.2.3}$$

Furthermore, $\zeta(\cdot, \omega)$ *is continuous at* x *for* $Q_A -$ *a. e.* ω*, for each* $x \in E$.
The resulting measurable random function is denoted by $(\Omega, \mathscr{F}, Q_A, \{\zeta(x)\}_{x \in E})$. *(A standard probability space is one which is isomorphic to* $[0, 1]$ *with Lebesgue measure.)*

Proof (a) Given $x \in E$, the existence of P_x satisfying (1.2.2) is a standard result on the existence of a *càdlàg* version of the Feller process(e.g. [Ek1 Chapt. 4, Theorem 2.7]).
(b) It can also be shown that the mapping $x \to P_x$ from E to $M_1(D_E)$ is continuous when the latter is given the weak topology (see Seation 2.1). Since the map $x \to P_x$ is continuous, the existence of a representation $(\Omega, \mathscr{F}, Q_A, \{\zeta(x)\}_{x \in E})$, $\xi : \Omega \times E \to D_E$ such that
(i) for each $x \in E$, $\xi(\cdot, x)$ is measurable and has law P_x,
and
(ii) $\xi(\omega, \cdot)$ is continuous at x for $Q_A -$ *a. e.* ω, for each $x \in E$,
follows from the extension of Skorohod's almost sure representation theorem due to Blackwell and Dubins(1983). From this the existence of a jointly measurable version of ζ follows by a standard argument.

A system of a N independent particles $\{Z(t) : t \geqslant 0\} = \{Z_1(\cdot), \cdots, Z_N(\cdot)\}$ each undergoing an A – motion in E and with initial value $Z_i(0)$ having law $\mu \in M_1(E)$ is then realized on $((E \times \Omega)^N, (\mu \otimes Q_A)^N)$ by

$$Z_i(((e_1, \omega_1), \cdots, (e_N, \omega_N)), t) = \zeta(e_i, \omega_i)(t), \quad i = 1, 2, \cdots, N,$$
$$((e_1, \omega_1), \cdots, (e_N, \omega_N)) \in (E \times \Omega)^N.$$

Then $Z(t)$ is an E^N – valued Markov process with semigroup

$$S_t^N f(x_1, \cdots, x_N) = \int_E \cdots \int_E f(y_1, \cdots, y_N) P(t, x_1, dy_1) \cdots P(t, x_N, dy_N), \ f \in C(E^N).$$

The semigroup $\{S_t^N : t \geqslant 0\}$ is strongly continuous on the closure of D_0^N (= algebra generated by $\{f_1(x_1) \cdots f_N(x_N) : f_i \in D_0, i = 1, 2, \cdots, N\}$), which is $C(E^N)$, and hence S_t^N is a Feller semigroup associated with a process with valued in E^N. The corresponding generator is

$$A^{(N)} \doteq \sum_{i=1}^N A_i \text{ on } D(A^{(N)}) \subset C(E^N).$$

where A_i denotes the action of A on the ith variable. Furthermore it easily follows that $S_t^N : D_0^N \to D_0^N$ and therefore D_0^N is a core for $A^{(N)}$.

The associated empirical measure process is given by

$$X^N(t) = \Xi(Z_1(t), \cdots, Z_N(t)) \doteq N^{-1} \sum_{i=1}^N \delta_{Z_i(t)} \in M_1(E).$$

It will follow from Proposition 1.3.3 that $X^N(\cdot)$ is also a Feller process with state space

$M_{1,N}(E)$, the space of measure consisting of atoms whose masses are multiples of $1/N$ and contained in E. We will denote its generator by $\mathscr{B}_N^A$.

Let

$$\mathscr{D}_0(\mathscr{B}_N^A) \doteq \{F_{f,n}(\mu) = \langle \mu^n, f\rangle : f \in D_0^n, n \leq N\}.$$

For $F_{f,n} \in D_0(\mathscr{B}_N^A)$ and $\mu_N = N^{-1}\sum_{i=1}^N \delta_{Z_i}$,

$$F_{f,n}(\mu_N) = N^{-n}\sum_{i_1=1}^N \cdots \sum_{i_n=1}^N f(z_{i_1}, \cdots, z_{i_n})$$

$$= N^{-n}\sum_{k=1}^N \sum_{p\in P_k^n} \sum_{j_1=1}^N \cdots \sum_{j_k=1}^N f(z_{p1}, \cdots, z_{pn})$$

where for $1 \leq k < n$, P_k^n denote the set of partitions $p:\{1, \cdots, n\} \to \{1, \cdots, k\}$.

$$\mathscr{B}_N^A F_{f,n}(\mu_N) = \langle \mu_N^n, A^{(n)}f\rangle + N^{-n}\sum_{k=1}^N \sum_{p\in P_k^n} \sum_{j_{p1}=1}^N \cdots \sum_{j_{pk}=1}^N (A^{(k)}f^{(p)} - A^{(n)}f)(z_{jp1}, \cdots z_{jp_n})$$

$$= F_{A(n)f,n}(\mu_N) + R(N, n, f)(\mu_N)$$

and for $p \in P_k^n$, $f^{(p)}(z_{j_1}, \cdots, z_{j_n}) = f(z_{j_{p1}}, \cdots, z_{j_{pn}})$.

Since

$$R(N, n, f)(\mu_N) = N^{-n}\sum_{k=1}^N \sum_{p\in P_k^n} \sum_{j_{p1}=1}^N \cdots \sum_{j_{pk}=1}^N (A^{(k)}f^{(p)} - A^{(n)}f)(z_{j_{p1}}, \cdots, z_{j_{pn}}),$$

it follows that $|R(N, n, f)(\mu_N)| \leq c(n)\|f\|_{A,n}/N$ where for $f \in D_0^n$, $\|f\|_{A,n} \doteq \|f\| + \max_k \max_{p\in P_k^n} \|A^{(k)}f^{(p)}\|$. Note that $\|S_t^N f\|_{A,n} \leq \|f\|_{A,n}$ and therefore $S_t^N : D_0^n \to D_0^n$.

By the law of large numbers $X^N(0) \Rightarrow X(0) \doteq \mu$. Using the above expression for the generator we can then show that $X^N(t) \Rightarrow X(t)$, which is a deterministic $M_1(E)$ - valued process characterized as the unique solution of the weak equation

$$\langle X(t), f\rangle = \langle X(0), f\rangle + \int_0^t \langle X(s), Af\rangle ds, \ \forall f \in D(A),$$

that is, formally, $\frac{\partial X}{\partial t} = A^* X$ where A^* denotes the adjoint of A.

This implies that $<X(t), f> = <X(0), S_t f>$.

This is the simplest example of a dynamical law of large numbers and is a degenerate case of the Mckean - Vlasov limitof exchangeably interacting particle systems. For detailed developments on the Mckean Vlasov (or mean - field) limit of interacting particle systems and the related phenomenon of propagetion of chaos the reader is referred to *Gärtner*(1988), *Léonard*(1986), and *Sznitman*(1991).

1.3 Exchangeable particle systems

Let $\mathscr{P}(N)$ denote the set of permutations of $\{1, \cdots, N\}$. A continuous function $f: E^N \to R$ is said to be symmetric, $f \in C_{sym}(E^N)$, if $f = \tilde{\pi} f$, $\forall \pi \in \mathscr{P}(N)$, where $\tilde{\pi} f(z_1, \cdots, z_N) \doteq f(z_{\pi 1}, \cdots, z_{\pi N})$.

Given $z_1, \cdots, z_N \in E$ (not necessarily distinct) the associated empirical measure is defined by

$$\Xi_N(z_1, \cdots, z_N) \doteq N^{-1} \sum_{i=1}^{N} \delta_{Z_i} \in M_1(E).$$

The mapping $\Xi : E^N \to M_1(E)$ is clearly $\sigma(C_{sym}(E^N))$ - measurable. On the other hand, given a measure $\mu = \sum_{i=1}^{M} a_i \delta_{Z_i} + \nu \in M_1(E)$, with $z_1, \cdots, z_M$ distinct, and ν nonatomic, let $\sum(\mu) \doteq \{(z_1; a_1), \cdots, (z_M; a_M)\} \in (E \times \mathbb{R}_+)^M$, mod($\mathscr{P}(M)$). The mapping $\mu \to \sum(\mu)$ is measurable from $(M_1(E), \mathscr{B}(M_1(E)))$ to $\cup_{M=1}^{\infty}(E \times \mathbb{R}_+)^M$ where the latter is furnished with the smallest σ - algebra containing $\sigma(C_{sym}(E \times \mathbb{R}_+)^M)$ for each M (cf. Theorem2. 4. 1. 1 (d)). Consequently, if $\mu \in M_{1,N}(E)$, then the mapping $\mu \to ((z_1'; n_1), \cdots, (z_k'; n_k))$ where the $z_1', \cdots, z_k'$ are the distinct locations of the atoms and the n_k are their multiplicities is $(M(E), \mathscr{B}(M(E)))$-measurable. Then the unordered n - tuple $(z_1, \cdots, z_n)$ is given by listing the distinct $z_1', \cdots, z_k'$ with the appropriate multiplicities. Thus we obtain the following.

Lemma 1. 3. 1 *The sub - σ - algebras $\sigma(C_{sym}(E^N))$ and $\sigma(\Xi^N))$ of $\mathscr{B}(E^N)$ coincide. In particular, if $f \in C_{sym}(E^N)$, then $f(z_1, \cdots, z_N)$ is $\sigma(\Xi^N))$ - measurable.*

Proof If $\Sigma(\Xi^N(z_1, \cdots, z_N)) = (z_1'; n_1), \cdots, (z_k'; n_k)$, and $f \in \sigma(C_{sym}(E^N))$ then $f(z_1, \cdots, z_N) = f(z_1', \cdots, z_1', \cdots, z_k', \cdots, z_k')$ (with z_i' repeated n_i times for each $i = 1, \cdots, k$).

The E - valued random variables $Z_1, \cdots, Z_N$ are exchangeable if the joint distributions of $Z_1, \cdots, Z_N$ and $Z_{\pi 1}, \cdots, Z_{\pi N}$ are identical for any $\pi \in \mathscr{P}(N)$. The probability law P on $\mathscr{B}(E^N)$ of the exchangeable random variables $Z_1, \cdots, Z_N$ is uniquely determined by its restriction to the sub - σ - algebras $\sigma(C_{sym}(E^N))$. Let $M_{1,ex}(E^N)$ denote the family of exchangeable probability laws on E^N. Then $C_{sym}(E^N)$ is $M_{1,ex}(E^N)$ - determining, that is, if $\mu_1, \mu_2 \in M_{1,ex}(E^N)$ and $\int_{E^N} f(x)\mu_1(dx) = \int_{E^N} f(x)\mu_2(dx), \forall f \in C_{sym}(E^N)$, then $\mu_1 = \mu_2$. Moreover if $\mu \in M_{1,ex}(E^N)$, $g \in pC_{sym}(E^N)$, and $\langle \mu, g \rangle < \infty$, then $\mu_g(A) = \langle \mu, g1_A \rangle / \langle \mu, g \rangle \in M_{1,ex}(E^N)$.

Given a Polish space S let $D_S = D([o, \infty); S)$ denote the space of *càdlàg* functions from $[0, \infty)$ to S furnished with the Skorohod topology (cf. Ch. 2, Sect. 6). Given $\pi \in \mathscr{P}(N)$, let $\tilde{\pi}: E^N \to E^N$ be defined by $(\tilde{\pi}x)_i = x_{\pi i}$ for $x = (x_1, \cdots, x_N) \in E^N$ and $\tilde{\pi}: D_{E^N} \to D_{E^N}$ be defined by $(\tilde{\pi}x)_i(t) = x_{\pi i}(t)$.

An exchangeable system of N particles is defined by an exchangeable probability laws P on D_{E^N}, or equivalently,

(i) an exchangeable initial distribution $\pi_0 p$ on E^N, and

(ii) a family $\{P_y : y \in E^N\}$ of conditional distributions on D_{E^N} which satisfies $P_{\tilde{\pi}y} = P_y \circ \tilde{\pi}^{-1}$ or $P_{\tilde{\pi}y}(\tilde{\pi}A) = P_y(A)$ for every $y \in E^N$, $A \in \mathscr{D}$ and $\pi \in \mathscr{P}(N)$.

We next give a simple criterion which implies that an E^N - valued Markov process is exchangeable.

Lemma 1. 3. 2 *Let $Z = (Z_1, \cdots, Z_N)$ be an E^N - valued càdlàg Markov process with transition function $p(s, x; t, dy)$. Then Z is an exchangeable system provided that the marginal distri-*

butions $P(Z(t) \in \cdot)$, $t \in \mathbb{R}_+$ *are exchangeable and* $p(s, x; t, dy) = p(s, \tilde{\pi}y; t, \tilde{\pi}B)$ *for every* $\pi \in \mathscr{P}(N)$, $y \in E^N$, $B \in \mathscr{B}(E^N)$, *or equivalently*,

$$(S_t f(\tilde{\pi} \cdot))(\tilde{\pi}^{-1}x) = (S_t f)(x), f \in C(E^N). \tag{1.3.1}$$

Note that in the case of a time homogeneous Feller semigroup $\{S_t\}$ *and with generator A and core* $D_0(A)$ *the above criterion is implied by*

$$(Af(\tilde{\pi} \cdot))(\tilde{\pi}^{-1}y) = (Af(\cdot))(y), f \in D_0(A). \tag{1.3.2}$$

Proof Let $m \in \mathbb{Z}_+$, $t_1, \cdots, t_m \in \mathbb{R}_+$ and $\pi \in \mathscr{P}(N)$. Then for $B_i^j \in \mathscr{B}(E)$,

$$P_{y_0}(Z(t_i) \in \Pi_{j=1}^N B_i^{\pi^{-1}j}, i = 1, \cdots, m)$$

$$= \int_{\pi_j B_1^{\pi^{-1}j}} \cdots \int_{\pi_j B_m^{\pi^{-1}j}} p(0, y_0; t_1, dy_1) \Pi_{i=1}^{m-1} p(t_i, y_i; t_{i+1}, dy_{i+1})$$

$$= \int_{\pi_j B_1^{\pi^{-1}j}} \cdots \int_{\pi_j B_m^{\pi^{-1}j}} p(0, \tilde{\pi}y_0; t_1, \tilde{\pi}dy_1) \Pi_{i=1}^{m-1} p(t_i, \tilde{\pi}y_i; t_{i+1}, \tilde{\pi}dy_{i+1})$$

$$= \int_{\pi_j B_1^{\pi^{-1}j}} \cdots \int_{\pi_j B_m^{\pi^{-1}j}} p(0, \tilde{\pi}y_0; t_1, dy_1) \Pi_{i=1}^{m-1} p(t_i, y_i; t_{i+1}, dy_{i+1})$$

$$= P_{\tilde{\pi}y_0}(Z(t_i) \in \Pi_{j=1}^N B_i^j, i = 1, \cdots, m),$$

since by assumption $p(t_i, \tilde{\pi}y_i; t_{i+1}, \tilde{\pi}dy_{i+1}) = p(t_i, y_i; t_{i+1}, dy_{i+1})$.

Thus the finite-dimensional distributions of $P_{\tilde{\pi}y_0}$ and $P_{y_0} \circ \tilde{\pi}^{-1}$ coincide which yields the result.

Proposition 1.3.3 *Let* $Z = (Z_1, \cdots, Z_N)$ *be an* E^N*-valued càdlàg exchangeable Feller process. Then the empirical measure process* $X(t) \doteq \Xi_N(Z(t))$ *is a càdlàg* $M_1(E)$*-valued Feller Markov process.*

Proof For each $\phi \in C(E)$, $\int \phi(x) X(t, dx) = N^{-1} \sum_{i=1}^N \phi(Z_i(t))$ is *càdlàg* and hence $X(t) \in D([0, \infty); M_1(E))$, a. s.

Let $\mathscr{F}_t^Z = \sigma\{Z(s); 0 \leq s \leq t\}$. Then in order to prove the Markov process for $X(\cdot)$ it suffices to show that

$$P(X(t+s) \in \cdot \mid \sigma(X(t)) \vee \mathscr{F}_t^Z) = P(X(t+s) \in \cdot \mid \sigma(X(t))), \text{ a. s.}$$

It follows from the Markov property of Z and the inclusion $\mathscr{F}_t^Z \supset \sigma(X(t))$, that the left hang side equals $P(X(t+s) \in \cdot \mid \sigma(Z(t)))$ *a. s.* Hence it suffices to show that $P(\Xi_N(Z(t+s)) \in \cdot \mid \sigma(Z(t))) = P(\Xi_N(Z(t+s)) \in \cdot \mid \sigma(X(t)))$. Since $(Z(t), Z(t+s))$ forms N $(E \times E)$-valued exchangeable random variables by hypothesis, $P(\Xi_N(Z(t+s)) \in \cdot \mid \sigma(Z(t))) = P(\Xi_N(\tilde{\pi}Z(t+s)) \in \cdot \mid \sigma(\tilde{\pi}Z(t))) = P(\Xi_N(Z(t+s)) \in \cdot \mid \sigma(\tilde{\pi}Z(t)))$ *a. s.* Thus $P(\Xi_N(Z(t+s)) \in \cdot \mid \sigma(Z(t)))$ is a symmetric function of $Z(t)$ and by Lemma 1.3.1 we conclude that there is a $\sigma(\Xi_N(Z(t)))$-measurable version of $P(\Xi_N(Z(t+s)) \in \cdot \mid \sigma(Z(t)))$, and this yields the Markov property. Finally, note that the assumption that Z is *càdlàg* and Feller implies that Ξ_N is also *càdlàg*.

1.4 Random probability measures moment measures and exchangeable sequences

Let X be a random probability measure on E, E Polish. Then the *nth* moment measure is a probability measure defined on E^n as follows:

$$M_n(dx_1, \cdots, dx_n) = E(X(dx_1)\cdots X(dx_n)).$$

It is the probability law of n – exchangeable E – valued random variables, $\{Z_1, \cdots, Z_n\}$. Noting that this is a consistent family and using Kolmogorov's extension theorem we can associate with every random probability measure on E an exchangeable sequence of E – valued random variables, $\{Z_n : n \in \mathbb{N}\}$. The converse result is related to de Finetti's theorem.

Lemma 1.4.1 (a) *A random probability measure on E is uniquely determined by its moment measures of all orders.*

(b) *The sequence* $\{X_n\}$ *of random probability measure with moment measures* $\{M_{n,m}, n, m \in \mathbb{N}\}$ *converges weakly to a random probability measure X with moment measures* $\{M_m\}$ *if and only if* $M_{n,m} \Rightarrow M_m$ *as* $n \to \infty$ *for each* $m \in \mathbb{N}$.

Proof This follows from Lemma 1.1.2(a), Corollary 2.2.6 and Lemma 2.2.7.

1.5 Weak convergence and the martingale problem

In subsequent sections we will systematically develop the notions of weak convergence of measure-valued processes and measure-valued martingale problem. In this section we briefly introduce this approach by applying it to the Fieming-Viot process.

In particular we will show that in addition to weak convergence of finit dimensional distributions the laws of the measure-valued Maran processes X_N, their distributions $P_{\mu_N}^N$ are tight in the space of probability measures on $D([0, \infty); M_1(E))$ and consequently weak convergence of processes follows. This implies that the Fleming-Viot process can be realized as a *càdlàg* process.

We will now show that the Fleming-Viot process can also be characterized as the unique solution of the martingale problem for $(\mathscr{B}, \mathscr{D}_0(\mathscr{B}))$.

Since $\{X_N(t)\}$ is a Feller process with generator $\mathscr{B}_N$ and core $\mathscr{D}_0(\mathscr{B})$, it follows that

$$M_N(t) \doteq F(X_N(t)) - \int_0^t \mathscr{B}_N F(X_N(s))ds, \; F \in \mathscr{D}_0(\mathscr{B}),$$

is a bounded martingale under $P_{\mu_N}^N$.

Therefore for $t \in [0, T]$,

$$F_{f,n}(X_N(t)) - \int_0^t \mathscr{B}F_{f,n}(X_N(s))ds = M_N(t) + \int_0^t R(N, F, s)(X_N(s))ds \quad (1.5.1)$$

and $\sup_{0\le s\le T} |R(N, F, s)| \le c(F)/N$.

In order to prove the tightness of the $P_{\mu_N}^N$ on $D \doteq D([0,\infty); M_1(E))$ we will use Theorem 2.6.4 it suffices to show that for $\phi \in D_0(A)$, $\langle X_N(t), \phi\rangle$ are tight in $D([0,\infty); \mathbb{R})$. Applying (1.5.1) to $F_1(\mu) = \langle \mu, \phi\rangle$ and $F_2(\mu) = \langle \mu, \phi\rangle^2$, and then using Corollary 2.6.3 it follows that the laws of $< X_N(t), \phi >$ are tight in $D([0,\infty); \mathbb{R})$. Since using the convergence of the finite dimensional distributions, this yields the weak convergence of probability measures $P_{\mu_N}^N$ on D.

Now let $F \in D_0(\mathscr{B})$. In this case the real - valued functional $F(X(t)) - F(X(s)) - \int_s^t \mathscr{B} F(X(u))du$ is canonical process we get that

$$P_\mu[(F(X(t)) - F(X(s)) - \int_s^t \mathscr{B} F(X(u))du)H(X)]$$

$$= \lim_{N\to\infty} P_{\mu_N}^N[(F(X(t)) - F(X(s)) - \int_s^t \mathscr{B} F(X(u))du)H(X)]$$

$$= \lim_{N\to\infty} P_{\mu_N}^N[(F(X(t)) - F(X(s)) - \int_s^t \mathscr{B} F(X(u))du) + \int_s^t R(N, F, u)du)H(X)]$$

$$= \lim_{N\to\infty} P_{\mu_N}^N[(M_N(t) - M_N(s) + \int_s^t R(N, F, u)du)H(X)]$$

$$= \lim_{N\to\infty} P_{\mu_N}^N[(\int_s^t R(N, F, u)du)H(X)] = 0$$

This implies that

$$M_F(t) = F(X(t)) - F(X(s)) - \int_s^t \mathscr{B} F(X(u))du$$

is also a martingale for each $F \in \mathscr{D}_0(\mathscr{B})$ under P_μ. Therefore $\{P_\mu : \mu \in M_1(E)\}$ is a solution to the martingale problem for $(\mathscr{B}, \mathscr{D}_0(\mathscr{B}))$. In fact the family $\{P_\mu : \mu \in M_1(E)\}$ is uniquely characterized in this way since any solution to the martingale problem must have the same moment measures as the Fieming - Viot process. This can be verified by applying above. The details of this argument will be given in greater generality below.

1.6 Branching particle systems

Let us for the moment continue in the same spirit and consider a simple branching particle system on a compact metric space E. The main difference from the Moran model is that the total number of particles is no longer constant in time. For this reason the basic state space is now $M(E)$ the space of finite Borel measures on E. We will again follow the elementary approach based on moment measures to characterize the transition function for the limiting measure-valued process.

We consider a system of particles in the space E which move, die and produce offspring. We begin by assuming that during its lifetime each particle performs an A - moment independently of the other particle.

In the case of critical branching when particles die they produce k particles with probability p_k, $k = 0, 1, 2, \cdots$, $\sum_k kp_k = 1$. We will assume in this section that

$$m_2 = \sum k^2 p_k, and \sum k^3 p_k < \infty.$$

After branching the resulting set of particles evolve in the same way and independently of each other starting off from the parent particle's branching site. Let $N(t)$ denote the total number of particles at time t. We denote their locations by $\{x_i(t) : 1 \leq i \leq N(t)\}$.

In order to obtain a measure-valued process by use of an appropriate scaling we assume that particles have mass ε and branch at rate c/ε.

For $B \in \varepsilon$,define

$$X_\varepsilon(t, B) = \varepsilon\left(\sum_{i=1}^{N(t)} 1_B(x_i(t))\right).$$

Let C_F denote the class of function on $M(E)$ of the form $F_f(\mu) = f(\langle\mu, \phi\rangle)$ with $f \in C_b(\mathbb{R})$, $\phi \in C(E)$. Let $\mathscr{D}(\mathscr{B}) \doteq \{F_f(\mu) : F_f(\mu) = f(\langle\mu, \phi\rangle); f \in C_b^\infty(\mathbb{R}), \phi \in D_0(A)\}$ where $D_0(A)$ is in Sect. 1.2. Then $X_\varepsilon(\cdot)$ is an $M(E)$ – valued Feller process. The generator of $X_\varepsilon(\cdot)$ is defined on $\mathscr{D}(\mathscr{B})$, by

$$\mathscr{B}_\varepsilon F(\mu) \doteq G^A F_f(\mu) + c\varepsilon^2 \int \{\sum_k p_k[f(\langle\mu, \phi\rangle + \varepsilon(k-1)\phi(x)) - f(\langle\mu, \phi\rangle)]\}\mu(dx)$$

where G^A denote the generator of the empirical process associated to particles performing independent A – motions in E.

Then for $F \in \mathscr{D}(\mathscr{B})$, $\mathscr{B}_\varepsilon F(\mu) = \mathscr{B} F(\mu) + O(\varepsilon)$

where $\mathscr{B}F(\mu) = f'(\langle\mu, \phi\rangle)\langle\mu, A\phi\rangle + \frac{1}{2}c(m_2 - 1)f''(\langle\mu, \phi\rangle)\langle\mu, \phi^2\rangle$.

Letting $\varepsilon \to 0$ we obtain a measure-valued process with generator defined on $\mathscr{D}(\mathscr{B})$ by

$$\begin{aligned}\mathscr{B}\varepsilon F(\mu) &\doteq f'(\langle\mu, \phi\rangle)\langle\mu, A\phi\rangle + \frac{1}{2}c(m_2 - 1)f''(\langle\mu, \phi\rangle)\langle\mu, \phi^2\rangle \\ &= \int A(\delta F(\mu)/\delta\mu(x))\mu(dx) + \frac{1}{2}c(m_2 - 1)\iint(\delta^2 F(\mu)/\delta\mu(x)\delta\mu(y))\delta_x \\ &\quad (dy)\mu(dx)).\end{aligned}$$