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TOPICS ON DYNAMIC EPISTEMIC LOGIC

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Preface

Dynamic Epistemic Logic is the logic of knowledge, actions and the interrelation of them. This is not about one logic, but about a whole family of logics that allows us to specify static and dynamic aspects of rational agent systems. The book provides various logics to support such formal specifications.

Knowledge here is understood in a broad sense, that is, we see also doxastic logic as a sort of epistemic logic.

Knowledge is defined as a set of propositions that an agent knows in the classical dynamic epistemic logic. In other words, cognitive objects of the agent are propositions for such logic. But, in my opinion, cognitive objects of an agent can also be actions, agents, individuals and so on.

Hence, in this book, we will study such cognitive objects by logical methods.

In the book, except classical logical methods, we mostly use four methods: Semi-infinitary Method, Bounded-valuation Method, Self-substitution Method and Fixed-point Method. By Semi-infinitary Method we mean a method presented by de Lavalette, Kooi and Verbrugge [2004], where they used such a method to prove strong completeness of $\mathbf{PDL}_{\omega}$. But it seems to me that Bounded-valuation Method, Self-substitution Method and Fixed-point Method are new ones.

Contents

Chapter 1 Foundations

In this chapter, we will introduce epistemic logics, dynamic logic and dynamic epistemic logic we need later.

In Section 1, we will introduce multi-agent epistemic logics, and see simple-agent epistemic logics as the limited cases of the former.

In Section 2, we will introduce dynamic logic **PDL** which is a standard logic characterizing actions.

In Section 3, we will introduce semi-infinitary dynamic logic $\mathbf{PDL}_\omega$ and one generalization of it.

In Section 4, we will introduce dynamic epistemic logic.

The logics above are foundations of logics we shall present.

§ 1 Epistemic Logic

In this section we will introduce multi-agent epistemic logics, and see imple-agent epistemic logics as the limited cases of the former.

In this book we always use PV := $\{p_1,\ldots, p_n,\ldots\}$ as a countable set of propositional variables, and Agent as a finite set of (names for) agents.

Definition and Convention 1.1.1 A *multi-agent epistemic language* EL_M is a set of formulas φ, given by the following *formation rules*:

$\varphi := p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid \mathrm{K}_A\varphi$, where $p \in$ PV and $A \in$ Agent.

When Agent = $\{A\}$, we use Kφ as $\mathrm{K}_A\varphi$, *EL* as EL_M, and call *EL* a *simple-agent epistemic language*.

For every agent A, $\mathrm{K}_A\varphi$ is interpreted as "agent A knows (that) φ".

Formulas $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, $(\varphi \leftrightarrow \psi)$, T and ⊥ are defined as follows:

$(\varphi \vee \psi) := \neg(\neg\varphi \wedge \neg\psi)$,

$(\varphi \rightarrow \psi) := \neg(\varphi \wedge \neg\psi)$,

$(\varphi \leftrightarrow \psi) := ((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$,

$\mathrm{T} := (p_1 \vee \neg p_1)$,

$\bot := \neg\mathrm{T}$,

$\mathrm{k}_A\varphi := \neg\mathrm{K}_A\neg\varphi$.

For convenience, we usually abbreviate "if...then...", "if and only if", "not", "for all", "there are/is" to ⇒, ⇔, ~, ∀ and ∃, respectively.

If not especially mentioned henceforth, in this book, we always use *metavariables* p, q, ... (with or without subscripts) as formulas in PV, *metavariables* φ, ψ, θ, ... (with or without subscripts) as formulas of the languages defined above or below, and Φ, Ψ, ... (with or without the subscript) as formula sets, namely, subsets of such languages.

φ is called a *compound formula* ⇔ $\varphi \in EL_M -$ PV.

As usual, we will omit the outside parentheses of a formula and the inside parentheses subject to the convention that each of the following symbols is more binding than the one in its right:

$\neg$, K_A, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$.

In this book we always use $\varphi_1 \to \ldots \to \varphi_n$ as $\varphi_1 \to (\ldots \to \varphi_n)\ldots)$, and Form_0 as the following set:

$\{\varphi \in EL_M \mid \varphi$ is a formula generated by propositional variables, $\neg$ and $\wedge$ such that φ is not an instantiation of some tautology or its negation$\}$. ⊣

Remark. A formula in Form_0 can be called intuitively a *factual proposition*.

Definition and Abbreviation 1.1.2

(1) Let **S** be an axiomatic system[①] we shall present in this book. φ is a *theorem* of **S**, denoted by $\vdash_{\mathbf{S}} \varphi$, if φ has a formal proof in **S**; that is, there are formulas $\varphi_1, \ldots, \varphi_n$ such that for every $1 \le i \le n$, φ_i is an axiom of **S** or φ_i is obtained from some formula(s) in front of it by a rule of **S**.

We use Th(**S**) as the set of all theorems of **S**, and $\nvdash_{\mathbf{S}} \varphi$ as $\varphi \notin$ Th(**S**). In the following we shall omit the subscript **S** if not confusing.

(2) Let **S** be a system. We use **S** + A/R as the system obtained by adding Axiom A or Rule R to **S**, and **S** – A/R as the system obtained by deleting Axiom A or Rule R from **S**.

We use $\mathbf{S}_1 + \mathbf{S}_2$ as the system formed by all the axioms and rules of $\mathbf{S}_1$ and $\mathbf{S}_2$.

Let $\varphi := \psi$ be an abbreviation. We write $\mathbf{S} + \varphi := \psi$ as $\mathbf{S} + \varphi \leftrightarrow \psi$.

A system is called an *abbreviative system* $\Leftrightarrow$ it is of the form $\mathbf{S} + \varphi := \psi$, otherwise it is called a *non-abbreviative system*.

Such a method for generating systems by abbreviations are called *Abbreviation Method*.

(3) A *multi-agent epistemic system* **EK** is defined as follows:

(Taut) all instantiations of tautologies,

(MP) $\varphi, \varphi \to \psi \,/\, \psi$ (*modus ponens*),

($\mathrm{K}_{\mathrm{K}A}$) $\mathrm{K}_A(\varphi \to \psi) \to \mathrm{K}_A\varphi \to \mathrm{K}_A\psi$,

($\mathrm{RN}_{\mathrm{K}A}$) $\varphi \,/\, \mathrm{K}_A\varphi$.

(4) A *multi-agent epistemic system* **ET** := **EK** +

($\mathrm{T}_{\mathrm{K}A}$) $\mathrm{K}_A\varphi \to \varphi$.

(5) A *multi-agent epistemic system* **ES5** := **ET** +

($5_{\mathrm{K}A}$) $\neg\mathrm{K}_A\varphi \to \mathrm{K}_A\neg\mathrm{K}_A\varphi$ (*Negative Introspection Axiom*).

(6) Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be two systems. $\mathbf{S}_1$ and $\mathbf{S}_2$ are *equivalent*, denoted by $\mathbf{S}_1 = \mathbf{S}_2$, if Th($\mathbf{S}_1$) = Th($\mathbf{S}_2$). ⊣

Lemma 1.1.3

(I) The following are derived rules and theorems of **EK**:

[①] In the following an axiomatic system is called simply a system.

(1) $\varphi \to \psi$ / $K_A\varphi \to K_A\psi$,

$\varphi \to \psi$ / $k_A\varphi \to k_A\psi$;

(2) $K_A\varphi \leftrightarrow \neg k_A\neg\varphi$;

(3) $\varphi_1 \wedge \ldots \wedge \varphi_n \to \varphi$ / $K_A\varphi_1 \wedge \ldots \wedge K_A\varphi_n \to K_A\varphi$ (denoted by RK_K);

(4) $k_A(\varphi_1 \wedge \ldots \wedge \varphi_n) \to k_A\varphi_1 \wedge \ldots \wedge k_A\varphi_n$.

(II) The following is a theorem of **ES5**:

($\mathbf{4}_K$) $K_A\varphi \to K_AK_A\varphi$ (*Positive Introspection Axiom*). ⊣

In this book we always use $f: X \to Y$ as "f is a mapping (function) from X into Y".

Definition and Abbreviation 1.1.4

(1) A *multi-agent epistemic frame* for EL_M is a tuple (W, R_K) such that W is a set of states such that $W \neq \varnothing$, and R_K is a mapping such that $R_K(A)$, abbreviated to $R_{K\,A}$ below, is a binary relation on W for each $A \in$ Agent.

(2) A *multi-agent epistemic model* for EL_M is a tuple $M = (W, R_K, V)$ such that

$F = (W, R_K)$

is an epistemic frame and V: PV $\to \wp(W)$, where $\wp(W)$ is the *power set* of W.

Here F is called the *frame underlying* M or the *underlying frame* of M or the *frame* of M, M is called a *model based on* F or a *model on* F, and V is called a *valuation* on F.

(3) In this section we always use Frame as the class of all epistemic frames and Model as the class of all epistemic models.

(4) We always use $wR_{K\,A}u$ as $(w, u) \in R_{K\,A}$ and $\sim wR_{K\,A}u$ as $(w, u) \notin R_{K\,A}$.

(5) For each $w \in W$, $R_{K\,A}(w) := \{u \in W \mid wR_{K\,A}u\}$. ⊣

Remark. If not especially mentioned henceforth, in this book we always use *metavariables* $w, u, v, \ldots$ (with or without subscripts) as elements of W, and *metavariables* $U, V, \ldots$ (with or without subscripts) as subsets of W.

Definition 1.1.5 (Truth Definition) Let $M = (W, R_K, V) \in$ Model. For every compound formula φ, the *truth set* $V(\varphi)$ of φ w.r.t. M is defined inductively as follows: for all $w \in W$,

(1) $w \in V(\neg\varphi) \Leftrightarrow w \notin V(\varphi)$.

(2) $w \in V(\varphi \wedge \psi) \Leftrightarrow w \in V(\varphi)$ and $w \in V(\psi)$.

(3) $w \in V(K_A\varphi) \Leftrightarrow R_{K\,A}(w) \subseteq V(\varphi)$. ⊣

Remark. As I see it, together a model definition and a truth definition of compound formulas of a language can be called a *semantics*, because it can give a meaning (a proposition) to every formula of the current language w.r.t. every model.

As usual, it is easy to prove the following lemma.

Lemma 1.1.6 Let $(W, R_K, V) \in$ Model. Then

(1) $V(\neg\varphi) = W - V(\varphi)$.
$V(\varphi \wedge \psi) = V(\varphi) \cap V(\psi)$.
$V(\varphi \vee \psi) = V(\varphi) \cup V(\psi)$.
$V(\bot) = \varnothing$, $V(\top) = W$.

(2) $V(\varphi) \cap V(\varphi \to \psi) \subseteq V(\psi)$.

(3) $V(\varphi \to \psi) = W \Leftrightarrow V(\varphi) \subseteq V(\psi)$.

(4) $V(\varphi \leftrightarrow \psi) = W \Leftrightarrow V(\varphi) = V(\psi)$. ⊣

Definition 1.1.7 (Validity Definition) Let $F = (W, R_K) \in$ Frame and $M = (W, R_K, V) \in$ Model.

(1) φ is *valid* in M, denoted by $M \vDash \varphi$, $\Leftrightarrow V(\varphi) = W$;
otherwise, φ is *invalid* in M, denoted by $M \nvDash \varphi$.

(2) φ is *valid* in F, denoted by $F \vDash \varphi$, $\Leftrightarrow V(\varphi) = W$ for each valuation V on F;
otherwise, φ is *invalid* in F, denoted by $F \nvDash \varphi$.

(3) Rule $\varphi_1, \ldots, \varphi_n / \psi$ *preserves validity* w.r.t. $M \Leftrightarrow$
$V(\varphi_1) = \ldots = V(\varphi_n) = W \Rightarrow V(\psi) = W$.

(4) φ is *satisfiable* $\Leftrightarrow$ there is a model $(W, R_K, V) \in$ Model and $w \in W$ such that $w \in V(\varphi)$.

⊣

Let **S** be a system which has been defined or will be defined.

Definition 1.1.8 (Soundness Definition and Completeness Definition) Let $\boldsymbol{C}$ be a model class or a frame class.

(1) **S** is *sound* w.r.t. $\boldsymbol{C} \Leftrightarrow$ every theorem of **S** is valid in all elements in $\boldsymbol{C}$.

(2) **S** is *complete* w.r.t. $\boldsymbol{C} \Leftrightarrow$ every formula valid in all elements in $\boldsymbol{C}$ is a theorem of **S**.

⊣

Definition 1.1.9 Let $F = (W, R_K) \in$ Frame. The following conditions are *frame conditions* on F:

(ref) $\forall w \in W(wR_{K\,A}w)$ (*Reflexivity*).

(euc) $\forall wuv \in W(wR_{K\,A}u$ and $wR_{K\,A}v \Rightarrow uR_{K\,A}v)$ (*Euclideanness*).

Let Frame(ref) be the class of all epistemic frames satisfying (ref) and Frame(ref, euc) the class of all epistemic frames satisfying (ref) and (euc), respectively. ⊣

As usual, we get easily:

Theorem 1.1.10 (Frame Soundness Theorem)

(1) **EK** is sound w.r.t. Frame.

(2) **ET** is sound w.r.t. Frame(ref).

(3) **ES5** is sound w.r.t. Frame(ref, euc). ⊣

In the following we will prove the frame completeness of the systems above. We first give the following necessary definitions and lemmas to do it.

Let w be a finite formula set. We use $\wedge w$ as a conjunction of all elements of w in some fixed order.

Let $\wedge w := \top$ if $w = \varnothing$, and $\wedge w := \varphi$ if $w = \{\varphi\}$.

Definition 1.1.11 Let w be a formula set.

(1) w is **S**-*consistent* $\Leftrightarrow$ for all finite subset $u \subseteq w$, $\nvdash_{\mathbf{S}} \neg\wedge u$.

(2) w is *maximal* $\Leftrightarrow$ for every $\varphi \in EL_M$, $\varphi \in w$ or $\neg\varphi \in w$.

(3) w is *maximal* **S**-*consistent* $\Leftrightarrow$ w is maximal and **S**-consistent.

(4) φ is **S**-*consistent* $\Leftrightarrow$ $\{\varphi\}$ is **S**-consistent.

(5) $w \vdash^{\mathbf{S}} \varphi \Leftrightarrow$ there is a finite subset $u \subseteq w$ such that $\vdash_{\mathbf{S}} \wedge u \to \varphi$. ⊣

Remark. In the following we will omit the parameter and subscript **S** if not confusing, and abbreviate "maximal **S**-consistent set" to **S**-MCS and "maximal consistent set" to MCS. ⊣

As usual, it is easy to prove the following two lemmas.

Lemma 1.1.12 Let w be an **S**-MCS. Then

(1) $\neg\varphi \in w \Leftrightarrow \varphi \notin w$.

$\varphi \wedge \psi \in w \Leftrightarrow \varphi \in w$ and $\psi \in w$.

$\varphi \vee \psi \in w \Leftrightarrow \varphi \in w$ or $\psi \in w$.

$\varphi \in w$ and $\vdash_{\mathbf{S}} \varphi \to \psi \Rightarrow \psi \in w$.

$\varphi \in w$ and $\varphi \to \psi \in w \Rightarrow \psi \in w$.

(2) Th(**S**) $\subseteq w$.

(3) $w \vdash^{\mathbf{S}} \varphi \Leftrightarrow \varphi \in w$. ⊣

Lemma 1.1.13 Let W be the set of all **S**-MCSs, and let $|\varphi| := \{w \in W \mid \varphi \in w\}$. Then

(1) $|\neg\varphi| = W - |\varphi|$.

$|\varphi \wedge \psi| = |\varphi| \cap |\psi|$.

$|\varphi \vee \psi| = |\varphi| \cup |\psi|$.

$|\bot| = \varnothing$, $\quad |\top| = W$.

(2) $|\varphi| \cap |\varphi \to \psi| \subseteq |\psi|$.

(3) $|\varphi \to \psi| = W \Leftrightarrow |\varphi| \subseteq |\psi| \Leftrightarrow\ \vdash_{\mathbf{S}} \varphi \to \psi$.

(4) $|\varphi \leftrightarrow \psi| = W \Leftrightarrow |\varphi| = |\psi| \Leftrightarrow \vdash_S \varphi \leftrightarrow \psi$.

(5)(**Lindenbaum Lemma**) Let w be a **S**-consistent set. Then there is some $u \in W$ such that $w \subseteq u$.

(6) If $\nvdash_S \varphi$, then there is a $u \in W$ such that $\varphi \notin u$.

(7)(**Anti-chain Property**) Let $w, u \in W$. Then

$w \subseteq u \Leftrightarrow w = u$.

(8) $\vdash_S \neg\varphi \Leftrightarrow \varphi$ is not **S**-consistent.

(9) Let Φ be a set of formulas. Then

$\Phi \vdash^S \neg\varphi \Leftrightarrow \Phi \cup \{\varphi\}$ is not **S**-consistent.

(10) Let $\Phi \subseteq \mathrm{Th}(\mathbf{S})$.

$\nvdash_S \neg\varphi \Rightarrow \Phi \cup \{\varphi\}$ is **S**-consistent. ⊣

Given any formula set w, let

$K_A^- w := \{\varphi \mid K_A\varphi \in w\}$ and

$k_A^+ w := \{k_A\varphi \mid \varphi \in w\}$.

Definition 1.1.14 (Canonical model for S) The *canonical model* for **S** is a tuple (W, R_K, V) such that:

$W = \{w \mid w$ is maximal **S**-consistent$\}$,

$R_{K\,A} = \{(w, u) \in W^2 \mid K_A^- w \subseteq u\}$ for all $A \in$ Agent,

$V(p) = |p|$ for all $p \in$ PV.

We call (W, R_K) the *canonical frame* for **S**. ⊣

Lemma 1.1.15 (Main lemma for the canonical frame) Let (W, R_K) be the canonical frame for **S**. Then for all $w \in W$,

(1) $K_A\varphi \in w \Leftrightarrow \forall u \in W\,(wR_{K\,A}u \Rightarrow \varphi \in u)$.

(2) $k_A^+ w \subseteq u \Leftrightarrow K_A^- u \subseteq w$ for all $u \in W$.

(3) $k_A\varphi \in w \Leftrightarrow \exists u \in W\,(wR_{K\,A}u$ and $\varphi \in u)$.

Proof. (1) "⇒": Straightforward.

"⇐": Let $K_A\varphi \notin w$. It suffices to show that

(#) there is some $u \in W$ such that $K_A^- w \subseteq u$ and $\varphi \notin u$.

Hypothesize that $K_A^- w \cup \{\neg\varphi\}$ were not consistent, then there are $\varphi_1, \ldots, \varphi_n \in K_A^- w$ such that

$\vdash \varphi_1 \wedge \ldots \wedge \varphi_n \rightarrow \varphi$.

So, by RK_K in Lemma 1.1.3,

$\vdash K_A\varphi_1 \wedge \ldots \wedge K_A\varphi_n \rightarrow K_A\varphi$.

Since $\varphi_1, \ldots, \varphi_n \in K_A^- w$, it follows that $K_A\varphi_1, \ldots, K_A\varphi_n \in w$, so $K_A\varphi \in w$ by Lemma

1.1.12(1), contradicting that $K_A\varphi \notin w$. So $K_A^- w \cup \{\neg\varphi\}$ is consistent, and thus (#) holds by Lindenbaum Lemma.

(2) Straightforward.

By (1), we get (3) easily. ⊣

Lemma 1.1.16 (Truth Lemma) Let (W, R_K, V) be the canonical model for **S**. For all $w \in W$ and $\varphi \in EL_M$, we have

$$\varphi \in w \Leftrightarrow w \in V(\varphi).$$

Proof. Induction over φ. The propositional variables and Boolean connectives $\neg$ and $\wedge$ cases are standard.

Let $\varphi = K_A\psi$. We have

$$\begin{aligned} K_A\psi \in w &\Leftrightarrow \forall u \in W\,(wR_{K\,A}u \Rightarrow \psi \in u) && \text{by the previous lemma} \\ &\Leftrightarrow \forall u \in W\,(wR_{K\,A}u \Rightarrow u \in V(\psi)) && \text{by the induction hypothesis} \\ &\Leftrightarrow w \in V(K_A\psi) && \text{by Truth Definition 1.1.5(3).} \end{aligned}$$ ⊣

As usual, we can prove:

Lemma 1.1.17 Let (W, R_K) be the canonical frame for **S**.

(1) If **S** = **EK**, then $(W, R_K) \in$ Frame.

(2) If **S** = **ET**, then $(W, R_K) \in$ Frame(ref).

(3) If **S** = **ES5**, then $(W, R_K) \in$ Frame(ref, euc). ⊣

Theorem 1.1.18 (Frame Completeness Theorem)

(1) **EK** is complete w.r.t. Frame.

(2) **ET** is complete w.r.t. Frame(ref).

(3) **ES5** is complete w.r.t. Frame(ref, euc).

Proof. Take (3) for example. It suffices to show that

$$\nvdash_{\mathbf{ES5}} \varphi \Rightarrow \text{there is some } F \in \text{Frame(ref, euc) such that } F \nvDash \varphi.$$

Let (W, R_K, V) be the canonical model for **ES5**. Assume that $\nvdash_{\mathbf{ES5}} \varphi$. By Lemma 1.1.13(6), there is a $w \in W$ such that $\varphi \notin w$, so $w \notin V(\varphi)$ by Truth Lemma 1.1.16, so $(W, R_K, V) \nvDash \varphi$, and hence $(W, R_K) \nvDash \varphi$.

By the previous lemma (3), $(W, R_K) \in$ Frame(ref, euc). ⊣

Definition 1.1.19 Let **S** and $\mathbf{S}_1$ be two systems. **S** is a *subsystem* of $\mathbf{S}_1$ or $\mathbf{S}_1$ is an *extension system* of **S**, denoted by $\mathbf{S} \subseteq \mathbf{S}_1$, $\Leftrightarrow$ Th(**S**) $\subseteq$ Th($\mathbf{S}_1$). ⊣

The system consisting of Taut and MP is called the *classical propositional calculus*, denoted by **PC**. In the following we will present a lot of axiomatic systems **S** such that **S** is an extension system of **PC**. Since most of the above definitions and lemmas hold for such systems **S** and the corresponding semantic concepts, in the following we mention them simply at most.

§ 2 Dynamic Logic PDL

Dynamic Logic is a formal system for reasoning about actions. This branch of logic was started by Pratt [1976]. The propositional part of dynamic logic (**PDL**) became an object of study in itself. **PDL** was introduced by Fischer and Ladner [1979]. The standard reference is Harel, Kozen and Tiuryn [2000].

We first give the definition of language expressing **PDL**.

Definition 1.2.1 In this book we always let AA be a countable set of atomic actions.

(1) A *dynamic language* DL^- consists of a set Action of actions α and a set Form of formulas φ, given by the following *formation rules*:

$\alpha := a \mid (\alpha \cup \beta) \mid (\alpha \,;\, \beta) \mid \alpha^*$, where $a \in$ AA, and

$\varphi := p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid [\alpha]\varphi$, where $p \in$ PV.

(2) A *dynamic language DL* consists of a set Action of actions α and a set Form of formulas φ, given by the following *simultaneous induction* (*mutual induction*):

$\alpha := a \mid (\alpha \cup \beta) \mid (\alpha \,;\, \beta) \mid \alpha^* \mid \varphi?$, where $a \in$ AA;

$\varphi := p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid [\alpha]\varphi$, where $p \in$ PV.

(3) Let N be the *set of all natural numbers* and $n \in$ N. $[\alpha^n]\varphi$ is defined as follows:

$[\alpha^0]\varphi = \varphi, \ldots, [\alpha^{n+1}]\varphi = [\alpha][\alpha^n]\varphi$.

(4) $\langle\alpha\rangle\varphi := \neg[\alpha]\neg\varphi$. ⊣

Remark. If not especially mentioned henceforth, in this book, we always use *metavariables a, b, …* (with or without subscripts) as actions in AA, and *metavariables* α, β, γ, … (with or without subscripts) as actions in Action.

The intended interpretation of $[\alpha]\varphi$ is that it is necessary that after (some agent) doing α, φ is true.

Actions of the form $\alpha \cup \beta$, $\alpha \,;\, \beta$, α^* and $\varphi?$ are called *compound actions*. They have the following intuitive meanings:

$\alpha \cup \beta$ "Do α or β nondeterministically."

$\alpha ; \beta$ "Do α, then do β."

α^* "Do α any finite number of times, 0 or more, nondeterministically."

$\varphi?$ "Test φ; proceed if true, fail if false."

Here $\cup$, ;, * and ? are called *nondeterministic choice* operator, *sequential composition* operator, *iteration* operator and *test* operator, respectively.

Definition 1.2.2

(1) (Blackburn, de Rijke and Venema [2001] (p. 203)) *Regular Dynamic System* $\mathbf{PDL}^-$ (expressed by DL^-) is defined as follows: for all $\alpha, \beta \in$ Action and $\varphi, \psi \in$ Form,

(Taut)	all instantiations of tautologies,	
(K_α)	$[\alpha](\varphi \to \psi) \to [\alpha]\varphi \to [\alpha]\psi$,	
($Ax_\cup$)	$[\alpha \cup \beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi$	(Choice Axiom),
($Ax_;$)	$[\alpha ; \beta]\varphi \leftrightarrow [\alpha][\beta]\varphi$	(Composition Axiom),
(Ax_*)	$[\alpha^*]\varphi \leftrightarrow \varphi \wedge [\alpha][\alpha^*]\varphi$	(Iteration Axiom),
(IAx_*)	$\varphi \wedge [\alpha^*](\varphi \to [\alpha]\varphi) \to [\alpha^*]\varphi$	(Induction Axiom),
(MP)	$\varphi, \varphi \to \psi / \psi$	(*modus ponens*),
(RN_α)	$\varphi / [\alpha]\varphi$.	

(2) (Harel, Kozen and Tiuryn [2000] (p. 22 and p. 240)) *Regular Dynamic System* **PDL** (expressed by DL) := $\mathbf{PDL}^-$ +

($Ax_?$) $[\varphi?]\psi \leftrightarrow \varphi \to \psi$ (Test Axiom). ⊣

Lemma 1.2.3

(Ⅰ) The following are derived rules and theorems of $\mathbf{PDL}^-$ and **PDL**:

(1) $\varphi \to \psi / [\alpha]\varphi \to [\alpha]\psi$, $\varphi \to \psi / \langle\alpha\rangle\varphi \to \langle\alpha\rangle\psi$.

(2) $[\alpha]\varphi \leftrightarrow \neg\langle\alpha\rangle\neg\varphi$.

(3) $\varphi_1 \wedge \ldots \wedge \varphi_n \to \varphi / [\alpha]\varphi_1 \wedge \ldots \wedge [\alpha]\varphi_n \to [\alpha]\varphi$ (denoted by RK_α).

(4) $\neg(\langle\alpha\rangle\varphi_1 \wedge \ldots \wedge \langle\alpha\rangle\varphi_n) \to \neg\langle\alpha\rangle(\varphi_1 \wedge \ldots \wedge \varphi_n)$.

(5) $[\alpha^*]\varphi \to [\alpha^n]\varphi$ for all $n \in \mathbb{N}$.

(6) $\langle\alpha \cup \beta\rangle\varphi \leftrightarrow \langle\alpha\rangle\varphi \vee \langle\beta\rangle\varphi$ (by $Ax_\cup$).

(7) $\langle\alpha ; \beta\rangle\varphi \leftrightarrow \langle\alpha\rangle\langle\beta\rangle\varphi$ (by $Ax_;$).

(8) $\varphi \vee \langle\alpha\rangle\langle\alpha^*\rangle\varphi \leftrightarrow \langle\alpha^*\rangle\varphi$ (by Ax_*).

(Ⅱ) The following is theorem of **PDL**:

$\langle\psi?\rangle\varphi \leftrightarrow \psi \wedge \varphi$ (by $Ax_?$).

(Ⅲ) In **PDL** without the induction axiom, the following axioms and rules are interderivable: