面向 “十二五”高等教育课程改革项目研究成果

ELASTICITY

Zhang-jian Wu Hai-jun Wu Feng Han

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北京理工大学出版社
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Abstract

The purpose of this book is to introduce the basic knowledge about the classic elasticity theories and the associated research achievements by the authors. The whole book is constructed on the basis of the course syllabuses and the contents of elasticity used in the past few years at Beijing Institute of Technology, China and the University of Manchester, UK. In order to meet the requirement of bilingual pedagogic development in higher education, and with reference to some classic textbooks on elasticity and newly-obtained teaching and learning outputs, such a content arrangement of this book can currently be more appropriate and convenient for readers to study elasticity under the dual-language environment.

By reading this book as well as other relevant Chinese-version textbooks, the readers should be able to command the fundamental knowledge of elasticity, comprehend some related standard technical terms and enhance their level of professional English. The book is intended for senior undergraduate and postgraduate engineering students, especially for engineering mechanics students, of higher education engineering institutes. It can also be considered as an English reference for engineers, researchers and novices.

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PREFACE

This book is, to a large extent, an outgrowth of the lecture notes used by the authors during the past few years in courses on Elasticity at Beijing Institute of Technology and the University of Manchester. It is assumed that the students and readers have already had some acquaintance with the theory of rods and beams, concepts of stress and strain, and so forth, such as may be found in basic texts on the strength of materials or "Mechanics of Materials." It is intended to give advanced undergraduate and graduate students sound foundations on which to build advanced courses such as thermal elasticity, the theory of plasticity, plates and shells, mechanics of composite materials, wave propagation, the finite element method, and those branches of mechanics which require the analysis of stress and strain. Chapter 1 includes the fundamental assumptions adopted in the course of elasticity theory and, for ready reference, certain mathematic preliminaries. The main content of the book begins with stresses and equilibrium in Chapter 2. The theory of deformation is presented in Chapter 3. These two chapters emphasize the independence of stress and strain and also show their mathematical similarity. They are independent of material behaviour. Stress and strain are linked in Chapter 4 by the introduction of three-dimensional stress-strain constitutive relations — generalised Hooke's law. Also in this chapter presentations include the strain energy density function, St. Venant's principle and the assembled basic field equations from Chapters 2 and 3 in order to facilitate the explanation of general solution techniques in boundary value problems of elasticity. From Chapters 5 to 7, two-dimensional boundary value problems, including plane stress and plane strain, torsion and bending of bars, are selected for the application of these solution techniques. The methods of solution are classical and elementary, but they show clearly what concepts and methods are adopted. The more sophisticated methods for solving the boundary value problems of elasticity, such as Love's strain function, Galerkin method, complex potentials and integral transforms, have been deliberately omitted. They are covered extensively in most traditional treatises on the mathematical theory of elasticity. Readers are encouraged to appreciate some of these specialised approaches after they are familiar with the contents of Chapters 5 through 7.

Chapter 8 involves an innovative hybrid technique to the exact solution of three-dimensional elasticity — state space method. This method has been used in the area of automatic control theory for many years. To the best knowledge of the authors, this kind of method and relevant materials are here, for the first time, introduced into a text book on elasticity. It is believed that the method will have wide application in anisotropic elasticity and laminated systems.

Chapter 9 provides the elementary material for classical Kirchhoff plate theory, mainly applicable to the analysis of isotropic thin plates.

Chapter 10 is concerned with formulating energy principles and their application to solid continuum mechanics. The emphasis is on showing how the relevant equations of equilibrium and compatibility can be deduced for this "approximate" theory of elasticity in a systematic and consistent manner. The readers should bear in mind that this alternative approximate solution approach is actually the theoretical foundation of the powerful finite element method (which has been excluded from this book due to the limitation of book size). For the same reason, Cartesian tensor notations have been adopted in this chapter only in order to eliminate tedious and lengthy algebraic and integral computation.

A few special topics of elasticity in engineering such as thermal elasticity, propagation of elastic waves, are shown in Chapter 11. Various strength theories and fracture criteria are listed too for potential engineering applications. This chapter is intended to be an entry to the material for the interests of some readers. Lecturers can decide if they are delivered or not.

Most of the material presented in the book is classical (except for the state space method of Chapter 8), and for this reason very limited references are provided at the end of the book. All important citations have been clearly indicated in the book. Hopefully, we will be forgiven any omission deemed important. S.I. units are consistently used throughout the book.

The development of this textbook in the English language was strongly influenced by some factors, such as the importance of English language due to the globalization of the world economy. The bilingual teaching provides an excellent tool for those who wish to learn about elasticity using the standard terminology of mechanics in both English and Chinese. We also believe that the inclusion of large numbers of examples and exercises or Problems/Tutorial Questions makes this book suitable for self-study. Students, researchers, or practitioners, novices and experts, will profit much from reading the book and having it for reference in the years to come.

We wish to express our gratitude to Prof. Feng-lei Huang and Qing-ming Zhang from Beijing Institute of Technology (BIT), Mr. Mike Maidens and Dr. Qing-ming Li from the University of Manchester who read the manuscript and made useful suggestions and to Prof. Jia-rang Fan from Hefei University of Technology with whom many sections were discussed. Sincere thanks are due to Prof. Zhuo-cheng Ou from BIT for his critical review and positive comments, Ms. Yan-li Wang, and Ms. Tong-hua Liang, from the Press of BIT, for their patience, proofreading and other supports during the preparation of the final manuscript. Special thanks are due to BIT Fund of Teaching Material for financial support and to the publisher, the Press of BIT, for their interest, cooperation, and help in publishing this book in a timely fashion.

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CHAPTER 1 ● ● ● ●

BASIC ASSUMPTIONS AND MATHEMATICAL PRELIMINARIES

1.1 Introduction

Elasticity is a subject associated with the determination of the stresses and displacements in a body as a result of applied mechanical or thermal loads, for those cases in which the body reverts to its original state on the removal of the loads. In this text book, we shall further restrict attention to the case of linearly infinitesimal elasticity, in which the stresses and displacements are linearly proportional to the applied loads and the displacements are small in comparison with the characteristic length dimensions of the body. Detailed assumptions and restrictions on the subject can refer to Section 1.2.

Elastic theory was once a problem solver on its own with closed form solutions which are still admirable and challenging even today. However, extensive and successful use of numerical approaches in engineering, such as finite elements, have diminished to a great extent the need to solve problems analytically, which usually requires enormous, sometimes overwhelming, mathematical skills. The challenge to engineers today appears to be to make correct and effective use of available problem solvers and to assess results obtained from the solvers. This can be achieved only when the engineer understands the formulation of the problem before the solvers are employed.

The main intention of our study of Elasticity is to promote the understanding of the concepts and formulation of elasticity problems. Problem solving skills and techniques are relegated to an adjunct role, being used to help in understanding or in making sense of a theoretical development. Citation of some solvable problems in Elasticity is necessary in this book for us to overview its theoretical frame. It is assumed that the readers possess preliminary knowledge of Structural and Stress Analysis/Mechanics of Solids and Structures. Sometimes such knowledge comes from the prerequisite course called Strength of Materials which will be referred to in the present book.

Strength of Materials deals with bars subjected to loads such as axial tension, compression, torsion about the longitudinal axis and lateral bending. A common assumption is usual for all these cases: the plane section assumption, i.e. a cross-section perpendicular to the axis of the bar deforms into a plane section perpendicular to the axis of the deformed bar. Other restrictions are also needed in order to validate the theoretical analysis, e.g. the bar should be sufficiently long compared with its other dimensions.

Questions arise at this point. How to quantify these restrictions? What happens if a problem goes beyond these restrictions? How to deal with objects other than bars, e.g. those shaped like a two-dimensional plate or three-dimensional laminated structures? These questions set the scene and the main content of the present study of Elasticity.

1.2 Basic Assumptions

All the assumptions in Elasticity have appeared in Strength of Materials (not the wrong way around!) and they will be put in a more rigorous and more explicit manner below.

For the material:

(1) ***Homogeneity*** means that material properties are the same at any point in the material. Microscopically, no material can be homogeneous when one realises that materials are all composed of molecules or atoms, in the case of metals, a larger unit is a grain. However, if an "infinitesimal" volume contains a large enough number of these units, homogeneity is a reasonable assumption on a statistical basis.

Exceptions: reinforced concrete, sandwich laminates, etc.

(2) ***Continuity*** means that every material point is connected and infinitesimally close to the one next to it. All materials carry discontinuous nature in one way or another microscopically. When engineers describe the behaviour of a material at a point, they do not mean a mathematical point (a mathematical point has zero area and zero volume). What they really mean is a small but finite area or volume. They are sufficiently small that they can be regarded as "infinitesimal" without losing any macroscopic feature of the material. If such small areas and volumes are much larger than the characteristic dimension of the discontinuities involved (e.g. the grain size or the distance between atoms), the material can be reasonably assumed to be continuous. Therefore, whenever a small segment dx, a small area dA, etc. are mentioned, they are in fact a continued segment and a continued area, etc.

Exceptions: porous material, granular materials, material with cracks which could not be treated as parts of total boundary of the material, etc.

(3) ***Isotropy*** means that material properties at a point are the same in any direction. It depends on the arrangement of the microscopic material units within an "infinitesimal" volume. If they are arranged at random, isotropy is statistically true.

Exceptions: fibre reinforced composites, wood, etc.

(4) ***Elasticity*** requires that the internal state, such as the strains in the material, reverts to the original state after loads are removed, and, also, that the unloading have to follow the same path as when it was loaded up. It is reasonable for many materials when only small deformation is involved.

Exceptions: problems involving large strain into plastic regime, some plastics of viscosity, etc.

(5) ***Linearity*** requires that the loading path both in terms of load-deformation and stress-strain relation, be a straight line. It is usually reasonable for small deformation problems. Elasticity is a necessary condition of linearity but not sufficient. There are materials which are nonlinear but elastic.

Exceptions: some nonlinear elastic rubbers, materials after yielding, etc.

For the deformation:

Small (infinitesimal) deformation is always assumed in Elasticity. Because of this, higher terms of deformation (gradients of displacements) are negligible relative to non-vanishing lower order terms. Also when equilibrium conditions are established, changes in geometry of the object as a result of deformation can be ignored. It is reasonable in most engineering applications when the deformation is very small relative to a relevant dimension of the object. The geometrical linearity is hence applicable and linear superposition and a wide range of series and transform techniques can be used.

Exceptions: finite deformation, buckling, etc.

For the load:

Only ***Conservative force*** and the force which is not the function of deformation are considered. This is always reasonable when deformation is small.

Exceptions: pressure in large deformation problems, etc.

Note that there is not a priori assumption made on deformation pattern such as the plane section assumption in Strength of Materials. This is why Elasticity is a more general theory.

1.3 Coordinate Systems and Transformations

Throughout the book unless otherwise specified, all references will be made with respect to a rectangular Cartesian coordinate system, x, y and z. Other coordinate systems, such as polar coordinates (2-D) and cylindrical coordinates (3-D) or spherical coordinates, are referred to for the sake of simplicity for the particular problem under consideration. Necessary derivations will be dealt with when they are required. However, for reference purposes, a table of transformations of the derivatives and operators between rectangular Cartesian and cylindrical coordinate systems is provided here as Table 1.1.

Table 1.1 Transformation of Coordinates and Various Mathematical Operations

Coordinate Systems	Rectangular Cartesian Coordinates (3-D): x- y-z Rectangular Cartesian Coordinates (2-D): x-y	Cylindrical Coordinates (3-D): r-θ-z ($r \geqslant 0$) Polar Coordinates (2-D): r-θ ($r \geqslant 0$)
Coordinates	$x = r\cos\theta$ $y = r\sin\theta$ $z = z$	$r = (x^2+y^2)^{1/2}$ $\theta = \tan^{-1}(y/x)$ $z = z$
Differentiation of Coordinates	$\mathrm{d}x = \cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta$ $\mathrm{d}y = \sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta$ $\mathrm{d}z = \mathrm{d}z$	$\mathrm{d}r = \cos\theta\,\mathrm{d}x + \sin\theta\,\mathrm{d}y$ $\mathrm{d}\theta = -(\sin\theta\,\mathrm{d}x + \cos\theta\,\mathrm{d}y)/r$ $\mathrm{d}z = \mathrm{d}z$
Infinitesimal Segment	$\mathrm{d}S^2 = \mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2$	$\mathrm{d}S^2 = \mathrm{d}r^2+r^2\,\mathrm{d}\theta^2+\mathrm{d}z^2$
Base Vectors	$\mathbf{i}, \mathbf{j}, \mathbf{k}$	$\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$
Vector	$V = \mathbf{i}v_x + \mathbf{j}v_y + \mathbf{k}v_z = \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix}$	$V = \mathbf{e}_r v_r + \mathbf{e}_\theta v_\theta + \mathbf{e}_z v_z = \begin{Bmatrix} v_r \\ v_\theta \\ v_z \end{Bmatrix}$
Tensor (2nd order)	$T = [T] = \begin{bmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{bmatrix}$	$T = [T] = \begin{bmatrix} t_{rr} & t_{r\theta} & t_{rz} \\ t_{\theta r} & t_{\theta\theta} & t_{\theta z} \\ t_{zr} & t_{z\theta} & t_{zz} \end{bmatrix}$
Transformation of Base Vectors	$\mathbf{i} = \mathbf{e}_r\cos\theta - \mathbf{e}_\theta\sin\theta$ $\mathbf{j} = \mathbf{e}_r\sin\theta + \mathbf{e}_\theta\cos\theta$ $\mathbf{k} = \mathbf{e}_z$	$\mathbf{e}_r = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$ $\mathbf{e}_\theta = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta$ $\mathbf{e}_z = \mathbf{k}$
Differentiations of the Base Vectors	$\mathrm{d}\mathbf{i}/\mathrm{d}\theta = 0$ $\mathrm{d}\mathbf{j}/\mathrm{d}\theta = 0$ $\mathrm{d}\mathbf{k}/\mathrm{d}\theta = 0$	$\mathrm{d}\mathbf{e}_r/\mathrm{d}\theta = \mathbf{e}_\theta$ $\mathrm{d}\mathbf{e}_\theta/\mathrm{d}\theta = -\mathbf{e}_r$ $\mathrm{d}\mathbf{e}_z/\mathrm{d}\theta = 0$

(continued)

Coordinate Systems	Rectangular Cartesian Coordinates (3-D): x-y-z Rectangular Cartesian Coordinates (2-D): x-y	Cylindrical Coordinates (3-D): r-θ-z ($r \geqslant 0$) Polar Coordinates (2-D): r-θ ($r \geqslant 0$)
Gradient of a Scalar Field	$\nabla\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z} = \begin{Bmatrix} \frac{\partial\phi}{\partial x} \\ \frac{\partial\phi}{\partial y} \\ \frac{\partial\phi}{\partial z} \end{Bmatrix}$	$\nabla\phi = \mathbf{e}_r\frac{\partial\phi}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial\phi}{\partial\theta} + \mathbf{e}_z\frac{\partial\phi}{\partial z} = \begin{Bmatrix} \frac{\partial\phi}{\partial r} \\ \frac{1}{r}\frac{\partial\phi}{\partial\theta} \\ \frac{\partial\phi}{\partial z} \end{Bmatrix}$
Gradient of a Vector Field	$\nabla V = \text{tensor} = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_y}{\partial x} & \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} & \frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial y} \\ \frac{\partial v_x}{\partial z} & \frac{\partial v_y}{\partial z} & \frac{\partial v_z}{\partial z} \end{bmatrix}$	$\nabla V = \text{tensor} = \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} & \frac{\partial v_z}{\partial r} \\ \frac{1}{r}\frac{\partial v_r}{\partial\theta} & \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r} & \frac{1}{r}\frac{\partial v_z}{\partial\theta} \\ \frac{\partial v_r}{\partial z} & \frac{\partial v_\theta}{\partial z} & \frac{\partial v_z}{\partial z} \end{bmatrix}$
Divergence of a Vector Field	$\nabla \cdot V = \text{scalar} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$	$\nabla \cdot V = \text{scalar} = \frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{\partial v_z}{\partial z}$
Divergence of a Tensor Field	$\nabla \cdot T = \text{vector} = \begin{Bmatrix} \frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{xy}}{\partial y} + \frac{\partial t_{xz}}{\partial z} \\ \frac{\partial t_{yx}}{\partial x} + \frac{\partial t_{yy}}{\partial y} + \frac{\partial t_{yz}}{\partial z} \\ \frac{\partial t_{zx}}{\partial x} + \frac{\partial t_{zy}}{\partial y} + \frac{\partial t_{zz}}{\partial z} \end{Bmatrix}$	$\nabla \cdot T = \text{vector}$ $= \begin{Bmatrix} \frac{\partial t_{rr}}{\partial r} + \frac{t_{rr} - t_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial t_{r\theta}}{\partial\theta} + \frac{\partial t_{rz}}{\partial z} \\ \frac{\partial t_{\theta r}}{\partial r} + \frac{t_{\theta r} + t_{r\theta}}{r} + \frac{1}{r}\frac{\partial t_{\theta\theta}}{\partial\theta} + \frac{\partial t_{\theta z}}{\partial z} \\ \frac{\partial t_{zr}}{\partial r} + \frac{1}{r}\frac{\partial t_{z\theta}}{\partial\theta} + \frac{\partial t_{zz}}{\partial z} \end{Bmatrix}$
Laplacean of a Scalar Field	$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$	$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$

1.4 Vector and Matrix Notations and Their Operations

In order to understand the expressions and formulas listed in Table 1.1, we are going to discuss scalar, vector and tensor fields in much more details in this section.

By a scalar field we mean a function ϕ that assigns to each point in a three-dimensional

Euclidean space (say a x-y-z rectangular Cartesian coordinate system) a scalar $\phi(x,y,z)$. A vector in a rectangular Cartesian coordinate system, x, y and z, is denoted by

$$V=\{V\}=\begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} \tag{1.1}$$

In Mathematics, a vector is also called a first order tensor.

Second order tensors are important in Elasticity. They can be represented by 3×3 matrices, which is denoted by

$$T=[T]=\begin{bmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{bmatrix} \tag{1.2}$$

The above vector and matrix notations will be employed in the book, mainly for notation purposes.

From time to time, coordinate transformations are required from the system x, y, z to another rectangular Cartesian system of different orientation, x', y', z'. A vector transforms as follows

$$V'=AV \quad \text{or} \quad \{V'\}=[A]\{V\}$$

that is

$$\begin{Bmatrix} v'_x \\ v'_y \\ v'_z \end{Bmatrix}=[A]\begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} \tag{1.3}$$

and a second order tensor $[T]$ is transformed as

$$T'=ATA^{\mathrm{T}} \quad \text{or} \quad [T']=[A][T][A]^{\mathrm{T}}$$

or

$$\begin{bmatrix} t'_{xx} & t'_{xy} & t'_{xz} \\ t'_{yx} & t'_{yy} & t'_{yz} \\ t'_{zx} & t'_{zy} & t'_{zz} \end{bmatrix}=[A]\begin{bmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{bmatrix}[A]^{\mathrm{T}} \tag{1.4}$$

where

$$[A]=\begin{bmatrix} \cos(x',x) & \cos(x',y) & \cos(x',z) \\ \cos(y',x) & \cos(y',y) & \cos(y',z) \\ \cos(z',x) & \cos(z',y) & \cos(z',z) \end{bmatrix} \tag{1.5}$$

is an matrix of the direction cosines between the two Cartesian coordinate systems. The superscript symbol, T, represents transpose of a matrix. When system x', y' and z' is obtained from x, y and z by rotating through an angle α about the z-axis

$$[A]=\begin{bmatrix}\cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix} \tag{1.6}$$

Several other operations of these (scalar, vector and tensor) fields are very important in the establishment of fundamental equations and their solutions in Elasticity. Such mathematical operations include, typically, gradient, divergence, etc.

For a differentiable scalar function $\phi(x,y,z)$, we define the gradient of ϕ at any point (x, y, z) by $\nabla\phi$, in which the symbol ∇ is called the del operator and it can be expressed in terms of the basis $\{\mathbf{e}_i\}$ of a coordinate system by

$$\nabla=\sum_i \mathbf{e}_i\frac{\partial}{\partial x_i} \tag{1.7}$$

Obviously, ∇ is a vector operator. Under the rectangular Cartesian system x, y, z, Eq.(1.7) becomes

$$\nabla=\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z} \tag{1.8}$$

since the base vector $\{\mathbf{e}_i\}$ of this system is (**i**, **j**, **k**).

For a differentiable vector function $V(x,y,z)$, we define the gradient of V at any point (x, y, z) by a second-order tensor ∇V or in components

$$(\nabla V)_{ij}=V_{i,j}=\frac{\partial V_i}{\partial x_j} \tag{1.9}$$

∇V can be written in a matrix form based on Eq.(1.9),

$$\nabla V=\begin{bmatrix}\frac{\partial V_x}{\partial x} & \frac{\partial V_y}{\partial x} & \frac{\partial V_z}{\partial x}\\ \frac{\partial V_x}{\partial y} & \frac{\partial V_y}{\partial y} & \frac{\partial V_z}{\partial y}\\ \frac{\partial V_x}{\partial z} & \frac{\partial V_y}{\partial z} & \frac{\partial V_z}{\partial z}\end{bmatrix} \tag{1.10}$$

It is the gradient of the vector $V(x,y,z)$.

This definition of the gradient for scalar and vector may be extended to include the gradient of a second-order tensor field (and a resultant third-order tensor is expected), etc. Readers can work them out if interested.

It is noticed that the dot product of the del operator