Springer 大学数学图书——影印版

Symmetries

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TSINGHUA UNIVERSITY PRESS

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内 容 提 要

本书研究空间几何中的各种对称，介绍相关的对称群；以通俗易懂的方式讲述几何与群的本质，以及两者之间的联系（即对称）。书中有大量习题并附部分习题答案或提示。

本书是一本优秀的数学教材，适用于数学系本科生和其他专业对数学有兴趣的本科生用作数学参考书或课外读物。

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  Britton, Nicholas F.
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序　言

在学校教书多年，当学生（特别是本科生）问有什么好的参考书时，我们所能推荐的似乎除了教材还是教材，而且不同教材之间的差别并不明显、特色也不鲜明。所以多年前我们就开始酝酿，希望为本科学生引进一些好的参考书，为此清华大学数学科学系的许多教授与清华大学出版社共同付出了很多心血。

这里首批推出的十余本图书，是从 Springer 出版社的多个系列丛书中精心挑选出来的。在丛书的筹划过程中，我们挑选图书最重要的标准并不是完美，而是有特色并包容各个学派（有些书甚至有争议，比如从数学上看也许不够严格），其出发点是希望我们的学生能够吸纳百家之长；同时，在价格方面，我们也做了很多工作，以使得本系列丛书的价格能让更多学校和学生接受，使得更多学生能够从中受益。

本系列图书按其定位，大体有如下四种类型（一本书可以属于多类，但这里限于篇幅不能一一介绍）。

一、适用面比较广、有特色并可以用作教材或参考书的图书。例如：

● Lovász et al.: Discrete Mathematics. 2003

该书是离散数学的入门类型教材。与现有的教材（包括国外的教材）相比，它涵盖了离散数学新颖而又前沿的研究课题，同时还涉及信息科学方面既基本又有趣的应用；在着力打好数学基础的同时，也强调了数学与信息科学的关联。姚期智先生倡导和主持的清华大学计算机科学试验班，已经选择该书作为离散数学课程的教材。

二、在目前国内的数学教育中，课程主要以学科的纵向发展为主线，而对数学不同学科之间的联系讨论很少。学生缺乏把不同学科视为一个数学整体的训练，这方面的读物尤其欠缺。这是本丛书一个重要的着力点。最典型的是：

● Fine/Rosenberger: The Fundamental Theorem of Algebra. 1997

该书对数学中最重要的定理——代数基本定理给出了六种证明，方法涉及到分析、代数与拓扑；附录中还给出了 Gauss 的证明和 Cauchy 的证明。全书以一个数学问题为主线展开，纵横数学的核心领域；结构严谨、文笔流畅、浅显易懂、引人入胜，是一本少见的能够让读者入迷的好读物，用它来引导学生欣赏和领会“数学的美”绝对不会落于空谈。该书适于自学、讨论，也是极好的短学期课程教材。

● Baker: Matrix Groups. 2001

就内容而言，本书并不超出我国大学线性代数、抽象代数和一般拓扑学课程的内容，但是本书所讲的是李群和李代数的基础理论——这是现代数学和物理学非常重要的工具。各种矩阵群和矩阵代数是李群和李代数最典型和

最重要的例子，同时也能帮助学生建立数学不同学科之间的联系。从矩阵出发，既能把握李群和李代数的实质，又能学会计算和运用，所以这是一本不可多得的好书。

三、科学与技术的发展不断为数学提出新的研究课题，因此在数学学科的发展过程中，来自其他学科的推动力是不能忽视的。本系列中第三种类型的读物就是强调数学与其他学科的联系。例如：

- Woodhouse: Special Relativity. 2003

  该书将物理与数学有机结合，体现了物理学家伽利略的名言："大自然是一部用数学语言写成的巨著。" 不仅如此，本书作者还通过对线性代数、微积分、场论等数学的运用进一步强调并贯穿这样的观点：数学的真谛和发展存在并产生于物理或自然规律及其发现中。精读此书有助于理解物理学和数学的整体关系。

- Britton: Essential Mathematical Biology. 2003

  生命科学在本世纪一定会有很大发展，其对数学的需求和推动是可以预见的。因此生物数学在应用数学中占有日益重要的地位，数学系培养的学生至少一部分人应当对这个领域有所了解。随着生命科学的迅速发展，生物数学也发展很快。本书由浅入深，从经典的问题入手，最后走向学科前沿和近年的热点问题。该书至少可以消除学生对生物学的神秘感。

四、最后一类是适合本科学生的课外读物。这类图书对激发和引导学生学习数学的兴趣会非常有帮助，而且目前国内也急需这样的图书。例如：

- Daepp/Gorkin: Reading, Writing and Proving. 2003

  该书对初学高等数学的读者来说特别有意义。它的基本出发点是引导读者以研究的心态去学习，让读者养成独立思考的习惯，并进而成为研究型的学习者。该书将一个学习数学的过程在某种意义下程序化，努力让学习者养成一个好的学习习惯，以及学会如何应对问题。该书特色鲜明，类似的图书确实很少。

- Brzezniak/Zastawniak: Basic Stochastic Processes. 1998

  随机过程理论在数学、科学和工程中有越来越广泛的应用，本书适合国内的需要。其主要特点是：书中配有的习题是巩固和延伸学习内容的基本手段，而且有十分完整的解答，非常适合自学和作为教学参考书。这是一本难得的好书，它 1999 年出版，到 2000 年已经是第 3 次印刷，到 2003 年则第 6 次重印。

- Anglin/Lambek: The Heritage of Thales. 1995

  该书的基本内容是数学的历史和数学的哲学。数学历史是该书的线索，数学是内容的主体，引申到的是数学哲学。它不是一本史论型的著作，而是采用专题式编写方式，每个专题相对独立，所以比较易读、易懂，是本科生学习数学过程中非常好的课外读物。

本系列丛书中的大部分图书还将翻译为中文出版，以适应更多读者的需要。丛书筹划过程中，冯克勤、郑志勇、卢旭光、郑建华、王殿军、杨利军、叶俊、扈志明等很多清华大学的教授都投入了大量精力。他们之中很多人也将是后面中文版的译者。此外，我们今后还将不断努力丰富引进丛书的种类，同时也会将选书的范围在可能情况下进一步扩大到其他高水平的出版机构。

教育是科学和技术发展的基石，数学教育更是基石的基础。因为是基础所以它重要；也因为是基础所以它显示度不高，容易不被重视。只有将人才培养放到更高的地位上，中国成为创新型国家的目标才会成为可能。

本系列丛书的正式推出，圆了一个我们多年的梦，但这无疑仅仅是开始。

白峰杉

2006 年 6 月于清华园

Preface

"... many eminent scholars, endowed with great geometric talent, make a point of never disclosing the simple and direct ideas that guided them, subordinating their elegant results to abstract general theories which often have no application outside the particular case in question. Geometry was becoming a study of algebraic, differential or partial differential equations, thus losing all the charm that comes from its being an art."

H. Lebesgue, *Leçons sur les Constructions Géométriques*, Gauthier-Villars, Paris, 1949.

This book is based on lecture courses given to final-year students at the University of Nottingham and to M.Sc. students at the University of the West Indies in an attempt to reverse the process of expurgation of the geometry component from the mathematics curricula of universities. This erosion is in sharp contrast to the situation in research mathematics, where the ideas and methods of geometry enjoy ever-increasing influence and importance. In the other direction, more modern ideas have made a forceful and beneficial impact on the geometry of the ancients in many areas. Thus trigonometry has vastly clarified our concept of angle, calculus has revolutionised the study of plane curves, and group theory has become the language of symmetry.

To illustrate this last point at a fundamental level, consider the notion of congruence in plane geometry: two triangles are congruent if one can be moved onto the other so that they coincide exactly. This property is guaranteed by each of the familiar conditions SSS, SAS, SAA and RHS. So congruent triangles are just copies of the same triangle appearing in (possibly) different places. This makes it clear that congruence is an equivalence relation, whose three defining properties correspond to properties of the moves mentioned above:

- reflexivity – the identity move,
- symmetry – inverse moves,
- transitivity – composition of moves.

Thinking of these moves as transformations, for which the associative law holds automatically, we have precisely the four axioms for a group: closure, associativity, identity and inverses.

The group just described underlies and in a sense determines plane geometry. It is called the Euclidean group and occupies a dominant position in this book. Its elements are isometries, as defined in Chapter 1, and a detailed study of these occupies Chapters 2 and 4. The rather bulky Chapters 3 and 5 are intended as crash courses on the theory of groups and group presentations respectively, and both lay emphasis on groups that are semidirect products. Such groups arise in the classification of discrete subgroups of the Euclidean group in Chapters 6, 7 and 8, and corresponding tessellations (or tilings) appear in Chapter 9. Regular tessellations of the sphere are classified in Chapter 10, and tessellations of other spaces, such as the hyperbolic plane, form the subject of Chapter 11. Finally, the notions of polygon in 2-space and polyhedron in 3-space are generalised in Chapter 12 to that of a polytope in n-dimensional Euclidean space. Regular polytopes are then defined using group theory and classified in all dimensions. The classification contains some surprises in dimension 4 and is achieved by as elegant a piece of mathematics as you might imagine.

The exercises at the end of each chapter form an integral part of the book, being designed to reinforce your grasp of the material. A large majority are more or less routine, but a handful of more challenging problems are included for good measure. Solutions to most of them, or at least generous hints, are given later, and suggestions for background, alternative and further reading appear towards the end of the book.

It is a pleasure to acknowledge my gratitude to a number of people: to J.A. Green, B.H. Neumann, J.H. Conway and R.C. Lyndon for influence and guidance over the years, and likewise to John Humphreys, Bob Laxton, Jim Wiegold and Geoff Smith for valuable encouragement; to Maxine Francis, Kate MacDougall and Aaron Wilson for skilful preparation of the typescript and diagrams; to all at Springer-Verlag, especially David Anderson, Nick Wilson, Susan Hezlet, David Ireland and Karen Borthwick, for efficient handling of matters connected with production; and last but not least to the students who provided much useful feedback on my lectures.

D.L.J.

Contents

1
Metric Spaces and their Groups

In many of the physical sciences a fundamental role is played by the concept of length: units of length are used to measure the distance between two points. In mathematics the idea of distance, as a function that assigns a real number to a given pair of points in some space, is formalised in terms of a few reasonable-looking properties, or axioms, and the result is called a metric on that space. Having defined a structure such as this on a set, it is natural to study those transformations, or maps, of such sets which preserve that structure. The requirement that these maps be invertible then leads naturally into the theory of groups.

Many types of groups arise in this way. Important examples are permutation groups, linear groups, Galois groups and symmetry groups. The story of the last of these begins as follows.

1.1 Metric Spaces

Our intuitive conception of distance is made precise in the following definition.

Definition 1.1

A **metric** on a set X is a map $d\colon X \times X \to \mathbb{R}$ with the following properties:

M1) $d(x,y) \geq 0\ \forall x,y \in X$, with equality if and only if $x = y$;

M2) $d(x,y) = d(y,x)\ \forall x,y \in X$;
M3) $d(x,y) + d(y,z) \geq d(x,z)\ \forall x,y,z \in X$.

A set X with a metric d is called a **metric space**, written (X,d).

Each of these axioms is in accordance with our intuition. Thus, referring to elements of X as points, M1 says that the distance between two distinct points is positive, and the distance from a point to itself is zero. M2 says that d is symmetric: the distance between two points is the same in either direction. M3 is the famous **triangle inequality**: the direct route between two points is the shortest. Time for some examples.

Example 1.1

Let X be any set and for $x,y \in X$ define $d(x,y) = 0$ if $x = y$, $d(x,y) = 1$ if $x \neq y$.

To check the axioms, observe that all three parts of M1 are trivial consequences of the definition of d, and the same goes for M2. As to M3, the triangle inequality can only fail if the right-hand side is 1 and both terms on the left are equal to zero. But this means that

$$x \neq z, \quad x = y \quad \text{and} \quad y = z,$$

and this is a contradiction. Hence, M3 holds and d is indeed a metric on X, sometimes referred to as the **discrete metric**.

Example 1.2

Let $X = \mathbb{R}$ and for $x,y \in \mathbb{R}$ define $d(x,y) = |x-y|$, where the modulus $|x|$ of a real number x is defined in the usual way:

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

The axioms correspond to obvious properties of the modulus function, and their verification, as in the next example, is left as an exercise. Both of the next two examples represent attempts to generalise this example from the real line $\mathbb{R}$ to the Cartesian plane $\mathbb{R}^2$.

Example 1.3

Let $X = \mathbb{R}^2$ and for $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2) \in \mathbb{R}$ define

$$d(\mathbf{x},\mathbf{y}) = |x_1 - y_1| + |x_2 - y_2|.$$

This metric is sometimes referred to as the **Manhattan metric** on $\mathbb{R}^2$.

Example 1.4

Again let $X = \mathbb{R}^2$ and for $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$ define

$$d(\mathbf{x}, \mathbf{y}) = +\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}, \tag{1.1}$$

the non-negative root. Unwieldy though it looks, this is the one we want. It is called the **Pythagorean metric**, and when we refer to $\mathbb{R}^2$ as a metric space it is this metric we have in mind.

As usual, M1 and M2 are pretty obvious, being consequences of simple facts about real numbers such as

$$x^2 = 0 \iff x = 0, \quad x^2 = (-x)^2.$$

M3 on the other hand is equivalent to the assertion that the length of one side of a plane triangle is less than or equal to the sum of the lengths of the other two, whence its epithet. For a formal proof in terms of coordinates, take any three points

$$\mathbf{x} = (x_1, x_2), \quad \mathbf{y} = (y_1, y_2), \quad \mathbf{z} = (z_1, z_2)$$

in the plane, and define real numbers a_1, a_2, b_1, b_2 by setting

$$\begin{aligned}(y_1, y_2) &= (x_1 + a_1, x_2 + a_2),\\ (z_1, z_2) &= (y_1 + b_1, y_2 + b_2)\\ &= (x_1 + a_1 + b_1, x_2 + a_2 + b_2).\end{aligned}$$

Then it is required to prove that

$$\sqrt{a_1^2 + a_2^2} + \sqrt{b_1^2 + b_2^2} \geq \sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2}. \tag{1.2}$$

Since squares of real numbers are non-negative, we have

$$(a_1 b_2 - a_2 b_1)^2 \geq 0,$$

which implies that

$$a_1^2 b_2^2 + a_2^2 b_1^2 \geq 2 a_1 b_1 a_2 b_2.$$

Adding $a_1^2 b_1^2 + a_2^2 b_2^2$ to both sides and taking square roots,

$$\sqrt{a_1^2 + a_2^2}\,\sqrt{b_1^2 + b_2^2} \geq a_1 b_1 + a_2 b_2.$$

Multiplying by 2, adding $a_1^2 + a_2^2 + b_1^2 + b_2^2$ to both sides and again taking square roots, we obtain (1.2).

1.2 Isometries

Definition 1.2

An **isometry** of a metric space (X, d) is a bijective map $u: X \to X$ that preserves distance:

$$d(xu, yu) = d(x, y) \quad \forall x, y \in X. \tag{1.3}$$

The set of all isometries of (X, d) is denoted by $\mathrm{Isom}(X, d)$, or just $\mathrm{Isom}(X)$ where d is taken for granted.

Notice that maps are written on the right: the image of x under u is denoted by xu rather than $u(x)$. This is standard practice in those parts of mathematics where the main interest in maps is centred on their composition, for then uv denotes the composite of two maps u and v in this order (first u, then v).

Another point to note is that this definition involves some redundancy. A bijection is by definition a map that is both injective and surjective. It is left to the Exercises to show that any map $u: X \to X$ satisfying (1.3) is necessarily injective (but not necessarily surjective).

Now recall that bijective maps are precisely those that have inverses: there is a map u^{-1} such that

$$uu^{-1} = 1 = u^{-1}u, \tag{1.4}$$

where 1 denotes the identity map, $x1 = x \ \forall x \in X$. Then, assuming that u satisfies (1.3), we compute

$$d(xu^{-1}, yu^{-1}) = d(xu^{-1}u, yu^{-1}u) = d(x1, y1) = d(x, y)$$

for all $x, y \in X$. This shows that if u is an isometry of X, then so is u^{-1}.

Next observe that the identity map $1: X \to X$ is obviously an isometry and that composition of maps is always associative, that is, independent of the bracketing.

Finally, consider the composite uv of two isometries $u, v \in \mathrm{Isom}(X)$. Since uv has an inverse, namely $v^{-1}u^{-1}$, it is a bijection. And since v, u satisfy (1.3), we have

$$d(xuv, yuv) = d(xu, yu) = d(x, y)$$

for all $x, y \in X$. Hence, uv is again an isometry of X. We say that $\mathrm{Isom}(X)$ is closed under composition of maps.

The last three paragraphs prompt the next definition and constitute a proof of our first theorem.