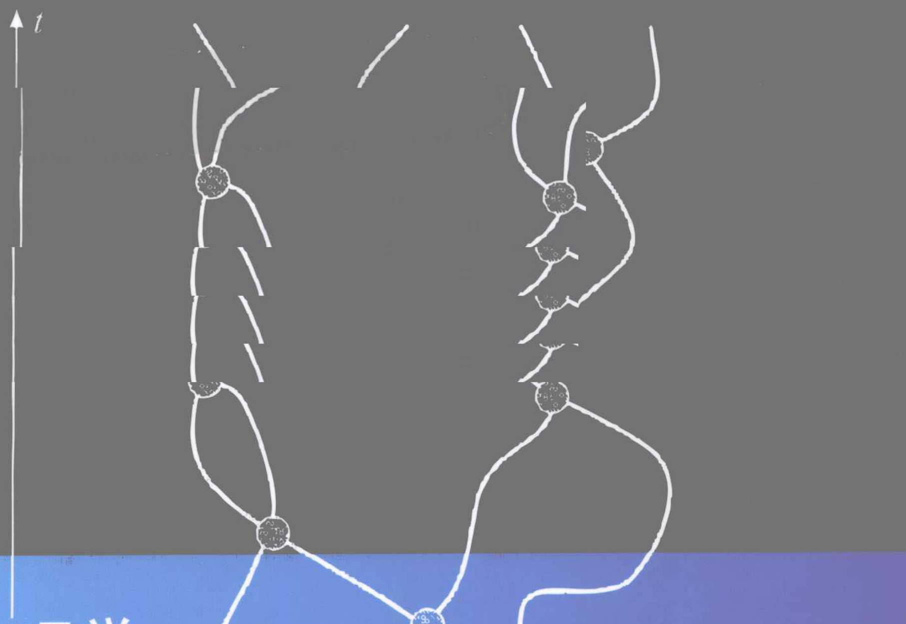
Mehran Kardar

Statistical Physics of Fields



场的统计物理学

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Statistical Physics of Fields

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Statistical Physics of Fields

While many scientists are familiar with fractals, fewer are cognizant of the concepts of scale-invariance and universality which underlie the ubiquity of such fascinating shapes. These inherent properties emerge from the collective behavior of simple fundamental constituents. The initial chapters smoothly connect the particulate perspective developed in the companion volume, Statistical Physics of Particles, to the coarse grained statistical fields studied in this textbook. It carefully demonstrates how such theories are constructed from basic principles such as symmetry and locality, and studied by innovative methods like the renormalization group. Perturbation theory, exact solutions, renormalization, and other tools are employed to demonstrate the emergence of scale invariance and universality. The book concludes with chapters related to the research of the author on non-equilibrium dynamics of interfaces, and directed paths in random media.

Covering the more advanced applications of statistical mechanics, this textbook is ideal for advanced graduate students in physics. It is based on lectures for a course in statistical physics taught by Professor Kardar at Massachusetts Institute of Technology (MIT). The large number of integrated problems introduce the reader to novel applications such as percolation and roughening. The selected solutions at the end of the book are ideal for self-study and honing calculation methods. Additional solutions are available to lecturers on a password protected website at www.cambridge.org/9780521873413.

MEHRAN KARDAR is Professor of Physics at MIT, where he has taught and researched in the field of Statistical Physics for the past 20 years. He received his B.A. in Cambridge, and gained his Ph.D. at MIT. Professor Kardar has held research and visiting positions as a junior fellow at Harvard, a Guggenheim fellow at Oxford, UCSB, and at Berkeley as a Miller fellow.

In this much-needed modern text, Kardar presents a remarkably clear view of statistical mechanics as a whole, revealing the relationships between different parts of this diverse subject. In two volumes, the classical beginnings of thermodynamics are connected smoothly to a thoroughly modern view of fluctuation effects, stochastic dynamics, and renormalization and scaling theory. Students will appreciate the precision and clarity in which difficult concepts are presented in generality and by example. I particularly like the wealth of interesting and instructive problems inspired by diverse phenomena throughout physics (and beyond!), which illustrate the power and broad applicability of statistical mechanics.

Statistical Physics of Particles includes a concise introduction to the mathematics of probability for physicists, an essential prerequisite to a true understanding of statistical mechanics, but which is unfortunately missing from most statistical mechanics texts. The old subject of kinetic theory of gases is given an updated treatment which emphasizes the connections to hydrodynamics.

As a graduate student at Harvard, I was one of many students making the trip to MIT from across the Boston area to attend Kardar's advanced statistical mechanics class. Finally, in *Statistical Physics of Fields* Kardar makes his fantastic course available to the physics community as a whole! The book provides an intuitive yet rigorous introduction to field-theoretic and related methods in statistical physics. The treatment of renormalization group is the best and most physical I've seen, and is extended to cover the often-neglected (or not properly explained!) but beautiful problems involving topological defects in two dimensions. The diversity of lattice models and techniques are also well-illustrated and complement these continuum approaches. The final two chapters provide revealing demonstrations of the applicability of renormalization and fluctuation concepts beyond equilibrium, one of the frontier areas of statistical mechanics.
Leon Balents, Department of Physics, University of California, Santa Barbara

Statistical Physics of Particles is the welcome result of an innovative and popular graduate course Kardar has been teaching at MIT for almost twenty years. It is a masterful account of the essentials of a subject which played a vital role in the development of twentieth century physics, not only surviving, but enriching the development of quantum mechanics. Its importance to science in the future can only increase with the rise of subjects such as quantitative biology.

Statistical Physics of Fields builds on the foundation laid by the Statistical Physics of Particles, with an account of the revolutionary developments of the past 35 years, many of which were facilitated by renormalization group ideas. Much of the subject matter is inspired by problems in condensed matter physics, with a number of pioneering contributions originally due to Kardar himself. This lucid exposition should be of particular interest to theorists with backgrounds in field theory and statistical mechanics.
David R Nelson, Arthur K Solomon Professor of Biophysics, Harvard University

If Landau and Lifshitz were to prepare a new edition of their classic Statistical Physics text they might produce a book not unlike this gem by Mehran Kardar. Indeed, Kardar is an extremely rare scientist, being both brilliant in formalism and an astoundingly careful and thorough teacher. He demonstrates both aspects of his range of talents in this pair of books, which belong on the bookshelf of every serious student of theoretical statistical physics.

Kardar does a particularly thorough job of explaining the subtleties of theoretical topics too new to have been included even in Landau and Lifshitz's most recent Third Edition (1980), such as directed paths in random media and the dynamics of growing surfaces, which are not in any text to my knowledge. He also provides careful discussion of topics that do appear in most modern texts on theoretical statistical physics, such as scaling and renormalization group.
H Eugene Stanley, Director, Center for Polymer Studies, Boston University

This is one of the most valuable textbooks I have seen in a long time. Written by a leader in the field, it provides a crystal clear, elegant and comprehensive coverage of the field of statistical physics. I'm sure this book will become "the" reference for the next generation of researchers, students and practitioners in statistical physics. I wish I had this book when I was a student but I will have the privilege to rely on it for my teaching.
Alessandro Vespignani, Center for Biocomplexity, Indiana University

Preface

Many scientists and non-scientists are familiar with fractals, abstract self-similar entities which resemble the shapes of clouds or mountain landscapes. Fewer are familiar with the concepts of scale-invariance and universality which underlie the ubiquity of these shapes. Such properties may emerge from the collective behavior of simple underlying constituents, and are studied through statistical field theories constructed easily on the basis of symmetries. This book demonstrates how such theories are formulated, and studied by innovative methods such as the renormalization group.

The material covered is directly based on my lectures for the second semester of a graduate course on statistical mechanics, which I have been teaching on and off at MIT since 1988. The first semester introduces the student to the basic concepts and tools of statistical physics, and the corresponding material is presented in a companion volume. The second semester deals with more advanced applications – mostly collective phenomena, phase transitions, and the renormalization group, and familiarity with basic concepts is assumed. The primary audience is physics graduate students with a theoretical bent, but also includes postdoctoral researchers and enterprising undergraduates. Since the material is comparatively new, there are fewer textbooks available in this area, although a few have started to appear in the last few years. Starting with the problem of phase transitions, the book illustrates how appropriate statistical field theories can be constructed on the basis of symmetries. Perturbation theory, renormalization group, exact solutions, and other tools are then employed to demonstrate the emergence of scale invariance and universality. The final two chapters deal with non-equilibrium dynamics of interfaces, and directed paths in random media, closely related to the research of the author.

An essential part of learning the material is doing problems; and in teaching the course I developed a large number of problems (and solutions) that have been integrated into the text. Following each chapter there are two sets of problems: solutions to the first set are included at the end of the book, and are intended to introduce additional topics and to reinforce technical tools. There are no solutions provided for a second set of problems which can be used in assignments.

I am most grateful to my many former students for their help in formulating the material, problems, and solutions, typesetting the text and figures, and pointing out various typos and errors. The final editing of the book was accomplished during visits to the Kavli Institute for Theoretical Physics. The support of the National Science Foundation through research grants is also acknowledged.

Contents

1
Collective behavior, from particles to fields

1.1 Introduction

One of the most successful aspects of physics in the twentieth century was revealing the atomistic nature of matter, characterizing its elementary constituents, and describing the laws governing the interactions and dynamics of these particles. A continuing challenge is to find out how these underlying elements lead to the myriad of different forms of matter observed in the real world. The emergence of new collective properties in the *macroscopic* realm from the dynamics of the *microscopic* particles is the topic of statistical mechanics.

The microscopic description of matter is in terms of the many degrees of freedom: the set of positions and momenta $\{\vec{p}_i, \vec{q}_i\}$, of particles in a gas, configurations of spins $\{\vec{s}_i\}$, in a magnet, or occupation numbers $\{n_i\}$, in a grand canonical ensemble. The evolution of these degrees of freedom is governed by classical or quantum equations of motion derived from an underlying Hamiltonian $\mathcal{H}$.

The macroscopic description usually involves only a few phenomenological variables. For example, the equilibrium properties of a gas are specified by its pressure P, volume V, temperature T, internal energy E, entropy S. The laws of thermodynamics constrain these equilibrium state functions.

A step-by-step derivation of the macroscopic properties from the microscopic equations of motion is generally impossible, and largely unnecessary. Instead, statistical mechanics provides a *probabilistic connection* between the two regimes. For example, in a canonical ensemble of temperature T, each microstate, μ, of the system occurs with a probability $p(\mu) = \exp\left(-\beta\mathcal{H}(\mu)\right)/Z$, where $\beta = (k_B T)^{-1}$. To insure that the total probability is normalized to unity, the *partition function* $Z(T)$ must equal $\sum_\mu \exp\left(-\beta\mathcal{H}(\mu)\right)$. Thermodynamic information about the *macroscopic* state of the system is then extracted from the *free energy* $F = -k_B T \ln Z$.

While circumventing the dynamics of particles, the recipes of statistical mechanics can be fully carried out only for a small number of simple systems; mostly describing non-interacting collections of particles where the partition function can be calculated exactly. Some effects of interactions can be included by perturbative treatments around such exact solutions. However, even for the relatively simple case of an imperfect gas, the perturbative approach breaks down close to the condensation point. On the other hand, it is precisely the multitude of new phases and properties resulting from interactions that renders macroscopic physics interesting. In particular, we would like to address the following questions:

(1) In the thermodynamic limit ($N \to \infty$), strong interactions lead to new phases of matter such as solids, liquid crystals, magnets, superconductors, etc. How can we describe the emergence of such distinct macroscopic behavior from the interactions of the underlying particles? What are the thermodynamic variables that describe the macroscopic state of these phases; and what are their identifying signatures in measurements of bulk response functions (heat capacity, susceptibility, etc.)?
(2) What are the characteristic low energy excitations of the system? As in the case of phonons in solids or in superfluid helium, low energy excitations are typically *collective modes*, which involve the coordinated motions of many microscopic degrees of freedom (particles). These modes are easily excited by thermal fluctuations, and probed by scattering experiments.

The underlying microscopic Hamiltonian for the interactions of particles is usually quite complicated, making an *ab initio* particulate approach to the problem intractable. However, there are many common features in the macroscopic behavior of many such systems that can still be fruitfully studied by the methods of statistical mechanics. Although the interactions between constituents are quite specific at the microscopic scale, one may hope that averaging over sufficiently many particles leads to a simpler description. (In the same sense that the central limit theorem ensures that the sum over many random variables has a simple Gaussian probability distribution function.) This expectation is indeed justified in many cases where the collective behavior of the interacting system becomes more simple at long wavelengths and long times. (This is sometimes called the *hydrodynamic limit* by analogy to the Navier–Stokes equations for a fluid of particles.) The averaged variables appropriate to these length and time scales are no longer the discrete set of particle degrees of freedom, but slowly varying continuous *fields*. For example, the velocity field that appears in the Navier–Stokes equations is quite distinct from the velocities of the individual particles in the fluid. Hence the appropriate method for the study of collective behavior in interacting systems is the statistical mechanics of fields. Accordingly, the aims of this book are as follows:

- **Goal:** To learn to describe and classify states of matter, their collective properties, and the mechanisms for transforming from one phase to another.
- **Tools:** Methods of classical field theories; use of symmetries, treatment of nonlinearities by perturbation theory, and the renormalization group (RG) method.
- **Scope:** To provide sufficient familiarity with the material to follow the current literature on such subjects as phase transitions, growth phenomena, polymers, superconductors, etc.

1.2 Phonons and elasticity

The theory of elasticity represents one of the simplest examples of a field theory. We shall demonstrate how certain properties of an elastic medium can be obtained, either by the complicated method of starting from first principles, or by the much simpler means of appealing to symmetries of the problem. As such, it represents a prototype of how much can be learned from a phenomenological approach. The actual example has little to do with the topics that will be covered later on, but it fully illustrates the methodology that will be employed. The task of computing the low temperature heat capacity of a solid can be approached by either *ab initio* or *phenomenological* methods.

Particulate approach

Calculating the heat capacity of a solid material from first principles is rather complicated. We breifly sketch some of the steps:

- The *ab initio* starting point is the Schrödinger equation for electrons and ions which can only be treated approximately, say by a density functional formalism. Instead, we start with a many-body potential energy for the *ionic* coordinates $\mathcal{V}(\vec{q}_1, \vec{q}_2, \cdots, \vec{q}_N)$, which may itself be the outcome of such a quantum mechanical treatment.
- Ideal lattice positions at zero temperature are obtained by minimizing $\mathcal{V}$, typically forming a lattice $\vec{q}^*(\ell, m, n) = [\ell\hat{a} + m\hat{b} + n\hat{c}] \equiv \vec{q}^*_{\vec{r}}$, where $\vec{r} = \{\ell, m, n\}$ is a triplet of integers, and $\hat{a}$, $\hat{b}$, and $\hat{c}$ are unit vectors.
- Small fluctuations about the ideal positions (due to finite temperature or quantum effects) are included by setting $\vec{q}_{\vec{r}} = \vec{q}^*_{\vec{r}} + \vec{u}(\vec{r})$. The cost of deformations in the potential energy is given by

$$\mathcal{V} = \mathcal{V}^* + \frac{1}{2}\sum_{\substack{\vec{r},\vec{r}' \\ \alpha,\beta}} \frac{\partial^2 \mathcal{V}}{\partial q_{\vec{r},\alpha} \partial q_{\vec{r}',\beta}} u_\alpha(\vec{r})\, u_\beta(\vec{r}') + \mathcal{O}(u^3), \tag{1.1}$$

  where the indices α and β denote spatial components. (Note that the first derivative of $\mathcal{V}$ vanishes at the equilibrium position.) The full Hamiltonian for small deformations is obtained by adding the kinetic energy $\sum_{\vec{r},\alpha} p_\alpha(\vec{r})^2/2m$ to Eq. (1.1), where $p_\alpha(\vec{r})$ is the momentum conjugate to $u_\alpha(\vec{r})$.
- The next step is to find the normal modes of vibration (phonons) by diagonalizing the matrix of derivatives. Since the ground state configuration is a regular lattice, the elements of this matrix must satisfy various translation and rotation symmetries. For example, they can only depend on the difference between the position vectors of ions $\vec{r}$ and $\vec{r}'$, i.e.

$$\frac{\partial^2 \mathcal{V}}{\partial q_{\vec{r},\alpha} \partial q_{\vec{r}',\beta}} = K_{\alpha\beta}(\vec{r} - \vec{r}'). \tag{1.2}$$

  This translational symmetry allows us to at least partially diagonalize the Hamiltonian by using the Fourier modes,

$$u_\alpha(\vec{r}) = \sum_{\vec{k}}{}' \frac{e^{i\vec{k}\cdot\vec{r}}}{\sqrt{N}} u_\alpha(\vec{k}). \tag{1.3}$$

(The above sum is restricted, in that only *wavevectors* $\vec{k}$ inside the first *Brillouin zone* contribute to the sum.) The Hamiltonian then reads

$$\mathcal{H} = V^* + \frac{1}{2}\sum_{\vec{k},\alpha,\beta}\left[\frac{|p_\alpha(\vec{k})|^2}{m} + K_{\alpha\beta}(\vec{k})u_\alpha(\vec{k})u_\beta(\vec{k})^*\right], \tag{1.4}$$

where $u_\beta(\vec{k})^*$ is the complex conjugate of $u_\beta(\vec{k})$. While the precise form of the Fourier transformed matrix $K_{\alpha\beta}(\vec{k})$ is determined by the microscopic interactions, it has to respect the underlying symmetries of the crystallographic point group. Let us assume that diagonalizing this 3×3 matrix yields eigenvalues $\left\{\kappa_\alpha(\vec{k})\right\}$. The quadratic part of the Hamiltonian is now decomposed into a set of independent (non-interacting) harmonic oscillators.

- The final step is to quantize each oscillator, leading to

$$\mathcal{H} = V^* + \sum_{\vec{k},\alpha}\hbar\omega_\alpha(\vec{k})\left(n_\alpha(\vec{k}) + \frac{1}{2}\right), \tag{1.5}$$

where $\omega_\alpha(\vec{k}) = \sqrt{\kappa_\alpha(\vec{k})/m}$, and $\{n_\alpha(\vec{k})\}$ are the set of occupation numbers. The average energy at a temperature T is given by

$$E(T) = V^* + \sum_{\vec{k},\alpha}\hbar\omega_\alpha(\vec{k})\left(\left\langle n_\alpha(\vec{k})\right\rangle + \frac{1}{2}\right), \tag{1.6}$$

where we know from elementary statistical mechanics that the average occupation numbers are given by $\langle n_\alpha(\vec{k})\rangle = 1/\left(\exp(\frac{\hbar\omega_\alpha}{k_BT}) - 1\right)$. Clearly $E(T)$, and other macroscopic functions, have a complex behavior, dependent upon microscopic details through $\left\{\kappa_\alpha(\vec{k})\right\}$. Are there any features of these functions (e.g. the functional dependence as $T \to 0$) that are independent of microscopic features? The answer is positive, and illustrated with a one-dimensional example.

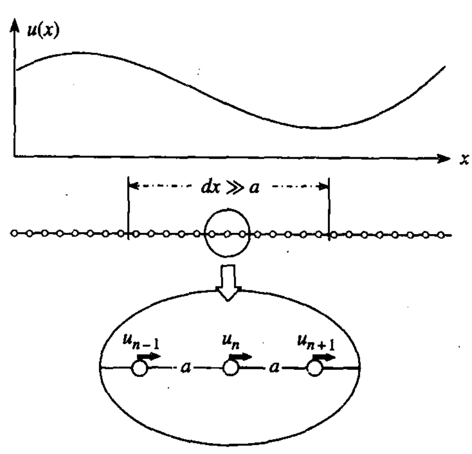

Fig. 1.1 Displacements $\{u_n\}$ of a one-dimensional chain of particles, and the coarse-grained field $u(x)$ of the continuous string.

Consider a chain of particles, constrained to move in one dimension. A most general quadratic potential energy for deformations $\{u_n\}$, around an average separation of a, is

$$\mathcal{V} = \mathcal{V}^* + \frac{K_1}{2}\sum_n (u_{n+1} - u_n)^2 + \frac{K_2}{2}\sum_n (u_{n+2} - u_n)^2 + \cdots, \tag{1.7}$$

where $\{K_i\}$ can be regarded as the Hookian constants of springs connecting particles that are i-th neighbors. The decomposition to normal modes is achieved via

$$u_n = \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}k}{2\pi} \mathrm{e}^{-\mathrm{i}kna} u(k), \quad \text{where} \quad u(k) = a\sum_n \mathrm{e}^{\mathrm{i}kna} u_n . \tag{1.8}$$

(Note the difference in normalizations from Eq. 1.3.) The potential energy,

$$\mathcal{V} = \mathcal{V}^* + \frac{K_1}{2}\sum_n \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}k}{2\pi}\frac{\mathrm{d}k'}{2\pi} (\mathrm{e}^{\mathrm{i}ka} - 1)(\mathrm{e}^{\mathrm{i}k'a} - 1)\mathrm{e}^{-\mathrm{i}(k+k')na} u(k) u(k') + \cdots, \tag{1.9}$$

can be simplified by using the identity $\sum_n \mathrm{e}^{-\mathrm{i}(k+k')na} = \delta(k+k')2\pi/a$, and noting that $u(-k) = u^*(k)$, to

$$\mathcal{V} = \mathcal{V}^* + \frac{1}{2a}\int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}k}{2\pi} \left[K_1(2 - 2\cos ka) + K_2(2 - 2\cos 2ka) + \cdots\right] |u(k)|^2. \tag{1.10}$$

A typical frequency spectrum of normal modes, given by $\omega(k) = \sqrt{[2K_1(1-\cos ka)+\cdots]/m}$, is depicted in Fig. 1.2. In the limit $k \to 0$, the dispersion relation becomes linear, $\omega(k) \to v|k|$, and from its slope we can identify a 'sound velocity' $v = a\sqrt{\overline{K}/m}$, where $\overline{K} = K_1 + 4K_2 + \cdots$.

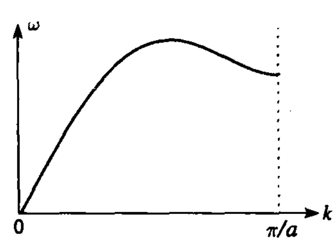

Fig. 1.2 Typical dispersion relation for phonons along a chain.

The internal energy of these excitations, for a chain of N particles, is

$$E(T) = \mathcal{V}^* + Na\int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}k}{2\pi} \frac{\hbar\omega(k)}{\exp\left(\hbar\omega(k)/k_\mathrm{B}T\right) - 1}. \tag{1.11}$$

As $T \to 0$, only modes with $\hbar\omega(k) < k_\mathrm{B}T$ are excited. Hence only the $k \to 0$ part of the excitation spectrum is important and $E(T)$ simplifies to

$$E(T) \approx \mathcal{V}^* + Na\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \frac{\hbar v|k|}{\exp(\hbar v|k|/k_\mathrm{B}T) - 1} = \mathcal{V}^* + Na\frac{\pi^2}{6\hbar v}(k_\mathrm{B}T)^2. \tag{1.12}$$

Note

(1) While the full spectrum of excitation energies can be quite complicated, as $k \to 0$,

$$\frac{K(k)}{2} = K_1(1-\cos ka) + K_2(1-\cos 2ka) + \cdots \to \frac{\bar{K}}{2}k^2 \quad \text{where,} \qquad (1.13)$$
$$\overline{K} = K_1 + 4K_2 + \cdots .$$

Thus, further neighbor interactions change the speed of sound, but not the form of the dispersion relation as $k \to 0$.

(2) The heat capacity $C(T) = \mathrm{d}E/\mathrm{d}T$ is proportional to T. This dependence is a *universal* property, i.e. not material specific, and independent of the choice of the interatomic interactions.

(3) The T^2 dependence of energy comes from excitations with $k \to 0$ (or $\lambda \to \infty$), i.e. from collective modes involving many particles. These are precisely the modes for which statistical considerations may be meaningful.

Phenomenological (field) approach

We now outline a mesoscopic approach to the same problem, and show how it provides additional insights and is easily generalized to higher dimensions. Typical excitations at low temperatures have wavelengths $\lambda > \lambda(T) \approx (\hbar v/k_B T) \gg a$, where a is the lattice spacing. We can eliminate the unimportant short wavelength modes by an averaging process known as **coarse graining**. The idea is to consider a point x, and an interval of size $\mathrm{d}x$ around it (Fig. 1.1). We shall choose $a \ll \mathrm{d}x \ll \lambda(T)$, i.e. the interval is large enough to contain many lattice points, but much shorter than the characteristic wavelength of typical phonons. In this interval all the displacements u are approximately the same; and we can define an average deformation field $u(x)$. By construction, the function $u(x)$ varies slowly over $\mathrm{d}x$, and despite the fact that this interval contains many lattice points, from the perspective of the function it is infinitesimal in size. We should always keep in mind that while $u(x)$ is treated as a continuous function, it does not have any variations over distances comparable to the lattice spacing a.

- By examining the displacements as a function of time, we can define a velocity field $\dot{u}(x) \equiv \partial u/\partial t$. The kinetic energy is then related to the mass density $\rho = m/a$ via $\rho \int \mathrm{d}x \dot{u}(x)^2/2$.
- What is the most general potential energy functional $\mathcal{V}[u]$, for the chain? A priori, we don't know much about the form of $\mathcal{V}[u]$, but we can construct it by using the following general principles:

  Locality: In most situations, the interactions between particles are short range, allowing us to define a potential energy *density* Φ at each point x, with $\mathcal{V}[u] = \int \mathrm{d}x \Phi\big(u(x), \partial u/\partial x, \cdots\big)$. Naturally, by including all derivatives we can also describe long-range interactions. In this context, the term *locality* implies that the higher derivative terms are less significant.

Translational symmetry: A uniform translation of the chain does not change its internal energy, and hence the energy density must satisfy the constraint $\Phi[u(x)+c]=\Phi[u(x)]$. This implies that Φ cannot depend directly on $u(x)$, but only on its derivatives $\partial u/\partial x$, $\partial^2 u/\partial x^2, \cdots$.

Stability: Since the fluctuations are around an *equilibrium* solution, there can be no linear terms in u or its derivatives. (Stability further requires that the quadratic part of $\mathcal{V}[u]$ must be positive definite.)

The most general potential consistent with these constraints can be expanded as a power series

$$\mathcal{V}[u]=\int \mathrm{d}x\left[\frac{K}{2}\left(\frac{\partial u}{\partial x}\right)^2+\frac{L}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)^2+\cdots+M\left(\frac{\partial u}{\partial x}\right)^2\left(\frac{\partial^2 u}{\partial x^2}\right)+\cdots\right], \tag{1.14}$$

which after Fourier transformation gives

$$\begin{aligned}\mathcal{V}[u]=&\int \frac{\mathrm{d}k}{2\pi}\left[\frac{K}{2}k^2+\frac{L}{2}k^4+\cdots\right]|u(k)|^2\\ &-\mathrm{i}M\int \frac{\mathrm{d}k_1}{2\pi}\frac{\mathrm{d}k_2}{2\pi}k_1k_2(k_1+k_2)^2u(k_1)u(k_2)u(-k_1-k_2)+\cdots.\end{aligned} \tag{1.15}$$

As $k\to 0$, higher order gradient terms (such as the term proportional to L) become unimportant. Also, for small deformations we may neglect terms beyond second order in u (such as the cubic term with coefficient M). Another assumption employed in constructing Eq. (1.14) is that the mirror image deformations $u(x)$ and $u(-x)$ have the same energy. This may not be valid in more complicated lattices without inversion symmetry.

Adding the kinetic energy, we get a simple one-dimensional field theory, with a Hamiltonian

$$\mathcal{H}=\frac{\rho}{2}\int \mathrm{d}x\left[\left(\frac{\partial u}{\partial t}\right)^2+v^2\left(\frac{\partial u}{\partial x}\right)^2\right].$$

This is a one-dimensional elastic (string) theory with material dependent constants ρ and $v=\sqrt{K/\rho}$. While the phenomenological approach cannot tell us the value of these parameters, it does show that the low energy excitations satisfy the dispersion relation $\omega=v|k|$ (obtained by examining the Fourier modes).

We can now generalize the elastic theory of the string to arbitrary dimensions d: The discrete particle deformations $\{\vec{u}_n\}$ are coarse grained into a continuous deformation field $\vec{u}(\vec{x})$. For an *isotropic* material, the potential energy $\mathcal{V}[\vec{u}]$ must be invariant under both rotations and translations (described by $u_\alpha(\vec{x})\mapsto R_{\alpha\beta}u_\beta(\vec{x})+c_\alpha$, where $R_{\alpha\beta}$ is a rotation matrix). A useful local quantity is the symmetric *strain field*,

$$u_{\alpha\beta}(\vec{x})=\frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta}+\frac{\partial u_\beta}{\partial x_\alpha}\right), \tag{1.16}$$