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Ernest B. Vinberg

Linear Representations of Groups

群的线性表示

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Preface

This book gives an exposition of the fundamentals of the theory of linear representations of finite and compact groups, as well as elements of the theory of linear representations of Lie groups. As an application we derive the Laplace spherical functions. The book is based on lectures that I delivered in the framework of the experimental program at the Mathematics-Mechanics Faculty of Moscow State University and at the Faculty of Professional Skill Improvement. My aim has been to give as simple and detailed an account as possible of the problems considered. The book therefore makes no claim to completeness. Also, it can in no way give a representative picture of the modern state of the field under study as does, for example, the monograph of A. A. Kirillov [3].

For a more complete acquaintance with the theory of representations of finite groups we recommend the book of C. W. Curtis and I. Reiner [2], and for the theory of representations of Lie groups, that of M. A. Naimark [6].

Contents

Introduction

The theory of linear representations of groups is one of the most widely applied branches of algebra. Practically every time that groups are encountered, their linear representations play an important role. In the theory of groups itself, linear representations are an irreplaceable source of examples and a tool for investigating groups.

In the introduction we discuss some examples and en route we introduce a number of notions of representation theory.

0. Basic Notions

0.1. The exponential function

$$t \mapsto e^{ta} \quad (t \in \mathbf{R})$$

is, for every fixed $a \in \mathbf{R}$, a homomorphism of the additive group $\mathbf{R}$ into the multiplicative group $\mathbf{R}^*$. Are there any other homomorphisms $f : \mathbf{R} \to \mathbf{R}^*$? Without attempting to answer this question in full generality, we require that the function f be differentiable. The condition that f be a homomorphism is written:

$$f(t+u) = f(t)f(u)$$

for all $t, u \in \mathbf{R}$. Differentiating with respect to u and putting $u = 0$ we get

$$f'(t) = f(t)a,$$

where $a = f'(0)$. The general solution of this differential equation is $f(t) = Ce^{ta}$, but the condition that f be a homomorphism forces $f(0) = 1$, whence $C = 1$. Thus, *every differentiable group homomorphism of* $\mathbf{R}$ *into* $\mathbf{R}^*$ *is an exponential function.* This is one of the reasons why exponential functions play such an important role in mathematics.

In solving systems of linear differential equations with constant coefficients one encounters the *matrix exponential function*

(1) $$t \mapsto e^{tA} \quad (t \in \mathbf{R},\ A \in \mathrm{L}_n(\mathbf{R})).$$

Recall that it is defined as the sum of the series

$$\sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$$

or, alternatively, as the solution of the matrix differential equation

(2) $\quad F'(t) = F(t)A \qquad (F : \mathbf{R} \to \mathrm{L}_n(\mathbf{R}))$

with initial condition $F(0) = E$. The exponential matrix function has the property that

$$e^{(t+u)A} = e^{tA}e^{uA} \quad \text{for all } t, u \in \mathbf{R},$$

i.e., it is a *homomorphism* of the group $\mathbf{R}$ into the group $\mathrm{GL}_n(\mathbf{R})$. As above, one can show that *every differentiable group homomorphism of* $\mathbf{R}$ *into* $\mathrm{GL}_n(\mathbf{R})$ *has the form* (1). This is a generalization of the preceding result, since $\mathbf{R}^* = \mathrm{GL}_1(\mathbf{R})$.

Example. It follows from the addition formulas for trigonometric functions that

(3) $$\begin{pmatrix} \cos(t+u) & -\sin(t+u) \\ \sin(t+u) & \cos(t+u) \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{pmatrix}.$$

This says that the map

(4) $$F : t \mapsto \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

is a group homomorphism of $\mathbf{R}$ into $\mathrm{GL}_2(\mathbf{R})$, and hence it has the form (1). Here

$$A = F'(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

0.2. Let S_n denote the symmetric group of degree n and let K be an arbitrary field. To each permutation $\sigma \in S_n$ we assign the matrix

(5) $\quad M(\sigma) = E_{\sigma(1),1} + \ldots + E_{\sigma(n),n},$

where E_{ij} designates the matrix whose (i, j)-entry is the identity element of the field K, while the remaining entries are zero.

We claim that for all $\sigma, \tau \in S_n$:

(6) $\quad M(\sigma\tau) = M(\sigma)M(\tau).$

In fact, one readily verifies the following multiplication rule for the matrices E_{ij}:

$$E_{ij}E_{\ell k} = \begin{cases} E_{ik} & \text{if } j = \ell, \\ 0 & \text{if } j \neq \ell. \end{cases}$$

Using it we find that

$$M(\sigma)M(\tau) = E_{\sigma\tau(1),\tau(1)}E_{\tau(1),1} + \ldots + E_{\sigma\tau(n),\tau(n)}E_{\tau(n),n}$$
$$= E_{\sigma\tau(1),1} + \ldots + E_{\sigma\tau(n),n} = M(\sigma\tau),$$

as claimed.

We remark that the matrices $M(\sigma)$ are nonsingular. More precisely,

$$\det M(\sigma) = \Pi(\sigma)$$

where $\Pi(\sigma) = \pm 1$ denotes the parity of the permutation σ. Equality (6) says that M is a group homomorphism of S_n into $\mathrm{GL}_n(K)$.

0.3. Definition. A MATRIX REPRESENTATION of the group G over the field K is a homomorphism

$$T: G \to \mathrm{GL}_n(K)$$

of G into the group $\mathrm{GL}_n(K)$ of nonsingular matrices of order n over K. The number n is called the DIMENSION of the representation T.

The word "representation" should suggest that once a matrix representation is given, the elements of the group can be viewed as matrices or, in other words, that there is an isomorphism of the given group with some group of matrices. In the cases where the group G has a rather complicated structure, it may very well turn out that such a representation is the only simple way of describing G. For instance, the group $\mathrm{GL}_n(K)$ is defined as a matrix group, i.e., the identity map $\mathrm{Id}: \mathrm{GL}_n(K) \to \mathrm{GL}_n(K)$ is a matrix representation of $\mathrm{GL}_n(K)$.

We should, however, emphasize from the very beginning that in reality a matrix representation does not always give an isomorphism between the group G and a subgroup of $\mathrm{GL}_n(K)$. The reason is that one and the same matrix might correspond to distinct elements of G. To the reader familiar with the general properties of homomorphisms it should be clear that this occurs if and only if there are elements of G different from its identity element which are mapped to the identity element of the group $\mathrm{GL}_n(K)$, i.e., to the identity matrix.

The set of all elements of G which are taken by T into the identity matrix is a normal subgroup of G, called the KERNEL of the representation T and denoted by $\mathrm{Ker}\, T$. For example, the kernel of the representation (4) of $\mathbf{R}$ consists of all numbers $2\pi k$ with $k \in \mathbf{Z}$.

If $\mathrm{Ker}\, T$ reduces to the identity element of G, the representation T is said to be FAITHFUL. In this case G is isomorphic to the subgroup $T(G)$ of $\mathrm{GL}_n(K)$.

The representation of the group S_n constructed above in 0.2 provides an example.

The other extreme case occurs when $\operatorname{Ker} T = G$, i.e., all elements of G are taken by T into the identity matrix. Such a T is called a TRIVIAL representation.

0.4. We next discuss the *geometric approach* to the notion of representation.

Every matrix A of order n with entries in the field K defines a linear transformation $x \mapsto Ax$ of the space K^n of column vectors. Moreover, this correspondence between matrices and linear transformations is bijective and turns the multiplication of matrices into the multiplication (composition) of linear transformations. In particular, the group $\mathrm{GL}_n(K)$ is canonically isomorphic to the group $\mathrm{GL}(K^n)$ of invertible linear transformations of the space K^n. Accordingly, every matrix representation of a group G can be regarded as a group homomorphism of G into $\mathrm{GL}(K^n)$.

Example. Let M be the matrix representation of the group S_n constructed above in 0.2. For each $\sigma \in S_n$ the matrix $M(\sigma)$ can be regarded as a linear transformation of K^n. Let $(e_1, \ldots, e_n)$ be the standard basis of K^n. Then

(7) $$M(\sigma)e_i = e_{\sigma(i)} \qquad \text{for} \quad i = 1, \ldots, n.$$

Since a linear operator is uniquely determined by its action on the basis vectors, equations (7) may be taken as the definition of the representation M.

Replacing K^n by an arbitrary vector space V over the field K, we now arrive at the following generalization of the notion of a matrix representation.

Definition. A LINEAR REPRESENTATION of the group G over the field K is a homomorphism of G into the group $\mathrm{GL}(V)$ of all invertible linear transformations (linear operators) of a vector space V over K. V is called the REPRESENTATION SPACE, and its dimension is called the DIMENSION or the DEGREE of the representation.

Suppose that the space V has a finite dimension n. Then with each linear representation T of the group G in V we can associate a class of n-dimensional matrix representations. To this end we pick some basis $(e) = (e_1, \ldots, e_n)$ of V. Every operator $T(g)$, for $g \in G$, is described with respect to the basis (e) by a matrix $T(g)_{(e)}$, and the map $T_{(e)} : g \mapsto T(g)_{(e)}$ that arises in this manner is a matrix representation of G. On choosing another basis $(f) = (e)C$ (where C is the transition matrix from (e) to (f)), we obtain another representation $T_{(f)}$, related to $T_{(e)}$ as follows:

$$T_{(f)}(g) = C^{-1} T_{(e)}(g) C.$$

Definition. We say that two matrix representations T_1 and T_2 are EQUIVALENT (and write $T_1 \simeq T_2$) if they have the same dimension and there exists a nonsingular matrix C such that

$$(8) \qquad T_2(g) = C^{-1}T_1(g)C$$

for all $g \in G$.

The foregoing discussion makes clear that to each finite-dimensional representation there corresponds a class of equivalent matrix representations.

0.5. Definition. We say that two linear representations,

$$T_1 \colon G \to \mathrm{GL}(V_1) \qquad \text{and} \qquad T_2 \colon G \to \mathrm{GL}(V_2),$$

are ISOMORPHIC (or EQUIVALENT), and write $T_1 \simeq T_2$, if there exists an isomorphism $\sigma \colon V_1 \to V_2$ such that

$$(9) \qquad \sigma T_1(g) = T_2(g)\sigma$$

for all $g \in G$.

In other words, $T_1 \simeq T_2$ if, upon identifying the spaces V_1 and V_2 by means of σ, the representations T_1 and T_2 become identical. In particular, if V_1 and V_2 are finite-dimensional, then in bases that correspond under σ the matrices of the operators $T_1(g)$ and $T_2(g)$ coincide for all $g \in G$. This means that the matrix representations associated with the linear representations T_1 and T_2 are identical for a compatible choice of bases, and equivalent for an arbitrary choice of bases.

Conversely, suppose that for some choice of bases of the spaces V_1 and V_2 the linear representations T_1 and T_2 determine equivalent matrix representations. Then, for an appropriate choice of bases, T_1 and T_2 determine identical matrix representations and hence are isomorphic.

Specifically, let the bases $(e)_1$ and and $(e)_2$ in V_1 and V_2 be such that $T_1(g)_{(e)_1} = T_2(g)_{(e)_2}$ for all $g \in G$. Then the isomorphism $\sigma \colon V_1 \to V_2$ which takes $(e)_1$ into $(e)_2$ satisfies condition (9).

Thus, *the matrix representations associated with the finite-dimensional linear representations T_1 and T_2 are equivalent if and only if T_1 and T_2 are isomorphic.*

0.6. Halting here this somewhat unexciting yet necessary discussion, let us see what benefits we can extract from it.

Recall that linear operators can be added, multiplied with one another, and multiplied by numbers (elements of the ground field).

In an arbitrary basis, to these operations there correspond the same operations on matrices; however, the definitions of the operations on linear operators are independent of the choice of a basis. Further, in a finite-dimensional space over **R** or **C**, one can define the limit of a sequence of linear operators. The act of passing to the limit is also compatible with the corresponding action on matrices, but again does not depend on the choice of a basis.

Let V be a finite-dimensional vector space over $K = \mathbf{R}$ or $\mathbf{C}$. The exponential operator-valued function

$$t \mapsto e^{t\alpha} \qquad (t \in K, \quad \alpha \in \mathrm{L}(V))$$

can be defined in the same manner as the matrix exponential function (see 0.1 above). It is a linear representation of the additive group K. The matrix representation that corresponds to this representation in an arbitrary basis is $t \mapsto e^{tA}$, where A denotes the matrix of the operator α in that basis. Since the choice of basis is at our disposal, we can attempt to make it so that A will take the simplest possible form. For example, in the case $K = \mathbf{C}$ we can arrange that A be in Jordan form. It is known that the Jordan form is uniquely determined up to a permutation of the blocks. This implies, in particular, that two linear representations,

$$t \mapsto e^{t\alpha} \qquad \text{and} \qquad t \mapsto e^{t\beta},$$

are isomorphic if and only if the matrices of the operators α and β have the same Jordan form.

This example illustrates the geometric approach to the notion of representation, which does not distinguish between equivalent matrix representations and permits us, in certain cases, to avoid computations in coordinates.

0.7. In order to describe a linear representation, it is not obligatory to choose a basis in the representation space. Alternatively, representations can be described geometrically.

Examples.

1. We specify a linear representation of the group **R** as follows: to the element $t \in \mathbf{R}$ we assign the rotation by angle t in the Euclidean plane. From geometric considerations it is obvious that the composition of the rotations by the angles t and u is the rotation by the angle $t + u$, and hence that the map thus constructed is indeed a linear representation. In an orthonormal basis we have the corresponding matrix representation (4). Formula (3) now follows automatically, and we can derive the addition formulas for trigonometric functions from it (and not conversely, as we did in 0.1).

2. Let V denote the linear space of all polynomials with real coefficients. To each $t \in \mathbf{R}$ we assign a linear operator $L(t) \in \mathrm{GL}(V)$ by the rule

$$(L(t)f)(x) = f(x - t). \tag{10}$$

It is readily checked that $L(t)$ is indeed a linear operator and that $L(t+u) = L(t)L(u)$, i.e., L is a linear representation of the additive group $\mathbf{R}$.

3. In the preceding example we can replace polynomials by any space of functions which is invariant under translations. Here are some examples:

a) the space of continuous functions;

b) the space of polynomials of degree $\leq n$;

c) the space of trigonometric polynomials, i.e., polynomials in $\cos x$ and $\sin x$;

d) the linear span of the functions $\cos x$ and $\sin x$.

Let us examine in more detail the last case. Since

$$\begin{aligned} L(t)\cos x &= \cos t \cos x + \sin t \sin x, \\ L(t)\sin x &= -\sin t \cos x + \cos t \sin x, \end{aligned} \tag{11}$$

the transformation $L(t)$ takes any linear combination of $\cos x$ and $\sin x$ again into such a linear combination. Formulas (11) show that in the basis $(\cos x, \sin x)$ the operator $L(t)$ is described by the matrix

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Therefore, the representation L of the group $\mathbf{R}$ in the space $\langle \cos x, \sin x \rangle$ is isomorphic to the representation constructed above geometrically.

0.8. Group theory is also concerned with more general "representations," called *actions*.

Let X be an arbitrary set and let $S(X)$ denote the group of all bijections of X onto itself. (If $X = \{1, 2, \ldots, n\}$, then $S(X) = S_n$).

Definition. An ACTION of the group G on X is a homomorphism $s: G \to S(X)$.

In other words, the action s assigns to each $g \in G$ a bijective map $s(g)$ of the set X onto itself in such a manner that $s(g_1 g_2) = s(g_1)s(g_2)$ for all $g_1, g_2 \in G$.

If it is clear to which action we are referring, we will simply write gx instead of $s(g)x$. The last equality above signified "associativity": $(g_1g_2)x = g_1(g_2x)$ for every $x \in X$.

We may regard a linear representation as a special kind of action. As a nonlinear example we give the action of $\mathbf{R}$ on itself defined by the rule $s(t)x = x + t$.

This last example can be generalized as follows. Let G be a group. Then G acts on itself by left translations,

(12) $$l(g)x = gx,$$

as well as by right translations,

(13) $$r(g)x = xg^{-1}.$$

Let us check that, say, formula (13) determines an action:

$$r(g_1g_2)x = x(g_1g_2)^{-1} = xg_2^{-1}g_1^{-1},$$

and

$$r(g_1)r(g_2)x = r(g_1)(xg_2^{-1}) = xg_2^{-1}g_1^{-1}.$$

(The proof explains why g appears in (13) with the exponent -1.)

0.9. With each action we can associate a *linear representation in a function space.*

Let K be a field and $K[X]$ the vector space of all K-valued functions on the set X. Each $\sigma \in S(X)$ defines a linear transformation σ_* in $K[X]$:

(14) $$(\sigma_* f)(x) = f(\sigma^{-1}x) \qquad (f \in K[X]).$$

We have $(\sigma\tau)_* = \sigma_*\tau_*$, i.e., (14) defines a linear representation of the group $S(X)$. In fact,

$$((\sigma\tau)_* f)(x) = f((\sigma\tau)^{-1}x) = f(\tau^{-1}\sigma^{-1}x),$$

and

$$(\sigma_*\tau_* f)(x) = (\tau_* f)(\sigma^{-1}x) = f(\tau^{-1}\sigma^{-1}x).$$

In an analogous manner we can define a linear representation of the group $S(X)$ in a space of functions of several variables:

$$(\sigma_* f)(x_1, \ldots, x_k) = f(\sigma^{-1}x_1, \ldots, \sigma^{-1}x_k).$$

Now suppose we are given an action $s: G \to S(X)$. We define a linear representation s_* of the group G in the space $K[X]$, setting

$$s_*(g) = s(g)_*.$$

Representations of G in spaces of functions of several variables are defined similarly.

Examples.

1. Consider the group G of rotations of a cube (isomorphic, as is known, to S_4) and its natural action s on the set X of faces of the cube. Here the space $K[X]$ is six-dimensional. As a basis of $K[X]$ we can take the functions $f_1, \ldots, f_6$, each of which is equal to 1 on one of the faces and to 0 on the others. Relative to this basis the operators $s_*(g)$ are written as matrices of 0's and 1's such that in every row and every column there is exactly one 1. For example, let g be the rotation by $2\pi/3$ around an axis passing through the center of the cube and one of its vertices. Then for a suitable labeling of the basis functions $f_1, \ldots, f_6$, the operator $s_*(g)$ is given by the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

2. Let s be the natural action of the group S_n on the set $\{1, 2, \ldots, n\}$. Then the representation s_* is isomorphic to the representation M constructed above in 0.2 (see also the example in 0.4).

In general, if X is a finite set, then the space $K[X]$ is finite-dimensional and its dimension is equal to the number of elements of X. The functions δ_x, $x \in X$, defined as

$$(15) \qquad \delta_x(y) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

form a basis of $K[X]$. The operators $s_*(g)$, $g \in G$, simply permute the elements of this basis. Specifically,

$$(16) \qquad s_*(g)\delta_x = \delta_{gx}.$$

In fact,

$$(s_*(g)\delta_x)(y) = \delta_x(g^{-1}y) = \begin{cases} 1 & \text{if } y = gx, \\ 0 & \text{if } y \neq gx. \end{cases}$$

In those cases of interest where the set X is infinite, it usually possesses an additional structure (for example, it is a topological space or a differentiable manifold), and the given group action is compatible with that structure. Then one considers only functions that are "nice" in one sense or another (for example, continuous or differentiable), rather than arbitrary functions on X. For instance, in the case of the action s of the group $\mathbf{R}$ on itself by translations, we can take the space of polynomials for the representation space of s_*. We thus get the linear representation of $\mathbf{R}$ considered above in Example 2, 0.7.

Definition. The linear representation $l_* = L$ of an arbitrary group G (in a suitable class of functions on G) associated with the action l of G on itself by left translations is called the LEFT REGULAR REPRESENTATION of G. According to this definition

(17) $(L(g)f)(x) = f(g^{-1}x) \qquad (g, x \in G).$

The RIGHT REGULAR REPRESENTATION is defined in a similar manner. Needless to say, the class of functions in question must be specified exactly.

As we shall see, the study of regular representations is the key step towards the description of all representations of a given group.

0.10. Methods of producing new representations from one or several given ones play an important role in the theory of linear representations. One of the simplest of these methods is *the composition of a representation and a homomorphism.*

Let $T: G \to \mathrm{GL}(V)$ be a linear representation of the group G, and let $\phi: H \to G$ be a homomorphism. Then $T \circ \phi$ is a linear representation of the group H.

Let us examine two particular cases of this construction. If H is a subgroup of G and ϕ is the inclusion map of H into G, then $T \circ \phi$ is simply *the restriction of the representation T to H*. We denote the restriction operation by Res_H^G. According to the definition,

$$(\mathrm{Res}_H^G T)(h) = T(h) \qquad \text{for } h \in H.$$

If now ϕ is an *automorphism* of G, then $T \circ \phi$ is, like T, a representation of G. Such a "twisted" representation can be isomorphic or not to the original representation T.

Examples.

1. Let $\phi = a(h)$ be the *inner automorphism* defined by $h \in G$, i.e., $a(h)g = hgh^{-1}$. Then for every $g \in G$

$$(T \circ a(h))(g) = T(hgh^{-1}) = T(h)T(g)T(h)^{-1}$$

or, equivalently,

(18) $(T \circ a(h))(g)T(h) = T(h)T(g).$

Since $T(h)$ is an isomorphism of the vector space V onto itself, equality (18) shows that $T \circ a(H) \simeq T$.

2. Let ϕ be the automorphism of the group $\mathbf{C}$ acting as $\phi(x) = -x$. Let V be a finite-dimensional complex vector space. The map $F_\alpha: t \mapsto e^{t\alpha}$ is a linear

representation of $\mathbf{C}$ for every $\alpha \in \mathrm{L}(V)$ (see 0.6). Obviously, $F_\alpha \circ \phi = F_{-\alpha}$. The representations F_α and $F_{-\alpha}$ are isomorphic if and only if the matrices of the operators α and $-\alpha$ have the same Jordan form. (The latter in turn holds if and only if for any k and c in the Jordan form of the matrix of α, the number of Jordan blocks of order k with eigenvalue c is equal to the number of Jordan blocks of order k with eigenvalue $-c$).

Questions and Exercises

1.* Show that $\det e^A = e^{\operatorname{tr} A}$ for any matrix $A \in \mathrm{L}_n(\mathbf{R})$.

2. If F is as given below, show that F is a matrix representation of $\mathbf{R}$ and find a matrix A such that $F(t) = e^{tA}$:

a) $$F(t) = \begin{pmatrix} \cosh t & -\sinh t \\ \sinh t & \cosh t \end{pmatrix};$$

b) $$F(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

3. What is a one-dimensional matrix representation?

4. Let M be the matrix representation of the group S_n constructed in 0.2. Show that $\operatorname{tr} M(\sigma)$ is the number of fixed points of the permutation σ.

5. How many trivial matrix representations does an arbitrary group admit?

6. Show, without resorting to calculations, that $e^{C^{-1}AC} = C^{-1}e^A C$ for any matrices $A \in \mathrm{L}_n(\mathbf{R})$ and $C \in \mathrm{GL}_n(\mathbf{R})$.

7. Let $S: \mathbf{R} \to S(V)$ be one of the maps listed below, where V is the space of all polynomials with real coefficients and $t \in \mathbf{R}, f \in V$:

a) $(S(t)f)(x) = f(tx)$;

b) $(S(t)f)(x) = f(e^t x)$;

c) $(S(t)f)(x) = e^t f(x)$;

d) $(S(t)f)(x) = f(x) + t$;

e) $(S(t)f)(x) = e^t f(x + t)$.

Is S a linear representation?

8. Describe one of the equivalent matrix representations associated with the linear representation $S\colon \mathbf{R} \to \mathrm{GL}(V)$, where V is the space of polynomials of degree ≤ 3 and $(S(t)f)(x) = f(x-t)$ for $t \in \mathbf{R}, f \in V$.

9. Find all finite-dimensional linear representations of

a) $\mathbf{Z}$;

b) $\mathbf{Z}_m$.

10.* Find all differentiable finite-dimensional complex linear representations of

a) $G = \mathbf{R}^+$ (the multiplicative group of positive reals);

b) $G = \mathbf{T} = \{ z \in \mathbf{C}^* \mid |z| = 1 \}$.

11. Let s be an action of the group G on a set X and e the identity element of G. Show that $s(e)$ is the identity map of X.

12. Let $\hat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$. For any matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{R})$ put

$$s(A)x = \frac{ax+b}{cx+d} \qquad (x \in \hat{\mathbf{R}}),$$

with the convention that

$$\frac{a \cdot \infty + b}{c \cdot \infty + d} = \frac{a}{c} \quad \text{and} \quad \frac{u}{0} = \infty$$

for $u \neq 0$. Show that s is an action of $\mathrm{GL}_2(\mathbf{R})$ on $\hat{\mathbf{R}}$.

13. Write down an explicit formula for the right regular representation of the group G.

14.* Prove that the left regular representation of any group G is isomorphic to its right regular representation.

15. Show that every group admits a faithful linear representation.

16. Is every finite-dimensional complex representation of $\mathbf{Z}$ obtained by restricting a representation of $\mathbf{C}$ to $\mathbf{Z}$?

17. Let ϕ denote the automorphism of the group $\mathbf{Z}_m$ defined by the rule $\phi(x) = -x$. Find all complex finite-dimensional representations T of $\mathbf{Z}_m$ with the property that $T \circ \phi \simeq T$.