ALM 11

Advanced Lectures in Mathematics

Recent Advances in Geometric Analysis

几何分析最新进展

Editors: Yng-Ing Lee • Chang-Shou Lin • Mao-Pei Tsui

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ADVANCED LECTURES IN MATHEMATICS

Preface

"2007 International Conference in Geometric Analysis" was held in Taiwan University from June 18th to 22nd, 2007. This conference is sponsored by Mathematics Division, Center for Theoretical Sciences (NCTS) Taipei Office, Taida Institute for Mathematical Sciences (TIMS), Academia Sinica, Central University and Tsing-Hua University (Xinzu).

Geometric analysis studies functions, maps, tensors, and submanifolds governed by natural differential equations. A good understanding of these objects reveals important information of analytical and geometric structures, and has many implications in physics, algebraic geometry and topology.

In recent years, we have witnessed a great success of geometric analysis, the most important event being the solution of the Poincare conjecture by the Ricci flow. This shows the power of geometric partial differential equations in resolving some deepest problems in topology.

The aim of "2007 International Conference in Geometric Analysis" is to gather leading experts to discuss and exchange new progress and ideas on various topics in the field. This proceeding is an account on recent advances in geometric analysis and related equations, including Ricci flow, affine normal flow, geometric analysis on pseudoconvex hypersurfaces, Alexandrov space, manifolds with special holonomy, and singular plateau problem.

We would like to take this opportunity to thank Prof. Shing-Tung Yau for the support of publishing this proceeding. We also want to thanks all the authors for contributing this proceeding.

Yng-Ing Lee
Chang-Shou Lin
Mao-Pei Tsui
April 2009

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Recent Progress on Ricci Solitons[†]

Huai-Dong Cao*

Abstract

In recent years, there has seen much interest and increased research activities in Ricci solitons. Ricci solitons are natural generalizations of Einstein metrics. They are also special solutions to Hamilton's Ricci flow and play important roles in the singularity study of the Ricci flow. In this paper, we survey some of the recent progress on Ricci solitons.

2000 Mathematics Subject Classification: 53C21, 53C25.

Keywords and Phrases: Ricci soliton, singularity of Ricci flow, stability, Gaussian density.

The concept of *Ricci solitons* was introduced by Hamilton [65] in mid 80's. They are natural generalizations of Einstein metrics. Ricci solitons also correspond to self-similar solutions of Hamilton's Ricci flow [63] and often arise as limits of dilations of singularities in the Ricci flow [67, 11, 26, 92]. They can be viewed as fixed points of the Ricci flow, as a dynamical system, on the space of Riemannian metrics modulo diffeomorphisms and scalings. Ricci solitons are of interests to physicists as well and are called *quasi-Einstein* metrics in physics literature (see, e.g., [51]). In this paper, we survey some of the recent progress on Ricci solitons as well as the role they play in the singularity study of the Ricci flow. This paper can be regarded as an update of the article [14] written by the author a few years ago.

1 Ricci solitons

1.1 Ricci solitons

Recall that a Riemannian metric g_{ij} is *Einstein* if its Ricci tensor $R_{ij} = \rho g_{ij}$ for some constant ρ. A smooth n-dimensional manifold M^n with an Einstein metric

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g is an *Einstein manifold.* Ricci solitons, introduced by Hamilton [65], are natural generalizations of Einstein metrics.

Definition 1.1. A complete Riemannian metric g_{ij} on a smooth manifold M^n is called a *Ricci soliton* if there exists a smooth vector field $V = (V^i)$ such that the Ricci tensor R_{ij} of the metric g_{ij} satisfies the equation

$$R_{ij} + \frac{1}{2}(\nabla_i V_j + \nabla_j V_i) = \rho g_{ij}, \tag{1.1}$$

for some constant ρ. Moreover, if V is a gradient vector field, then we have a *gradient Ricci soliton*, satisfying the equation

$$R_{ij} + \nabla_i \nabla_j f = \rho g_{ij}, \tag{1.2}$$

for some smooth function f on M. For $\rho = 0$ the Ricci soliton is *steady*, for $\rho > 0$ it is *shrinking* and for $\rho < 0$ *expanding*. The function f is called a *potential function* of the Ricci soliton.

Since $\nabla_i V_j + \nabla_j V_i$ is the Lie derivative $L_V g_{ij}$ of the metric g in the direction of V, we also write the Ricci soliton equations (1.1) and (1.2) as

$$Rc + \frac{1}{2}L_V g = \rho g \quad \text{and} \quad Rc + \nabla^2 f = \rho g, \tag{1.3}$$

respectively.

When the underlying manifold is a complex manifold, we have the corresponding notion of Kähler-Ricci solitons.

Definition 1.2. A complete Kähler metric $g_{\alpha\bar{\beta}}$ on a complex manifold X^n of complex dimension n is called a *Kähler-Ricci soliton* if there exists a holomorphic vector field $V = (V^\alpha)$ on X such that the Ricci tensor $R_{\alpha\bar{\beta}}$ of the metric $g_{\alpha\bar{\beta}}$ satisfies the equation

$$R_{\alpha\bar{\beta}} + \frac{1}{2}(\nabla_{\bar{\beta}} V_\alpha + \nabla_\alpha V_{\bar{\beta}}) = \rho g_{\alpha\bar{\beta}} \tag{1.4}$$

for some (real) constant ρ. It is called a gradient Kähler-Ricci soliton if the holomorphic vector field V comes from the gradient vector field of a real-valued function f on X^n so that

$$R_{\alpha\bar{\beta}} + \nabla_\alpha \nabla_{\bar{\beta}} f = \rho g_{\alpha\bar{\beta}}, \quad \text{and} \quad \nabla_\alpha \nabla_\beta f = 0. \tag{1.5}$$

Again, for $\rho = 0$ the soliton is *steady*, for $\rho > 0$ it is shrinking and for $\rho < 0$ expanding.

Note that the case $V = 0$ (i.e., f being a constant function) is an Einstein (or Kähler-Einstein) metric. Thus Ricci solitons are natural extensions of Einstein metrics. In fact, we will see below that there are no non-Einstein compact steady or expanding Ricci solitons. Also, by a suitable scale of the metric g, we can normalize $\rho = 0, +1/2$, or $-1/2$.

Lemma 1.1. (Hamilton [68]) *Let g_{ij} be a complete gradient Ricci soliton with potential function f. Then we have*

$$R + |\nabla f|^2 - 2\rho f = C \tag{1.6}$$

for some constant C. Here R denotes the scalar curvature.

Proof. Let g_{ij} be a complete gradient Ricci soliton on a manifold M^n so that there exists a potential function f such that the soliton equation (1.2) holds. Taking the covariant derivatives and using the commutating formula for covariant derivatives, we obtain

$$\nabla_i R_{jk} - \nabla_j R_{ik} + R_{ijkl}\nabla_l f = 0.$$

Taking the trace on j and k, and using the contracted second Bianchi identity

$$\nabla_j R_{ij} = \frac{1}{2}\nabla_i R,$$

we get

$$\nabla_i R = 2R_{ij}\nabla_j f. \tag{1.7}$$

Thus

$$\nabla_i(R + |\nabla f|^2 - 2\rho f) = 2(R_{ij} + \nabla_i\nabla_j f - \rho g_{ij})\nabla_j f = 0.$$

Therefore

$$R + |\nabla f|^2 - 2\rho f = C$$

for some constant C. □

Proposition 1.1. (cf. Hamilton [68], Ivey [71]) *On a compact manifold M^n, a gradient steady or expanding Ricci soliton is necessarily an Einstein metric.*

Proof. Taking the trace in Equation (1.2), we get

$$R + \Delta f = n\rho. \tag{1.8}$$

Taking the difference of (1.6) in Lemma 1.1 and (1.8), we get

$$\Delta f - |\nabla f|^2 + 2\rho f = n\rho - C.$$

When M is compact and $\rho \le 0$, it follows from the maximum principle that f must be a constant and hence g_{ij} is a Einstein metric. □

More generally, we have

Proposition 1.2. *Any compact steady or expanding Ricci soliton must be Einstein.*

Proof. This follows from Proposition 1.1 and Perelman's results that any compact Ricci soliton is necessarily a gradient soliton (see Propositions 2.1 $\sim$ 2.4). □

For compact shrinking Ricci solitons in low dimensions, we have

Proposition 1.3. (Hamiton [65][1] for $n = 2$, Ivey [71][2] for $n = 3$) *In dimension $n \le 3$, there are no compact shrinking Ricci solitons other than those of constant positive curvature.*

[1] See alternative proofs in (Proposition 5.21, [32]) or (Proposition 5.1.10, [18]), and [31].
[2] See [45] for alternative proofs

1.2 Examples of Ricci solitons

When $n \geq 4$, there exist nontrivial compact gradient shrinking solitons. Also, there exist complete noncompact Ricci solitons (steady, shrinking and expanding) that are not Einstein. Below we list a number of such examples. It turns out most of the examples are rotationally symmetric and gradient, and all the known examples of nontrivial *shrinking* solitons so far are Kähler.

Example 1.1. (Compact gradient Kähler shrinkers) For real dimension 4, the first example of a compact shrinking soliton was constructed in early 90's by Koiso [73] and the author [11][3] on compact complex surface $\mathbb{CP}^2\#(-\mathbb{CP}^2)$, where $(-\mathbb{CP}^2)$ denotes the complex projective space with the opposite orientation. This is a gradient Kähler-Ricci soliton, has $U(2)$ symmetry and positive Ricci curvature. More generally, they found $U(n)$-invariant Kähler-Ricci solitons on twisted projective line bundle over $\mathbb{CP}^{n-1}$ for all $n \geq 2$.

Remark 1.1. If a compact Kähler manifold M admits a non-trivial Kähler shrinker then M is Fano (i.e., the first Chern class $c_1(M)$ of M is positive), and the Futaki-invariant [52] is nonzero.

Example 1.2. (Compact toric gradient Kähler shrinkers) In [97], Wang-Zhu found a gradient Kähler-Ricci soliton on $\mathbb{CP}^2\#2(-\mathbb{CP}^2)$ which has $U(1)\times U(1)$ symmetry. More generally, they proved the existence of gradient Kähler-Ricci solitons on all Fano toric varieties of complex dimension $n \geq 2$ with non-vanishing Futaki invariant.

Example 1.3. (Noncompact gradient Kähler shrinkers) Feldman-Ilmanen-Knopf [48] found the first complete noncompact $U(n)$-invariant shrinking gradient Kähler-Ricci solitons, which are cone-like at infinity and have quadratic decay in the curvature. It has positive scalar curvature but the Ricci curvature doesn't have a fixed sign.

Example 1.4. (The cigar soliton) In dimension two, Hamilton [65] discovered the first example of a complete noncompact steady soliton on $\mathbb{R}^2$, called the *cigar soliton*, where the metric is given by

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}$$

with potential function

$$f = -\log(1 + x^2 + y^2).$$

The cigar has positive (Gaussian) curvature and linear volume growth, and is asymptotic to a cylinder of finite circumference at ∞.

Example 1.5. (The Bryant soliton) In the Riemannian case, higher dimensional examples of noncompact gradient steady solitons were found by Robert Bryant [6] on $\mathbb{R}^n$ ($n \geq 3$). They are rotationally symmetric and have positive

[3]The author's construction was carried out in 1991 at Columbia University. When he told his construction to S. Bando that year in New York, he also learned the work of Koiso from Bando.

sectional curvature. Furthermore, the geodesic sphere S^{n-1} of radius s has the diameter on the order $\sqrt{s}$. Thus the volume of geodesic balls $B_r(0)$ grow on the order of $r^{(n+1)/2}$.

Example 1.6. (Noncompact gradient steady Kähler solitons) In the Kähler case, the author [11] found two examples of complete rotationally noncompact gradient steady Kähler-Ricci solitons

(a) On $\mathbb{C}^n$ (for $n = 1$ it is just the cigar soliton). These examples are $U(n)$ invariant and have positive sectional curvature. It is interesting to point out that the geodesic sphere S^{2n-1} of radius s is an S^1-bundle over $\mathbb{CP}^{n-1}$ where the diameter of S^1 is on the order 1, while the diameter of $\mathbb{CP}^{n-1}$ is on the order $\sqrt{s}$. Thus the volume of geodesic balls $B_r(0)$ grow on the order of r^n, n being the complex dimension. Also, the curvature $R(x)$ decays like $1/r$.

(b) On the blow-up of $\mathbb{C}^n/\mathbb{Z}_n$ at the origin. This is the same space on which Eguchi-Hansen [44] ($n = 2$) and Calabi [9] ($n \geq 2$) constructed examples of Hyper-Kähler metrics. For $n = 2$, the underlying space is the canonical line bundle over $\mathbb{CP}^1$.

Example 1.7. (Noncompact gradient expanding Kähler solitons) In [12], the author constructed a one-parameter family of complete noncompact expanding solitons on $\mathbb{C}^n$. These expanding Kähker-Ricci solitons all have $U(n)$ symmetry and positive sectional curvature, and are cone-like at infinity.

More examples of complete noncompact Kähler-Ricci expanding solitons were found by Feldman-Ilmanen-Knopf [48] on "blow-ups" of $\mathbb{C}^n/\mathbb{Z}_k$, $k = n+1, n+2, \ldots$. (See also Pedersen et al [84].)

Example 1.8. (Sol and Nil solitons) Non-gradient expanding Ricci solitons on Sol and Nil manifolds were constructed by J. Lauret [75] and Baird-Laurent [2].

Example 1.9. (Warped products) Using doubly warped product and multiple warped product constructions, Ivey [72] and Dancer-Wang [41] produced noncompact gradient steady solitons, which generalize the construction of Bryant's soliton. Also, Gastel-Kronz [56] produced a two-parameter family (doubly warped product metrics) of gradient expanding solitons on $\mathbb{R}^{m+1} \times N$, where N^n ($n \geq 2$) is an Einstein manifold with positive scalar curvature.

Example 1.10. Very recently, Dancer-Wang [40] produced new examples of gradient shrinking, steady and expanding Kähler solitons on bundles over the product of Fano Kähler-Einstein manifolds, generalizing those in Example 1.1, 1.3, 1.6, 1.7 and those by Pedersen et al [84].

We conclude our examples with

Example 1.11. (Gaussian solitons) $(\mathbb{R}^n, g_0)$ with the flat Euclidean metric can be also equipped with both shrinking and expanding gradient Ricci solitons, called the Gaussian shrinker or expander.

(a) $(\mathbb{R}^n, g_0, |x|^2/4)$ is a gradient shrinker with potential function $f = |x|^2/4$:

$$Rc + \nabla^2 f = \frac{1}{2} g_0.$$

(b) $(\mathbb{R}^n, g_0, -|x|^2/4)$ is a gradient expander with potential function $f = -|x|^2/4$:

$$Rc + \nabla^2 f = -\frac{1}{2} g_0.$$

Remark 1.2. We'll see later that the Gaussian shrinker is very special because it has the largest reduced volume $\tilde{V} = 1$ (see Section 4.2)

2 Variational structures

In this section we describe Perelman's $\mathcal{F}$-functional and $\mathcal{W}$-functional and the associated λ-energy and ν-energy respectively. The critical points of the λ-energy (respectively ν-energy) are precisely given by compact gradient steady (respectively shrinking) solitons. We also consider the $\mathcal{W}_-$-functional and the corresponding ν_--energy introduced by Feldman-Ilmanen-Ni [49] whose critical points are expanding solitons. Throughout this section we assume that M^n is a compact smooth manifold.

2.1 The F-functional and λ-energy

In [85] Perelman considered the functional

$$\mathcal{F}(g_{ij}, f) = \int_M (R + |\nabla f|^2) e^{-f} dV$$

defined on the space of Riemannian metrics and smooth functions on M. Here R is the scalar curvature and f is a smooth function on M^n. Note that when $f = 0$, F is simply the total scalar curvature of g, or the Einstein-Hilbert action on the space of Riemannian metrics on M.

Lemma 2.1. (First variation formula of F-functional, Perelman [85]) *If $\delta g_{ij} = v_{ij}$ and $\delta f = \phi$ are variations of g_{ij} and f respectively, then the first variation of $\mathcal{F}$ is given by*

$$\delta \mathcal{F}(v_{ij}, \phi) = \int_M \left[-v_{ij}(R_{ij} + \nabla_i \nabla_j f) + \left(\frac{v}{2} - \phi\right)(2\Delta f - |\nabla f|^2 + R) \right] e^{-f} dV$$

where $v = g^{ij} v_{ij}$.

Next we consider the associated energy

$$\lambda(g_{ij}) = \inf \left\{ \mathcal{F}(g_{ij}, f) : f \in C^\infty(M), \int_M e^{-f} dV = 1 \right\}.$$

Clearly $\lambda(g_{ij})$ is invariant under diffeomorphisms. If we set $u = e^{-f/2}$, then the functional $\mathcal{F}$ can be expressed as

$$\mathcal{F} = \int_M (Ru^2 + 4|\nabla u|^2) dV.$$

Thus

$$\lambda(g_{ij}) = \inf\left\{\int_M (Ru^2 + 4|\nabla u|^2)dV : \int_M u^2 dV = 1\right\},$$

the first eigenvalue of the operator $-4\Delta + R$. Let $u_0 > 0$ be a first eigenfunction of the operator $-4\Delta + R$ so that

$$-4\Delta u_0 + Ru_0 = \lambda(g_{ij})u_0.$$

Then $f_0 = -2\log u_0$ is a minimizer of $\lambda(g_{ij})$:

$$\lambda(g_{ij}) = \mathcal{F}(g_{ij}, f_0).$$

Note that f_0 satisfies the equation

$$-2\Delta f_0 + |\nabla f_0|^2 - R = \lambda(g_{ij}). \tag{2.1}$$

For any symmetric 2-tensor $h = h_{ij}$, consider the variation $g_{ij}(s) = g_{ij} + sh_{ij}$. It is an easy consequence of Lemma 2.1 and Eq. (2.1) that the first variation $\mathcal{D}_g\lambda(h)$ of $\lambda(g_{ij})$ is given by

$$\left.\frac{d}{ds}\right|_{s=0} \lambda(g_{ij}(s)) = \int -h_{ij}(R_{ij} + \nabla_i\nabla_j f)e^{-f}dV, \tag{2.2}$$

where f is a minimizer of $\lambda(g_{ij})$. In particular, *the critical points of λ are precisely steady gradient Ricci solitons.*

Note that, by diffeomorphism invariance of λ, $\mathcal{D}_g\lambda$ vanishes on any Lie derivative $h_{ij} = \frac{1}{2}L_V g_{ij}$, and hence on $\nabla_i\nabla_j f = \frac{1}{2}L_{\nabla f}g_{ij}$. Thus, by inserting $h = -2(Ric + \nabla^2 f)$ in Eq. (2.2) one recovers the following result of Perelman [85].

Proposition 2.1. *Suppose $g_{ij}(t)$ is a solution to the Ricci flow on a compact manifold M^n. Then $\lambda(g_{ij}(t))$ is nondecreasing in t and the monotonicity is strict unless we are on a steady gradient soliton. In particular, a (compact) steady Ricci soliton is necessarily a gradient soliton.*

We remark that by considering the quantity

$$\bar{\lambda}(g_{ij}) = \lambda(g_{ij})(\mathrm{Vol}(g_{ij}))^{\frac{2}{n}},$$

which is a scale invariant version of $\lambda(g_{ij})$, Perelman [85] also showed the following result.

Proposition 2.2. *$\bar{\lambda}(g_{ij})$ is nondecreasing along the Ricci flow whenever it is nonpositive; moreover, the monotonicity is strict unless we are on a gradient expanding soliton. In particular, any (compact) expanding Ricci soliton is necessarily a gradient soliton.*

2.2 The $\mathcal{W}$-functional and ν-energy

In order to study shrinking Ricci solitons, Perelman [85] introduced the $\mathcal{W}$-functional

$$\mathcal{W}(g_{ij}, f, \tau) = \int_M [\tau(R + |\nabla f|^2) + f - n](4\pi\tau)^{-\frac{n}{2}} e^{-f} dV,$$

where g_{ij} is a Riemannian metric, f a smooth function on M^n, and τ a positive scale parameter. Clearly the functional $\mathcal{W}$ is invariant under simultaneous scaling of τ and g_{ij} (or equivalently the parabolic scaling), and invariant under diffeomorphism. Namely, for any positive number a and any diffeomorphism φ we have

$$\mathcal{W}(a\varphi^* g_{ij}, \varphi^* f, a\tau) = \mathcal{W}(g_{ij}, f, \tau).$$

Lemma 2.2. (First variation of $\mathcal{W}$-functional, Perelman [85]) *If $v_{ij} = \delta g_{ij}$, $\phi = \delta f$, and $\eta = \delta\tau$, then*

$$\begin{aligned}\delta\mathcal{W}(v_{ij}, \phi, \eta) = &\int_M -\tau v_{ij}\left(R_{ij} + \nabla_i f \nabla_j f - \frac{1}{2\tau} g_{ij}\right)(4\pi\tau)^{-\frac{n}{2}} e^{-f} dV \\ &+ \int_M \left(\frac{v}{2} - \phi - \frac{n}{2\tau}\eta\right)[\tau(R + 2\Delta f - |\nabla f|^2) \\ &+ f - n - 1](4\pi\tau)^{-\frac{n}{2}} e^{-f} dV \\ &+ \int_M \eta\left(R + |\nabla f|^2 - \frac{n}{2\tau}\right)(4\pi\tau)^{-\frac{n}{2}} e^{-f} dV.\end{aligned}$$

Here $v = g^{ij} v_{ij}$ as before.

Similar to the λ-energy, we can consider

$$\mu(g_{ij}, \tau) = \inf\{\mathcal{W}(g_{ij}, f, \tau) : f \in C^\infty(M), (4\pi\tau)^{-\frac{n}{2}} \int_M e^{-f} dV = 1\}. \qquad (2.3)$$

Note that if we let $u = e^{-f/2}$, then the functional $\mathcal{W}$ can be expressed as

$$\mathcal{W}(g_{ij}, f, \tau) = \int_M [\tau(Ru^2 + 4|\nabla u|^2]) - u^2 \log u^2 - nu^2](4\pi\tau)^{-\frac{n}{2}} dV,$$

and the constraint $\int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV = 1$ becomes $\int_M u^2 (4\pi\tau)^{-\frac{n}{2}} dV = 1$. Therefore $\mu(g_{ij}, \tau)$ corresponds to the best constant of a logarithmic Sobolev inequality.

Since the nonquadratic term is subcritical (in view of Sobolev exponent), it is rather straightforward to show that $\mu(g_{ij}, \tau)$ is achieved by some nonnegative function $u \in H^1(M)$ which satisfies the Euler-Lagrange equation

$$\tau(-4\Delta u + Ru) - 2u \log u - nu = \mu(g_{ij}, \tau)u. \qquad (2.4)$$

One can further show that the minimizer u is positive and smooth (see Rothaus [89]). This is equivalent to say that $\mu(g_{ij}, \tau)$ is achieved by some minimizer f satisfying the nonlinear equation

$$\tau(2\Delta f - |\nabla f|^2 + R) + f - n = \mu(g_{ij}, \tau). \qquad (2.5)$$

Proposition 2.3. (Perelman [85]) *Suppose $g_{ij}(t)$, $0 \leq t < T$ is a solution to the Ricci flow on a compact manifold M^n. Then $\mu(g_{ij}(t), T-t)$ is nondecreasing in t; moveover, the monotonicity is strict unless we are on a shrinking gradient soliton. In particular, any (compact) shrinking Ricci soliton is necessarily a gradient soliton.*

Remark 2.1. Recently, Naber [78] has shown that if (M^n, g) is a complete non-compact shrinking Ricci soliton with bounded curvature $|Rm| < C$ with respect to some smooth vector field V, then there exists a smooth function f on M such that (M^n, g) is a gradient soliton with f as a potential function. This in particular means that $V = \nabla f + X$ for some Killing field X on M.

The associated ν-energy is defined by

$$\nu(g_{ij}) = \inf\left\{\mathcal{W}(g, f, \tau) : f \in C^\infty(M), \tau > 0, (4\pi\tau)^{-\frac{n}{2}} \int e^{-f} dV = 1\right\}.$$

One checks that $\nu(g_{ij})$ is realized by a pair (f, τ) that solve the equations

$$\tau(-2\Delta f + |Df|^2 - R) - f + n + \nu = 0, \qquad (4\pi\tau)^{-\frac{n}{2}} \int f e^{-f} = \frac{n}{2} + \nu. \quad (2.6)$$

Consider variations $g_{ij}(s) = g_{ij} + sh_{ij}$ as before. Using Lemma 2.2 and (2.6), one calculates the first variation $\mathcal{D}_g\nu(h)$ to be

$$\left.\frac{d}{ds}\right|_{s=0} \nu(g_{ij}(s)) = (4\pi\tau)^{-\frac{n}{2}} \int -h_{ij}[\tau(R_{ij} + \nabla_i\nabla_j f) - \frac{1}{2}g_{ij}]e^{-f} dV.$$

A stationary point of ν thus satisfies

$$R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij} = 0,$$

which says that g_{ij} is a gradient shrinking Ricci soliton.

As before, $\mathcal{D}_g\nu(h)$ vanishes on Lie derivatives. By scale invariance it also vanishes on multiplies of the metric. Inserting $h_{ij} = -2(R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij})$, one recovers Perelman's formula that finds that $\nu(g_{ij}(t))$ is monotone on a Ricci flow, and constant if and only if $g_{ij}(t)$ is a gradient shrinking Ricci soliton.

2.3 The $\mathcal{W}_-$-functional and ν_--energy

In [49], Feldman-Ilmanen-Ni introduced the dual $\mathcal{W}_-$-functional (corresponding to expanders)

$$\mathcal{W}_-(g_{ij}, f, \sigma) = \int_M [\sigma(R + |\nabla f|^2) - (f - n)](4\pi\sigma)^{-\frac{n}{2}} e^{-f} dV,$$

the μ_--energy

$$\mu_-(g_{ij}, \sigma) = \inf\left\{\mathcal{W}_-(g_{ij}, f, \tau) : f \in C^\infty(M), (4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f} dV = 1\right\},$$

and the corresponding ν_--entropy

$$\nu_-(g_{ij}) = \sup_{\sigma>0}\{\mu_-(g_{ij},\sigma)\}.$$

Here, σ is a positive parameter. They proved that

Proposition 2.4. (Feldman-Ilmanen-Ni [49])
(a) *$\mu_-(g_{ij},\sigma)$ is achieved by a unique f; $\mu_-(g_{ij}(t),t-t_0)$ is nondecreasing under the Ricci flow; moveover, the monotonicity is strict unless we are on an expanding gradient soliton.*
(b) *If $\lambda(g)<0$, then $\nu_-(g_{ij})$ is achieved by a unique σ; $\nu_-(g_{ij}(t))$ is nondecreasing under the Ricci flow, and is constant only on an expanding soliton.*

Furthermore, if $\lambda(g) < 0$ then ν_- is achieved by a unique pair (f,σ) that solve the equations

$$\sigma(-2\Delta f+|Df|^2-R)+f-n+\nu_-=0, \qquad (4\pi\tau)^{-\frac{n}{2}}\int fe^{-f}=\frac{n}{2}-\nu_-.$$

3 Ricci solitons and Ricci flow

3.1 Ricci solitons as self-similar solutions of the Ricci flow

Let us first examine how Einstein metrics behave under Hamilton's Ricci flow

$$\frac{\partial g_{ij}(t)}{\partial t} = -2R_{ij}(t).$$

If the initial metric is Ricci flat, so that $R_{ij}=0$ at $t=0$, then clearly the metric does not change under the Ricci flow: $g_{ij}(t)=g_{ij}(0)$. Hence any Ricci flat metric is a stationary solution. This happens, for example, on a flat torus or on any $K3$-surface with a Calabi-Yau metric.

If the initial metric $g_{ij}(0)$ is Einstein with positive scalar curvature, then the metric will shrink under the Ricci flow by a time-dependent factor. Indeed, if at $t=0$ we have

$$R_{ij}(0)=\frac{1}{2}g_{ij}(0).$$

Then

$$g_{ij}(t)=(1-t)g_{ij}(0), \tag{3.1}$$

which shrinks homothetically to a point as $t\to T=1$, while the scalar curvature $R\to\infty$ like $1/(T-t)$ as $t\to T$. Note that $g(t)$ exists for $t\in(-\infty,T)$, hence an *ancient solution.*

By contrast, if the initial metric is an Einstein metric of negative scalar curvature, the metric will expand homothetically for all times. Suppose

$$R_{ij}(0)=-\frac{1}{2}g_{ij}(0)$$

at $t = 0$. Then the solution to the Ricci flow is given by

$$g_{ij}(t) = (1+t)g_{ij}(0).$$

Hence the evolving metric $g_{ij}(t)$ exists and expands homothetically for all time, and the curvature will fall back to zero like $-1/t$. Note that now the evolving metric $g_{ij}(t)$ only goes back in time to -1, when the metric explodes out of a single point in a "big bang".

Now suppose we have a one-parameter group of diffeomorphisms φ_t, $-\infty < t < \infty$, which is generated by some vector field V on M, and suppose $g_{ij}(t) = \varphi_t^* \hat{g}_{ij}$ is a slution to the Ricci flow, called a *self-similar slution*, with initial metric $\hat{g}_{ij}$. Then

$$-2Rc = \mathcal{L}_V g$$

for all t. In particular, the initial metric $g_{ij}(0) = \hat{g}_{ij}$ satisfies the steady Ricci soliton equation in (1.3).

Conversely, suppose we have a steady Ricci soliton $\hat{g} = (\hat{g}_{ij})$ on a smooth manifold M^n so that

$$2\hat{Rc} + \mathcal{L}_V \hat{g} = 0,$$

for some smooth vector field $V = (V^i)$. *Assume the vector field V is complete* (i.e., V generates a one-parameter group of diffeomorphisms φ_t of M). Then clearly

$$g_{ij}(t) = \varphi_t^* \hat{g}_{ij} \qquad -\infty < t < \infty,$$

is a self-similar solution of the Ricci flow with $\hat{g}_{ij}$ as the initial metric.

More generally, we can consider self-similar solutions to the Ricci flow which move by diffeomorphisms and also shrinks or expands by a (time-dependent) factor at the same time. Such self-similar solutions correspond to either shrinking or expanding Ricci solitons $(M, \hat{g}, V)$ with the vector filed V being complete. For example, a shrinking gradient Ricci soliton satisfying the equation

$$\hat{R}_{ij} + \nabla_i \nabla_j \hat{f} - \frac{1}{2}\hat{g}_{ij} = 0,$$

with $V = \nabla \hat{f}$ complete, corresponds to the self-similar Ricci flow solution $g_{ij}(t)$ of the form

$$g_{ij}(t) := (1-t)\varphi_t^*(g_{ij}), \qquad t < 1, \tag{3.2}$$

where φ_t are the diffeomorphisms generated by $V/(1-t)$. (Compare Eq. (3.2) with Eq. (3.1) for $\rho = 1/2$.)

Thus, we see a complete gradient Ricci soliton with respect to some complete vector field corresponds to the self-similar solution of the Ricci flow it generates. For this reason we often do not distinguish the two.

Remark 3.1. If M^n is compact, then V is always complete. But if M is noncompact then V may not be complete in general. Recently Z.-H. Zhang [102] has observed that for any **complete** gradient (steady, shrinking, or expanding) Ricci soliton g_{ij} with potential function f, $V = \nabla f$ is a complete vector field on M.

In particular, *a complete gradient Ricci soliton always corresponds to the self-similar solution of the Ricci flow it generates*