天元基金影印数学丛书

KdV & KAM

KdV方程和KAM理论

（影印版）

高等教育出版社
HIGHER EDUCATION PRESS

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序　　言

为了更好地借鉴国外数学教育与研究的成功经验，促进我国数学教育与研究事业的发展，提高高等学校数学教育教学质量，本着“为我国热爱数学的青年创造一个较好的学习数学的环境”这一宗旨，天元基金赞助出版“天元基金影印数学丛书”。

该丛书主要包含国外反映近代数学发展的纯数学与应用数学方面的优秀书籍，天元基金邀请国内各个方向的知名数学家参与选题的工作，经专家遴选、推荐，由高等教育出版社影印出版。为了提高我国数学研究生教学的水平，暂把选书的目标确定在研究生教材上。当然，有的书也可作为高年级本科生教材或参考书，有的书则介于研究生教材与专著之间。

欢迎各方专家、读者对本丛书的选题、印刷、销售等工作提出批评和建议。

天元基金领导小组

2007 年 1 月

In
memory
of

JÜRGEN MOSER

teacher
mentor
friend

Preface

This book is concerned with two aspects of the theory of integrable partial differential equations. The first aspect is a *normal form theory* for such equations, which we exemplify by the periodic *K*orteweg *d*e *V*ries equation – undoubtedly one of the most important nonlinear, integrable pdes. This makes for the 'KdV' part of the title of the book.

The second aspect is a theory for Hamiltonian perturbations of such pdes. Its prototype is the so called *KAM theory*, developed for finite dimensional systems by *K*olmogorov, *A*rnold and *M*oser. This makes for the 'KAM' part of the title of the book.

To be more specific, our starting point is the periodic KdV equation considered as an infinite dimensional, integrable Hamiltonian system admitting a complete set of independent integrals in involution. We show that this leads to a single, global, real analytic system of Birkhoff coordinates – the cartesian version of action-angle coordinates –, such that the KdV Hamiltonian becomes a function of the actions alone. In fact, these coordinates work simultaneously for all Hamiltonians in the KdV hierarchy.

While the existence of *global* Birkhoff coordinates is a special feature of KdV, *local* Birkhoff coordinates may be constructed via our approach for many integrable pdes anywhere in phase space. Specifically this holds true for the defocusing nonlinear Schrödinger equation, for which parallel results were developed in [51].

The global coordinates make it evident that all solutions of the periodic KdV equation are periodic, quasi-periodic, or almost-periodic in time. It also provides a convenient handle to study small Hamiltonian perturbations, by applying a suitable generalization of the KAM theory to partial differential operators. To check the pertinent nondegeneracy conditions, we construct Birkhoff normal forms up to order six to gain sufficient control over the KdV frequencies as functions of the actions. In fact, these Birkhoff normal forms are just the first terms in the power series expansion of the KdV Hamiltonian in Birkhoff coordinates.

Finally, we describe the set up, assumptions and conclusions of a general infinite dimensional KAM theorem, that is applicable here and goes back to Kuksin. The situation differs from more conventional applications of KAM to pdes in that the perturbations are given by unbounded operators. This is only partially compensated by a smoothing effect of the small divisors. In addition, one has to modify the iteration scheme and use normal forms which also depend on angular variables.

Only recently, monographs on KAM theory for integrable pdes appeared, by Bourgain [17], Craig [29], and Kuksin [75]. Of these, the first two choose a different approach, setting up a functional equation and applying a Lyapunov-Schmidt decomposition scheme pioneered by Craig & Wayne [28]. The latter employs a normal form theory for Lax-integrable pdes near finite dimensional tori, which is based on the Its-Matveev formula. In contrast, the normal form theory presented in this book with its global features goes much further. It allows us to obtain a perturbation theory for KdV from an abstract KAM theorem of a particularly simple form, and to use Birkhoff normal forms to check the relevant nondegeneracy conditions. Moreover, this normal form might turn out to be useful for other long time stability results for perturbed integrable pdes such as Nekhoroshev estimates.

This book is not only intended for the handful of specialists working at the intersection of integrable partial differential equations and Hamiltonian perturbation theory, but also researchers farther away from these fields. In fact, it is our intention to reach out to graduate students as well. It is for this reason that first of all, we have included a chapter on the classical theory, describing the finite dimensional background of integrable Hamiltonian systems and their perturbation theory according to the theory initiated by Kolmogorov, Arnold and Moser.

Secondly, we made the book self-contained, omitting only those proofs which can be found in well known textbooks. We therefore included numerous appendices – some of them, we hope, of independent interest – on topics from complex analysis on Hilbert spaces, spectral theory of Schrödinger operators, Riemann surface theory, representation of holomorphic differentials, and certain aspects of the KdV equation such as the KdV hierarchy and new formulas for the KdV frequencies.

Thirdly, we wrote the book in a modular manner, where each of its five main chapters – chapters II to VI – as well as its appendices may be read independently of each other. Every chapter has its own introduction, and the notation is explained. As a result, there is some natural repetition and overlap among them. Moreover, the results of these chapters are summarized in the very first chapter, titled "The Beginning", and here too we took the liberty to quote from the introductions to the later chapters. We consider these repetitions a benefit for the reader rather than a nuisance, since it allows him, or her, to peruse the material in a nonlinear manner.

This book took many years to complete, and during this long time we benefitted from discussions and collaborations with many friends and colleagues. We would like to thank all of them, in particular Benoît Grébert, with whom we developped parallel results for the defocusing nonlinear Schrödinger equation in [51], and Jürg Kramer, for his contribution to the nondegeneracy result for the first KdV Hamiltonian. Most of all we are indebted to Jürgen Moser, who initiated this joint effort and never failed to encourage us as long as he was able to do so. We dedicate this book to him.

The second author also gratefully acknowledges the hospitality of the Forschungsinstitut at the ETH Zürich and the Institute of Mathematics at the University of Zürich during many periods of our collaborative efforts, as well as the support of the Deutsche Forschungsgemeinschaft, while the first author gratefully acknowledges the support of the Swiss National Science Foundaton and of the European Research Training Network HPRN-CT-1999-00118.

Finally we would like to thank Jules Hobbes for his never tiring TEXpertise from the very first lines through many, many revisions up to the final, press-ready output, and Jürgen Jost and Springer Verlag for their pleasant cooperation to make this book happen.

Last, but not least we thank our families for their patience and support during these many years.

Zürich/Stuttgart
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I

The Beginning

1 Overview

In this book we consider the Korteweg-de Vries (KdV) equation

$$u_t = -u_{xxx} + 6uu_x.$$

The KdV equation is an evolution equation in one space dimension which is named after the two Dutch mathematicians Korteweg and de Vries [66] – see also Boussinesq [18] and Rayleigh [113]. It was proposed as a model equation for long surface waves of water in a narrow and shallow channel. Their aim was to obtain as solutions solitary waves of the type discovered in nature by Russell [114] in 1834. Later it became clear that this equation also models waves in other homogeneous, weakly nonlinear and weakly dispersive media. Since the mid-sixties the KdV equation received a lot of attention in the aftermath of the computational experiments of Kruskal and Zabusky [69], which lead to the discovery of the interaction properties of the solitary wave solutions and in turn to the understanding of KdV as an infinite dimensional integrable Hamiltonian system.

Our purpose here is to study small Hamiltonian perturbations of the KdV equation with periodic boundary conditions. In the unperturbed system *all* solutions are periodic, quasi-periodic, or almost-periodic in time. The aim is to show that large families of periodic and quasi-periodic solutions persist under such perturbations. This is true not only for the KdV equation itself, but in principle for all equations in the KdV hierarchy. As an example, the second KdV equation will also be considered.

The KdV Equation

Let us recall those features of the KdV equation that are essential for our purposes. It was observed by Gardner [46], see also Faddeev & Zakharov [40], that the KdV equation can be written in the Hamiltonian form

$$\frac{\partial u}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial H}{\partial u}$$

with the Hamiltonian

$$H(u) = \int_{S^1} \left(\tfrac{1}{2}u_x^2 + u^3\right) \mathrm{d}x,$$

where $\partial H/\partial u$ denotes the L^2-gradient of H, representing the Fréchet derivative of H with respect to the standard scalar product on L^2. Since we are interested in *spatially periodic solutions*, we take as the underlying phase space the Sobolev space

$$\mathcal{H}^N = H^N(S^1; \mathbb{R}), \qquad S^1 = \mathbb{R}/\mathbb{Z},$$

of real valued functions with period 1, where $N \geq 1$ is an integer, and endow it with the Poisson bracket proposed by Gardner,

$$\{F, G\} = \int_{S^1} \frac{\partial F}{\partial u(x)} \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial G}{\partial u(x)} \, \mathrm{d}x.$$

Here, F and G are differentiable functions on $\mathcal{H}^N$ with L^2-gradients in $\mathcal{H}^1$. This makes $\mathcal{H}^N$ a Poisson manifold, on which the KdV equation may also be represented in the form $u_t = \{u, H\}$ familiar from classical mechanics.

We note that the initial value problem for the KdV equation on the circle S^1 is well posed on every Sobolev space $\mathcal{H}^N$ with $N \geq 1$: for initial data $u^0 \in \mathcal{H}^N$ it has been shown by Temam for $N = 1, 2$ [128] and by Saut & Temam for any real $N \geq 2$ [121] that there exists a unique solution evolving in $\mathcal{H}^N$ and defined globally in time. For further results on the initial value problem see for instance [78, 88, 126] as well as the more recent results [14, 15, 64].

The KdV equation admits infinitely many conserved quantities, or *integrals*, in involution, and there are many ways to construct such integrals [46, 94, 95]. Lax [77] obtained a set of Poisson commuting integrals in a particularly elegant way by considering the spectrum of an associated Schrödinger operator. For

$$u \in \mathcal{H}^0 = L^2 = L^2(S^1, \mathbb{R}),$$

consider the differential operator

$$L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u$$

on the interval $[0, 2]$ of *twice* the length of the period of u with periodic boundary conditions. It is well known [80, 82, 84] that its spectrum, denoted $\mathrm{spec}(u)$, is pure point and consists of an unbounded sequence of *periodic eigenvalues*

$$\lambda_0(u) < \lambda_1(u) \leq \lambda_2(u) < \lambda_3(u) \leq \lambda_4(u) < \cdots.$$

Equality or inequality may occur in every place with a '$\leq$'-sign, and one speaks of the *gaps* $(\lambda_{2n-1}(u), \lambda_{2n}(u))$ of the *potential* u and its *gap length*

$$\gamma_n(u) = \lambda_{2n}(u) - \lambda_{2n-1}(u), \qquad n \geq 1.$$

If some gap length is zero, one speaks of a *collapsed gap*, otherwise of an *open gap*.

For $u = u(t, \cdot)$ depending also on t define the corresponding operator

$$L(t) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u(t, \cdot).$$

Lax observed that u is a solution of the KdV equation if and only if

$$\frac{\mathrm{d}}{\mathrm{d}t}L = [B, L],$$

where $[B, L] = BL - LB$ denotes the commutator of L with the *anti-symmetric* operator

$$B = -4\frac{\mathrm{d}^3}{\mathrm{d}x^3} + 3u\frac{\mathrm{d}}{\mathrm{d}x} + 3\frac{\mathrm{d}}{\mathrm{d}x}u.$$

It follows by an elementary calculation that the solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}U = BU, \qquad U(0) = I,$$

defines a family of *unitary* operators $U(t)$ such that $U^*(t)L(t)U(t) = L(0)$. Consequently, the spectrum of $L(t)$ is independent of t, and so the periodic eigenvalues $\lambda_n = \lambda_n(u)$ are conserved quantities under the evolution of the KdV equation, a fact first observed by Gardner, Greene, Kruskal & Miura [47]. Thus, the flow of the KdV equation defines an *isospectral deformation* on the space of all potentials in $\mathcal{H}^N$.

From an analytical point of view, however, the periodic eigenvalues are not satisfactory as integrals, as λ_n is not a smooth function of u whenever the corresponding gap collapses. But McKean & Trubowitz [89] showed that the *squared gap lengths*

$$\gamma_n^2(u), \qquad n \geq 1,$$

together with the *average*

$$[u] = \int_{S^1} u(x)\,\mathrm{d}x$$

form another set of integrals, which are *real analytic* on all of L^2 and Poisson commute with each other. Moreover, the squared gap lengths together with the average determine uniquely the periodic spectrum of a potential [48].

The space L^2 thus decomposes into the *isospectral sets*

$$\mathrm{Iso}(u) = \left\{ v \in L^2 \colon \operatorname{spec}(v) = \operatorname{spec}(u) \right\},$$

which are invariant under the KdV flow and may also be characterized as

$$\mathrm{Iso}(u) = \left\{ v \in L^2 \colon \text{gap lengths}(v) = \text{gap lengths}(u),\ [v] = [u] \right\}.$$

As shown by McKean & Trubowitz [89], these are compact connected *tori*, whose dimension equals the number of positive gap lengths and is infinite generically. Moreover, as the asymptotic behavior of the gap lengths characterizes the regularity of a potential in exactly the same way as its Fourier coefficients do [84], we have

$$u \in \mathcal{H}^N \quad \Leftrightarrow \quad \mathrm{Iso}(u) \subset \mathcal{H}^N$$

for each $N \geq 1$. Hence also the phase space $\mathcal{H}^N$ decomposes into a collection of tori of varying dimension which are *invariant* under the KdV flow.

Angle-Action and Birkhoff Coordinates

In classical mechanics the existence of a foliation of the phase space into Lagrangian invariant tori is tantamount, at least locally, to the existence of angle-action coordinates. This is the content of the Liouville-Arnold-Jost theorem. In the infinite dimensional setting of the KdV equation, however, the existence of such coordinates is far less clear as the dimension of the foliation is *nowhere* locally constant. Invariant tori of infinite and finite dimension each form dense subsets of the foliation. Nevertheless, angle-action coordinates *can* be introduced *globally* in the form of Birkhoff coordinates as we describe now. They will form the basis of our study of perturbations of the KdV equation.

To formulate the statement we define the phase spaces more precisely. For any integer $N \geq 0$, let

$$\mathcal{H}^N = \left\{ u \in L^2(S^1, \mathbb{R}) : \; \|u\|_N < \infty \right\},$$

where

$$\|u\|_N^2 = \left|\hat{u}(0)\right|^2 + \sum_{k \in \mathbb{Z}} |k|^{2N} \left|\hat{u}(k)\right|^2$$

is defined in terms of the discrete Fourier transform $\hat{u}$ of u. The Poisson structure $\{\cdot, \cdot\}$ is degenerate on $\mathcal{H}^N$ and admits the average $[\cdot]$ as a Casimir function. The leaves of the corresponding symplectic foliation are given by $[u] = \text{const}$. Instead of restricting the KdV Hamiltonian to each leaf, it is more convenient to fix one such leaf, namely

$$\mathcal{H}_0^N = \left\{ u \in \mathcal{H}^N : \; [u] = 0 \right\},$$

which is symplectomorphic to each other leaf by a simple translation, and consider the mean value as a parameter. On $\mathcal{H}_0^N$ the Poisson structure is nondegenerate and induces a symplectic structure. Writing $u = v + c$ with $[v] = 0$ and $c = [u]$, the Hamiltonian then takes the form

$$H(u) = H_c(v) + c^3$$

with

$$H_c(v) = \int_{S^1} \left(\tfrac{1}{2} v_x^2 + v^3 \right) \mathrm{d}x + 6c \int_{S^1} \tfrac{1}{2} v^2 \, \mathrm{d}x.$$

We consider H_c as a 1-parameter family of Hamiltonians on $\mathcal{H}_0^N$.

We remark that

$$H^0 = \frac{1}{2} \int_{S^1} v^2 \, \mathrm{d}x$$

corresponds to translation and is the zero-th Hamiltonian of the KdV hierarchy, as described in appendix C.

To describe the angle-action variables on $\mathcal{H}_0^N$ we introduce the model space

$$h_r = \ell_r^2 \times \ell_r^2$$

with elements (x, y), where

$$\ell_r^2 = \left\{ x \in \ell^2(\mathbb{N}, \mathbb{R}) : \|x\|_r^2 = \sum_{n\geq 1} n^{2r} |x_n|^2 < \infty \right\}.$$

We endow h_r with the standard Poisson structure, for which $\{x_n, y_m\} = \delta_{nm}$, while all other brackets vanish.

The following theorem was first proven in [5] and [6]. A quite different approach for this result – and the one we expand on here – was first presented in [60]. For a related result for the nonlinear Schrödinger equation see [51].

Theorem 1.1. *There exists a diffeomorphism*

$$\Psi : \ h_{1/2} \to \mathcal{H}_0^0$$

with the following properties.

(i) *Ψ is one-to-one, onto, bi-analytic, and preserves the Poisson bracket.*

(ii) *For each $N \geq 0$, the restriction of Ψ to $h_{N+1/2}$, denoted by the same symbol, is a map*

$$\Psi : \ h_{N+1/2} \to \mathcal{H}_0^N,$$

which is one-to-one, onto, and bi-analytic as well.

(iii) *The coordinates (x, y) in $h_{3/2}$ are global Birkhoff coordinates for KdV. That is, for any $c \in \mathbb{R}$, the transformed Hamiltonian $H_c \circ \Psi$ depends only on $x_n^2 + y_n^2$, $n \geq 1$, with (x, y) being canonical coordinates.*

Thus, in the coordinates (x, y) the KdV Hamiltonian is a real analytic function of the actions alone:

$$H_c = H_c(I_1, I_2, \ldots), \qquad I_n = \frac{1}{2}(x_n^2 + y_n^2),$$

with equations of motion

$$\dot{x}_n = \omega_n(I) y_n, \qquad \dot{y}_n = -\omega_n(I) x_n,$$

where

$$\omega_n = \omega_{c,n} = \frac{\partial H_c}{\partial I_n}(I), \qquad I = (I_n)_{n\geq 1}.$$

The whole system appears now as an infinite chain of anharmonic oscillators, whose frequencies depend on their amplitudes in a nonlinear and real analytic fashion.

These results are not restricted to the KdV Hamiltonian. They simultaneously apply to *every* real analytic Hamiltonian in the Poisson algebra of all Hamiltonians which Poisson commute with all actions $I_1, I_2, \dots$. In particular, one obtains Birkhoff coordinates for every Hamiltonian in the KdV hierarchy defined in appendix C. As an example, we will later also consider the second KdV Hamiltonian.

The existence of Birkhoff coordinates makes it evident that every solution of the KdV equation is *almost-periodic* in time. In the coordinates of the model space every solution is given by

$$x_n(t) = \sqrt{2I_n^{\circ}}\, \sin(\theta_n^{\circ} + \omega_n(I^{\circ})t),$$
$$y_n(t) = \sqrt{2I_n^{\circ}}\, \cos(\theta_n^{\circ} + \omega_n(I^{\circ})t),$$

where $(\theta^{\circ}, I^{\circ})$ corresponds to the initial data u°. Hence, it winds around the underlying invariant torus

$$T_{I^{\circ}} = \left\{ (x,y) \colon x_n^2 + y_n^2 = 2I_n^{\circ},\ \ n \geq 1 \right\},$$

The solution in the original space $\mathcal{H}_0^N$ is thus winding around the embedded torus $\Psi(T_{I^{\circ}})$, and expanding Ψ into its Taylor series, it is of the form

$$\begin{aligned} u(t) &= \Psi(x(t), y(t)) \\ &= \sum_{k\in\mathbb{Z}^{\infty},\,|k|<\infty} \Psi_k(I^{\circ}, \theta^{\circ})\, \mathrm{e}^{\mathrm{i}\langle k,\omega(I^{\circ})\rangle t}. \end{aligned}$$

Here, $\langle k,\omega\rangle = \sum_n k_n\omega_n$, and each $\Psi_k(I^{\circ}, \theta^{\circ})$ is an element of $\mathcal{H}_0^N$. Thus, every solution is almost-periodic in time.

We remark that the solution above can also be represented in terms of the Riemann theta function. The corresponding formula is due to Its & Matveev [33].

Among all almost-periodic solutions there is a dense subset of quasi-periodic solutions, which are characterized by a *finite* number of frequencies and correspond to finite gap potentials. To describe them more precisely, let $A \subset \mathbb{N}$ be a *finite* index set, and consider the set of *A-gap potentials*

$$\mathcal{G}_A = \left\{ u \in \mathcal{H}_0^0 \colon \gamma_n(u) > 0 \Leftrightarrow n \in A \right\}.$$

That is, $u \in \mathcal{G}_A$ if and only if precisely the gaps $(\lambda_{2n-1}(u), \lambda_{2n}(u))$ with $n \in A$ are open. Clearly,

$$u \in \mathcal{G}_A \quad \Leftrightarrow \quad \mathrm{Iso}(u) \subset \mathcal{G}_A,$$

and all finite gap potentials are smooth, in fact real analytic, as almost all gap lengths are zero.

As might be expected there is a close connection between the set $\mathcal{G}_A$ and the subspace

$$h_A = \left\{ (x,y) \in h_0 \colon x_n^2 + y_n^2 > 0 \Leftrightarrow n \in A \right\}.$$