Graduate Texts in Mathematics

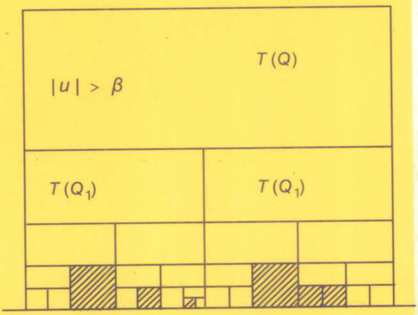
John B.Garnett

Bounded Analytic Functions

Revised First Edition

有界解析函数

修订版



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John B. Garnett

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Graduate Texts in Mathematics 236

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(continued after index)

To Dolores

Preface to Revised First Edition

This edition of *Bounded Analytic Functions* is the same as the first edition except for the corrections of several mathematical and typographical errors. I thank the many colleagues and students who have pointed out errors in the first edition. These include S. Axler, C. Bishop, A. Carbery, K. Dyakonov, J. Handy, V. Havin, H. Hunziker, P. Koosis, D. Lubinsky, D. Marshall, R. Mortini, A. Nicolau, M. O'Neill, W. Rudin, D. Sarason, D. Suárez, C. Sundberg, C. Thiele, S. Treil, I. Uriarte-Tuero, J. Väisälä, N. Varopoulos, and L. Ward.

I had planned to prepare a second edition with an updated bibliography and an appendix on results new in the field since 1981, but that work has been postponed for too long. In the meantime several excellent related books have appeared, including M. Andersson, *Topics in Complex Analysis*; G. David and S. Semmes, *Singular Integrals and Rectifiable Sets in* $\mathbb{R}^n$ and *Analysis of and on Uniformly Rectifiable Sets*; S. Fischer, *Function theory on planar domains*; P. Koosis, *Introduction to* H_p *spaces, Second edition*; N. Nikolski, *Operators, Functions, and Systems*; K. Seip, *Interpolation and Sampling in Spaces of Analytic Functions*; and B. Simon, *Orthogonal Polynomials on the Unit Circle*.

Several problems posed in the first edition have been solved. I give references only to Mathematical Reviews. The question page 167 on when $\mathcal{E}_\infty$ contains a Blaschke product was settled by A. Stray in MR 0940287. M. Papadimitrakis, MR 0947674, gave a counterexample to the conjecture in Problem 5 page 170. The late T. Wolff, MR 1979771, had a counterexample to the Baernstein conjecture cited on page 260. S. Treil resolved the g^2 problem on page 319 in MR 1945294. A constructive Fefferman-Stein decomposition of functions in BMO($\mathbb{R}^n$) was given by the late A. Uchiyama in MR 1007515, and C. Sundberg, MR 0660188, found a constructive proof of the Chang-Marshall theorem. Problem 5.3 page 420 was resolved by Garnett and Nicolau, MR 1394402, using work of Marshall and Stray MR 1394401. Problem 5.4. on page 420 remains a puzzle, but Hjelle and Nicolau (Pacific Journal of Mathematics, 2006) have an interesting result on approximation of moduli. P. Jones, MR 0697611, gave a construction of the P. Beurling linear operator of interpolation.

I thank Springer and F. W. Gehring for publishing this edition.

John Garnett

Contents

Preface vii
List of Symbols xiii

I. PRELIMINARIES

1. Schwarz's Lemma 1
2. Pick's Theorem 5
3. Poisson Integrals 10
4. Hardy–Littlewood Maximal Function 19
5. Nontangential Maximal Function and Fatou's Theorem 27
6. Subharmonic Functions 32

Notes 38
Exercises and Further Results 39

II. H^p SPACES

1. Definitions 48
2. Blaschke Products 51
3. Maximal Functions and Boundary Values 55
4. $(1/\pi)\int(\log|f(t)|/(1+t^2))dt > -\infty$ 61
5. The Nevanlinna Class 66
6. Inner Functions 71
7. Beurling's Theorem 78

Notes 83
Exercises and Further Results 84

III. CONJUGATE FUNCTIONS

1. Preliminaries 98
2. The L^p Theorems 106
3. Conjugate Functions and Maximal Functions 111

Notes 118
Exercises and Further Results 120

IV. SOME EXTREMAL PROBLEMS

1. Dual Extremal Problems 128
2. The Carleson–Jacobs Theorem 134
3. The Helson–Szegö Theorem 139
4. Interpolating Functions of Constant Modulus 145
5. Parametrization of K *151*
6. Nevanlinna's Proof 159
Notes 168
Exercises and Further Results 168

V. SOME UNIFORM ALGEBRA

1. Maximal Ideal Spaces 176
2. Inner Functions 186
3. Analytic Discs in Fibers 191
4. Representing Measures and Orthogonal Measures 193
5. The Space L^1/H_0^1 198
Notes 205
Exercises and Further Results 206

VI. BOUNDED MEAN OSCILLATION

1. Preliminaries 215
2. The John–Nirenberg Theorem 223
3. Littlewood–Paley Integrals and Carleson Measures 228
4. Fefferman's Duality Theorem 234
5. Vanishing Mean Oscillation 242
6. Weighted Norm Inequalities 245
Notes 259
Exercises and Further Results 261

VII. INTERPOLATING SEQUENCES

1. Carleson's Interpolation Theorem 275
2. The Linear Operator of Interpolation 285
3. Generations 289
4. Harmonic Interpolation 292
5. Earl's Elementary Proof 299
Notes 304
Exercises and Further Results 305

VIII. THE CORONA CONSTRUCTION

1. Inhomogeneous Cauchy–Riemann Equations 309
2. The Corona Theorem 314
3. Two Theorems on Minimum Modulus 323
4. Interpolating Blaschke Products 327
5. Carleson's Construction 333
6. Gradients of Bounded Harmonic Functions 338
7. A Constructive Solution of $\partial b/\partial\bar{z} = \mu$ 349
Notes 357
Exercises and Further Results 358

IX. DOUGLAS ALGEBRAS

1. The Douglas Problem 364
2. $H^\infty + C$ 367
3. The Chang-Marshall Theorem 369
4. The Structure of Douglas Algebras 375
5. The Local Fatou Theorem and an Application 379
Notes 384
Exercises and Further Results 385

X. INTERPOLATING SEQUENCES AND MAXIMAL IDEALS

1. Analytic Discs in $\mathfrak{M}$ 391
2. Hoffman's Theorem 401
3. Approximate Dependence between Kernels 406
4. Interpolating Sequences and Harmonic Separation 412
5. A Constructive Douglas–Rudin Theorem 418
Notes 428
Exercises and Further Results 428

BIBLIOGRAPHY 434

INDEX 453

List of Symbols

Symbol	Page
A_o	120
A^m	121
A^∞	121
A_α	57
A^{-1}	176
$\hat{A}$	179
$A^\perp$	196
$\mathfrak{A}_N$	57
B	237
B_0	274
BLO	271
BMO	216
BMOA	262
BMO(T)	218
BMO$_d$	266
BUC	242
$[B, H]$	270
$\mathscr{B} = \{f \text{ analytic}, \lvert f(z)\rvert \le 1\}$	7
$\mathscr{B}$, a set of inner functions	364
$\mathscr{B}_A$	364
$C = C(T)$	128
COP	432
C_A	367
Cl(f, z)	76
D	1
$\bar{D}$	1
dist(f, H^p)	129
dist$_*$	241
$\mathscr{D}$	391
$\partial/\partial z$	312
$\partial/\partial\bar{z}$	309
$\bar{\partial}$	354
E^*	323

Symbol	Page
$\lvert E\rvert$ = Lebesgue measure of E	23
$\mathscr{E}$	292
$f(m)$	180
$\hat{f}(m)$ = Gelfand transform	178
$\hat{f}(s)$ = Fourier transform	59
$f_x(t)$	15
$f^+(x)$	324
$f^\#(x)$	271
$f^*(t)$	55
$(\tilde{f})^*(\theta)$	106
G	401
$G_\delta(b)$	371
$Hf(\theta)$	102
$Hf(x)$	105
$H_\varepsilon f(x)$	123
$H^* f(x)$	123
$H^p = H^p(D)$	48
$H^p = H^p(dt)$	49
H^∞	48
H_0^q	128
H_d^1	267
$H_{\mathbb{R}}^1$	236
$(H^\infty)^{-1}$	63
$H^\infty + C$	132
$[H^\infty, \bar{\mathscr{B}}]$	364
$[H^\infty, \mathscr{U}_A]$	365
$\mathscr{H}$	5
$\tilde{I}$	221
J_m	404
$J(f_1, f_2, \ldots, f_n)$	319
$K(z_0, r)$	2
$K_\delta(B)$	401
$L_{\mathbb{C}}(x)$	235

Symbol	Page
L^1_{loc}	215
$L^p_{\mathbb{R}}$	236
$L^\infty_{\mathbb{R}}$	64
$L_z(\zeta)$	392
$\log^+|f(z)|$	33
$\ell(Q)$	278
$\mathscr{L}$	394
$m(f) = \hat{f}(m) = f(m)$	180
$m(\lambda)$	20
$M(T)$	128
$Mf(x)$	21
$M(d\mu)(t)$	28
$M_\delta(\varphi)$	242
$M_\mu f(x)$	45
$\mathfrak{M}$	183
$\mathfrak{M}_A$	178
$\mathfrak{M}_\zeta$	183
$\mathfrak{M}^{\mathscr{D}}$	394
N	66
$N(\sigma)$	30
N^+	68
$N_\varepsilon(x)$	339
norms: $\|f\|^p_{H^p}$	48, 49
$\|\varphi\|_*$	216
$\|\varphi\|'$	250
$\|u\|_{H_1}$	236
$P_{z_0}(\theta)$	10
$P_{z_0}(t)$	11
$P(m)$	393
$Q_r(\varphi)$	99
$Q_z(\varphi)$	98
QC	367
Q_A	366
$\mathbb{R}$ = real numbers	12
$\mathscr{R}(f, z_0)$	77
$S(\theta_0, h)$	371
$\tilde{S}_j$	334
$\mathscr{S}_h$	153
T	98

Symbol	Page
$T(Q)$	290
$u^+(t)$	116
$u^*(t)$	27
$\tilde{u}(z)$	98
UC	242
$\mathscr{U}_A$	365
$v_r(z)$	35
VMO	242
VMOA	274
VMO$_A$	375
X	181
X_ζ	207
$\beta\mathbb{N}$	180
$\delta(B)$	327
$\Delta(c, R)$	3
Δu	10
$\varepsilon(\rho)$	250, 258
$\Gamma_\alpha(t)$	21
$\Gamma_\alpha(e^{i\theta})$	23
$\Gamma^h_\alpha(t)$	379
λ_φ	233
λ^*	433
Λ_α	102
Λ^*	273
μ_φ	233
ν_f	371
$\rho(z, w)$	2
$\rho(m_1, m_2)$	392
$\rho(S, T)$	305
φ_I	216
$\omega(\delta)$	101
$\omega_f(\delta)$	101
∇_g	228
$|\nabla_g|^2$	228
$*$	13

$\bar{z}$ = complex conjugate of z

$\bar{f}$ = complex conjugate of f

$\bar{\mathfrak{F}} = \{\bar{f} : f \in \mathfrak{F}\}$, when $\mathfrak{F}$ is a set of functions

$\bar{E}$ = closure of E, when E is a point set

I

Preliminaries

As a preparation, we discuss three topics from elementary real or complex analysis which will be used throughout this book.

The first topic is the invariant form of Schwarz's lemma. It gives rise to the pseudohyperbolic metric, which is an appropriate metric for the study of bounded analytic functions. To illustrate the power of the Schwarz lemma, we prove Pick's theorem on the finite interpolation problem

$$f(z_j) = w_j, \qquad j = 1, 2, \ldots, n,$$

with $|f(z)| \leq 1$.

The second topic is from real analysis. It is the circle of ideas relating Poisson integrals to maximal functions.

The chapter ends with a brief introduction to subharmonic functions and harmonic majorants, our third topic.

1. Schwarz's Lemma

Let D be the unit disc $\{z : |z| < 1\}$ in the complex plane and let $\mathscr{B}$ denote the set of analytic functions from D into $\overline{D}$. Thus $|f(z)| \leq 1$ if $f \in \mathscr{B}$. The simple but surprisingly powerful Schwarz lemma is this:

Lemma 1.1. *If $f(z) \in \mathscr{B}$, and if $f(0) = 0$, then*

$$\begin{aligned} |f(z)| &\leq |z|, \quad z \neq 0, \\ |f'(0)| &\leq 1\,. \end{aligned} \tag{1.1}$$

Equality holds in (1.1) *at some point z if and only if $f(z) = e^{i\varphi}z$, φ a real constant.*

The proof consists in observing that the analytic function $g(z) = f(z)/z$ satisfies $|g| \leq 1$ by virtue of the maximum principle.

We shall use the invariant form of Schwarz's lemma due to Pick. A *Möbius transformation* is a conformal self-map of the unit disc. Every Möbius transformation can be written as

$$\tau(z) = e^{i\varphi}\frac{z - z_0}{1 - \bar{z}_0 z}$$

with φ real and $|z_0| \leq 1$. With this notation we have displayed $z_0 = \tau^{-1}(0)$.

Lemma 1.2. *If $f(z) \in \mathscr{B}$, then*

$$\frac{|f(z) - f(z_0)|}{|1 - \overline{f(z_0)}f(z)|} \leq \left|\frac{z - z_0}{1 - \bar{z}_0 z}\right|, \quad z \neq z_0, \tag{1.2}$$

and

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}. \tag{1.3}$$

Equality holds at some point z if and only if $f(z)$ is a Möbius transformation.

The proof is the same as the proof of Schwarz's lemma if we regard $\tau(z)$ as the independent variable and

$$\frac{f(z) - f(z_0)}{1 - \overline{f(z_0)}f(z)}$$

as the analytic function. Letting z tend to z_0 in (1.2) gives (1.3) at $z = z_0$, an arbitrary point of D.

The *pseudohyperbolic distance* on D is defined by

$$\rho(z, w) = \left|\frac{z - w}{1 - \bar{w}z}\right|.$$

Lemma 1.2 says that analytic mappings from D to D are Lipschitz continuous in the pseudohyperbolic distance:

$$\rho(f(z), f(w)) \leq \rho(z, w).$$

The lemma also says that the distance $\rho(z, w)$ is invariant under Möbius transformations:

$$\rho(z, w) = \rho(\tau(z), \tau(w)).$$

We write $K(z_0, r)$ for the noneuclidean disc

$$K(z_0, r) = \{z : \rho(z, z_0) < r\}, \quad 0 < r < 1.$$

Since the family $\mathscr{B}$ is invariant under the Möbius transformations, the study of the restrictions to $K(z_0, r)$ of functions in $\mathscr{B}$ is the same as the study of their restrictions to $K(0, r) = \{|w| < r\}$. In such a study, however, we must give $K(z_0, r)$ the coordinate function $w = \tau(z) = (z - z_0)/(1 - \bar{z}_0 z)$. For example, the set of derivatives of functions in $\mathscr{B}$ do not form a conformally invariant family, but the expression

$$|f'(z)|(1-|z|^2) \tag{1.4}$$

is conformally invariant. The proof of this fact uses the important identity

$$1-\left|\frac{z-z_0}{1-\bar{z}_0 z}\right|^2 = \frac{(1-|z|^2)(1-|\bar{z}_0|^2)}{|1-\bar{z}_0 z|^2} = (1-|z|^2)|\tau'(z)|, \tag{1.5}$$

which is (1.3) with equality for $f(z) = \tau(z)$. Hence if $f(z) = g(\tau(z)) = g(w)$, then

$$|f'(z)|(1-|z|^2) = |g'(w)||\tau'(z)|(1-|z|^2) = |g'(w)|(1-|w|^2)$$

and this is what is meant by the invariance of (1.4).

The noneuclidean disc $K(z_0, r)$, $0 < r < 1$, is the inverse image of the disc $|w| < r$ under

$$w = \tau(z) = \frac{z-z_0}{1-\bar{z}_0 z}.$$

Consequently $K(z_0, r)$ is also a euclidean disc $\Delta(c, R) = \{z : |z-c| < R\}$, and as such it has center

$$c = \frac{1-r^2}{1-r^2|z_0|^2} z_0 \tag{1.6}$$

and radius

$$R = r\frac{1-|z_0|^2}{1-r^2|z_0|^2}. \tag{1.7}$$

These can be found by direct calculation, but we shall derive them geometrically. The straight line through 0 and z_0 is invariant under τ, so that $\partial K(z_0, r) = \tau^{-1}(|w| = r)$ is a circle orthogonal to this line. A diameter of $K(z_0, r)$ is therefore the inverse image of the segment $[-rz_0/|z_0|, rz_0/|z_0|]$. Since $z = (w+z_0)/(1+\bar{z}_0 w)$, this diameter is the segment

$$[\alpha, \beta] = \left[\frac{|z_0|-r}{1-r|z_0|}\frac{z_0}{|z_0|}, \frac{|z_0|+r}{1+r|z_0|}\frac{z_0}{|z_0|}\right]. \tag{1.8}$$

The endpoints of (1.8) are the points of $\partial K(z_0, r)$ of largest and smallest modulus. Thus $c = (\alpha+\beta)/2$ and $R = |\beta-\alpha|/2$ and (1.6) and (1.7) hold. Note that if r is fixed and if $|z_0| \to 1$, then the euclidean radius of $K(z_0, r)$ is asymptotic to $1-|z_0|$.

Corollary 1.3. *If $f(z) \in \mathscr{B}$, then*

$$|f(z)| \le \frac{|f(0)|+|z|}{1+|f(0)||z|}. \tag{1.9}$$

Proof. By Lemma 1.2, $\rho(f(z), f(0)) \leq |z|$, so that $f(z) \in \overline{K(f(0), |z|)}$. The bound on $|f(z)|$ then follows from (1.8). Equality can hold in (1.9) only if f is a Möbius transformation and $\arg z = \arg f(0)$ when $f(0) \neq 0$. □

The pseudohyperbolic distance is a metric on D. The triangle inequality for ρ follows from

Lemma 1.4. *For any three points z_0, z_1, z_2 in D,*

$$\frac{\rho(z_0, z_2) - \rho(z_2, z_1)}{1 - \rho(z_0, z_2)\rho(z_2, z_1)} \leq \rho(z_0, z_1) \leq \frac{\rho(z_0, z_2) + \rho(z_2, z_1)}{1 + \rho(z_0, z_2)\rho(z_2, z_1)}. \tag{1.10}$$

Proof. We can suppose $z_2 = 0$ because ρ is invariant. Then (1.10) becomes

$$\frac{|z_0| - |z_1|}{1 - |z_0||z_1|} \leq \left|\frac{z_1 - z_0}{1 - \bar{z}_0 z_1}\right| \leq \frac{|z_0| + |z_1|}{1 + |z_0||z_1|}. \tag{1.11}$$

If $|z_1| = r$, then $z = (z_1 - z_0)/(1 - \bar{z}_0 z_1)$ lies on the boundary of the non-euclidean disc $K(-z_0, r)$, and hence $|z|$ lies between the moduli of the end-points of the segment (1.8). That proves (1.11). Of course (1.10) and especially (1.11) are easy to verify directly. □

Every Möbius transformation $w(z)$ sending z_0 to w_0 can be written

$$\frac{w - w_0}{1 - \bar{w}_0 w} = e^{i\varphi} \frac{z - z_0}{1 - \bar{z}_0 z}.$$

Differentiation then gives

$$|w'(z_0)| = \frac{1 - |w_0|^2}{|z_0|^2}. \tag{1.12}$$

This identity we have already encountered as (1.3) with equality. By (1.12) the expression

$$ds = \frac{2|dz|}{1 - |z|^2} \tag{1.13}$$

is a conformal invariant of the disc. We can use (1.13) to define the hyperbolic length of a rectifiable arc γ in D as

$$\int_\gamma \frac{2|dz|}{1 - |z|^2}.$$

We can then define the *Poincaré metric* $\psi(z_1, z_2)$ as the infimum of the hyperbolic lengths of the arcs in D joining z_1 to z_2. The distance $\psi(z_1, z_2)$ is then conformally invariant. If $z_1 = 0$, $z_2 = r > 0$, it is not difficult to see that

$$\psi(z_1, z_2) = 2\int_0^r \frac{dx}{1 - |x|^2} = \log \frac{1 + r}{1 - r}.$$

Since any pair of points z_1 and z_2 can be mapped to 0 and $\rho(z_1, z_2) = |(z_2 - z_1)/(1 - \bar{z}_1 z_2)|$, respectively, by a Möbius transformation, we therefore have

$$\psi(z_1, z_2) = \log \frac{1 + \rho(z_1, z_2)}{1 - \rho(z_1, z_2)}.$$

A calculation then gives

$$\rho(z_1, z_2) = \tanh\left(\frac{\psi(z_1, z_2)}{2}\right)$$

Moreover, because the shortest path from 0 to r is the radius, the geodesics, or paths of shortest distance, in the Poincaré metric consist of the images of the diameter under all Möbius transformations. These are the diameters of D and the circular arcs in D orthogonal to ∂D. If these arcs are called lines, we have a model of the hyperbolic geometry of Lobachevsky.

In this book we shall work with the pseudohyperbolic metric ρ rather than with ψ, although the geodesics are often lurking in our intuition.

Hyperbolic geometry is somewhat simpler in the upper half plane $\mathscr{H} = \{z = x + iy : y > 0\}$ In $\mathscr{H}$

$$\rho(z_1, z_2) = \left|\frac{z_1 - z_2}{z_1 - \bar{z}_2}\right|$$

and the element of hyperbolic arc length is

$$ds = \frac{|dz|}{y}.$$

Geodesics are vertical lines and circles orthogonal to the real axis. The conformal self-maps of $\mathscr{H}$ that fix the point at ∞ have a very simple form:

$$\tau(z) = az + x_0, \qquad a > 0, \qquad x_0 \in \mathbb{R}.$$

Horizontal lines $\{y = y_0\}$ can be mapped to one another by these self-maps of $\mathscr{H}$. This is not the case in D with the circles $\{|z| = r\}$. In $\mathscr{H}$ any two squares

$$\{x_0 < x < x_0 + h, h < y < 2h\}$$

are congruent in the noneuclidean geometry. The corresponding congruent figures in D are more complicated. For these and for other reasons, $\mathscr{H}$ is often the more convenient domain for many problems.

2. Pick's Theorem

A *finite Blaschke product* is a function of the form

$$B(z) = e^{i\varphi} \prod_{j=1}^{n} \frac{z - z_j}{1 - \bar{z}_j z}, \qquad |z_j| < 1.$$