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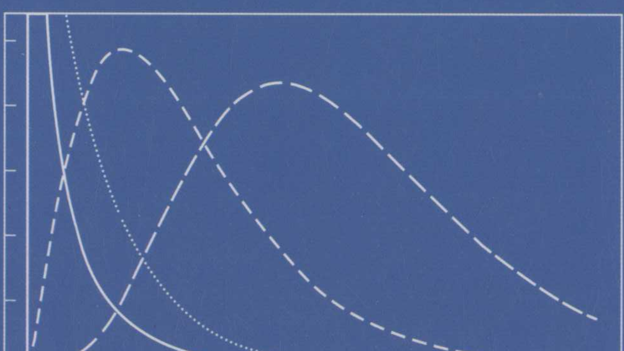
E. L. Lehmann

Elements of Large-Sample Theory

大样本理论基础



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Preface

The subject of this book, first order large-sample theory, constitutes a coherent body of concepts and results that are central to both theoretical and applied statistics. This theory underlies much of the work on such different topics as maximum likelihood estimation, likelihood ratio tests, the bootstrap, density estimation, contingency table analysis, and survey sampling methodology, to mention only a few. The importance of this theory has led to a number of books on the subject during the last 20 years, among them Ibragimov and Has'minskii (1979), Serfling (1980), Pfanzagl and Weflmeyer (1982), Le Cam (1986), Rüschendorf (1988), Barndorff-Nielson and Cox (1989, 1994), Le Cam and Yang (1990), Sen and Singer (1993), and Ferguson (1996).

These books all reflect the unfortunate fact that a mathematically complete presentation of the material requires more background in probability than can be expected from many students and workers in statistics. The present, more elementary, volume avoids this difficulty by taking advantage of an important distinction. While the proofs of many of the theorems require a substantial amount of mathematics, this is not the case with the understanding of the concepts and results nor of their statistical applications.

Correspondingly, in the present introduction to large-sample theory, the more difficult results are stated without proof, although with clear statements of the conditions of their validity. In addition, the mode of probabilistic convergence used throughout is convergence in probability rather than strong (or almost sure) convergence. With these restrictions it is possible to present the material with the requirement of only two years of calculus and, for the later chapters, some linear algebra. It is the purpose of the book, by these means, to make large-sample theory accessible to a wider audience.

It should be mentioned that this approach is not new. It can be found in single chapters of more specialized books, for example, Chapter 14 of Bishop, Fienberg, and Holland (1975) and Chapter 12 of Agresti (1990). However, it is my belief that students require a fuller, more extensive treatment to become comfortable with this body of ideas.

Since calculus courses often emphasize manipulation without insisting on a firm foundation, Chapter 1 provides a rigorous treatment of limits and order concepts which underlie all large-sample theory. Chapter 2 covers the basic probabilistic tools: convergence in probability and in law, the central limit theorem, and the delta method. The next two chapters illustrate the application of these tools to hypothesis testing, confidence intervals, and point estimation, including efficiency comparisons and robustness considerations. The material of these four chapters is extended to the multivariate case in Chapter 5.

Chapter 6 is concerned with the extension of the earlier ideas to statistical functionals and, among other applications, provides introductions to U-statistics, density estimation, and the bootstrap. Chapter 7 deals with the construction of asymptotically efficient procedures, in particular, maximum likelihood estimators, likelihood ratio tests, and some of their variants. Finally, an appendix briefly introduces the reader to a number of more advanced topics.

An important feature of large-sample theory is that it is nonparametric. Its limit theorems provide distribution-free approximations for statistical quantities such as significance levels, critical values, power, confidence coefficients, and so on. However, the accuracy of these approximations is not distribution-free but, instead, depends both on the sample size and on the underlying distribution. To obtain an idea of the accuracy, it is necessary to supplement the theoretical results with numerical work, much of it based on simulation. This interplay between theory and computation is a crucial aspect of large-sample theory and is illustrated throughout the book.

The approximation methods described here rest on a small number of basic ideas that have wide applicability. For specific situations, more detailed work on better approximations is often available. Such results are not included here; instead, references are provided to the relevant literature.

This book had its origin in a course on large-sample theory that I gave in alternate years from 1980 to my retirement in 1988. It was attended by graduate students from a variety of fields: Agricultural Economics, Biostatistics, Economics, Education, Engineering, Political Science, Psychology, Sociology, and Statistics. I am grateful to the students in these classes, and particularly to the Teaching Assistants who were in charge of the associated laboratories, for many corrections and other helpful suggestions. As the class notes developed into the manuscript of a book, parts were read

Contents

1

Mathematical Background

Preview

The principal aim of large-sample theory is to provide simple approximations for quantities that are difficult to calculate exactly. The approach throughout the book is to embed the actual situation in a sequence of situations, the limit of which serves as the desired approximation.

The present chapter reviews some of the basic ideas from calculus required for this purpose such as limit, convergence of a series, and continuity. Section 1 defines the limit of a sequence of numbers and develops some of the properties of such limits. In Section 2, the embedding idea is introduced and is illustrated with two approximations of binomial probabilities. Section 3 provides a brief introduction to infinite series, particularly power series. Section 4 is concerned with different rates at which sequences can tend to infinity (or zero); it introduces the o, $\asymp$, and O notation and the three most important growth rates: exponential, polynomial, and logarithmic. Section 5 extends the limit concept to continuous variables, defines continuity of a function, and discusses the fact that monotone functions can have only simple discontinuities. This result is applied in Section 6 to cumulative distribution functions; the section also considers alternative representations of probability distributions and lists the densities of probability functions of some of the more common distributions.

1.1 The concept of limit

Large-sample (or asymptotic*) theory deals with approximations to probability distributions and functions of distributions such as moments and quantiles. These approximations tend to be much simpler than the exact formulas and, as a result, provide a basis for insight and understanding that often would be difficult to obtain otherwise. In addition, they make possible simple calculations of critical values, power of tests, variances of estimators, required sample sizes, relative efficiencies of different methods, and so forth which, although approximate, are often accurate enough for the needs of statistical practice.

Underlying most large-sample approximations are limit theorems in which the sample sizes tend to infinity. In preparation, we begin with a discussion of limits. Consider a sequence of numbers a_n such as

$$a_n = 1 - \frac{1}{n}(n = 1, 2, \dots):\ 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \cdots, \tag{1.1.1}$$

and

$$a_n = 1 - \frac{1}{n^2}(n = 1, 2, \dots):\ 0, \frac{3}{4}, \frac{8}{9}, \frac{15}{16}, \frac{24}{25}, \frac{35}{36}, \cdots, \tag{1.1.2}$$

or, more generally, the sequences

$$a_n = a - \frac{1}{n} \text{ and } a_n = a - \frac{1}{n^2} \tag{1.1.3}$$

for some arbitrary fixed number a.

Two facts seem intuitively clear: (i) the members of both sequences in (1.1.3) are getting arbitrarily close to a as n gets large; (ii) this "convergence" toward a proceeds faster for the second series than for the first. The present chapter will make these two concepts precise and give some simple applications. But first, consider some additional examples.

The sequence obtained by alternating members of the two sequences (1.1.3) is given by

$$a_n = \begin{cases} a - \frac{1}{n} & \text{if } n \text{ is odd,} \\ a - \frac{1}{n^2} & \text{if } n \text{ is even:} \end{cases} \tag{1.1.4}$$

$$a - 1, a - \frac{1}{4}, a - \frac{1}{3}, a - \frac{1}{16}, a - \frac{1}{5}, a - \frac{1}{36}, \cdots.$$

*The term "asymptotic" is not restricted to large-sample situations but is used quite generally in connection with any limit process. See, for example, Definition 1.1.3. For some general discussion of asymptotics, see, for example, DeBruijn (1958).

For this sequence also, the numbers get arbitrarily close to a as n gets large. However, they do so without each member being closer to a than the preceding one. For a sequence $a_n, n = 1, 2, \ldots$, to tend to a limit a as $n \to \infty$, it is not necessary for each a_n to be closer to a than its predecessor a_{n-1}, but only for a_n to get arbitrarily close to a as n gets arbitrarily large.

Let us now formalize the statement that the members of a sequence $a_n, n = 1, 2, \ldots$, get arbitrarily close to a as n gets large. This means that for any interval about a, no matter how small, the members of the sequence will eventually, i.e., from some point on, lie in the interval. If such an interval is denoted by $(a - \epsilon, a + \epsilon)$ the statement says that from some point on, i.e., for all n exceeding some n_0, the numbers a_n will satisfy $a - \epsilon < a_n < a + \epsilon$ or equivalently

$$|a_n - a| < \epsilon \text{ for all } n > n_0. \tag{1.1.5}$$

The value of n_0 will of course depend on ϵ, so that we will sometimes write it as $n_0(\epsilon)$; the smaller ϵ is, the larger is the required value of $n_0(\epsilon)$.

Definition 1.1.1 The sequence $a_n, n = 1, 2, \ldots$, is said to tend (or converge) to a limit a; in symbols:

$$a_n \to a \text{ as n} \to \infty \quad \text{or} \quad \lim_{\mathrm{n}\to\infty} \mathrm{a_n} = \mathrm{a} \tag{1.1.6}$$

if, given any $\epsilon > 0$, no matter how small, there exists $n_0 = n_0(\epsilon)$ such that (1.1.5) holds.

For a formal proof of a limit statement (1.1.6) for a particular sequence a_n, it is only necessary to produce a value $n_0 = n_0(\epsilon)$ for which (1.1.5) holds. As an example consider the sequence (1.1.1). Here $a = 1$ and $a_n - a = -1/n$. For any given ϵ, (1.1.5) will therefore hold as soon as $\frac{1}{n} < \epsilon$ or $n > \frac{1}{\epsilon}$. For $\epsilon = 1/10, n_0 = 10$ will do; for $\epsilon = 1/100, n_0 = 100$; and, in general, for any ϵ, we can take for n_0 the smallest integer, which is $\geq \frac{1}{\epsilon}$.

In examples (1.1.1)–(1.1.4), the numbers a_n approach their limit from one side (in fact, in all these examples, $a_n < a$ for all n). This need not be the case, as is shown by the sequence

$$a_n = \left\{ \begin{array}{ll} 1 - \frac{1}{n} & \text{if } n \text{ is odd} \\ 1 + \frac{1}{n} & \text{if } n \text{ is even} \end{array} \right\} = 1 + (-1)^n \frac{1}{n}. \tag{1.1.7}$$

It may be helpful to give an example of a sequence which does not tend to a limit. Consider the sequence

$$0, 1, 0, 1, 0, 1, \ldots$$

given by $a_n = 0$ or 1 as n is odd or even. Since for arbitrarily large n, a_n takes on the values 0 and 1, it cannot get arbitrarily close to any a for all sufficiently large n.

The following is an important example which we state without proof.

Example 1.1.1 The exponential limit. For any finite number c,

$$\left(1+\frac{c}{n}\right)^n \to e^c \text{ as } n \to \infty. \tag{1.1.8}$$

To give an idea of the speed of the convergence of $a_n = \left(1+\frac{1}{n}\right)^n$ to its limit e, here are the values of a_n for a number of values of n, and the limiting value $e(n=\infty)$ to the nearest 1/100.

TABLE 1.1.1. $\left(1+\frac{1}{n}\right)^n$ to the nearest 1/100

n	1	3	5	10	30	50	100	500	∞
a_n	2.00	2.37	2.49	2.59	2.67	2.69	2.70	2.72	2.72

To the closest 1/1000, one has $a_{500} = 2.716$ and $e = 2.718$. □

The idea of limit underlies all of large-sample theory. Its usefulness stems from the fact that complicated sequences $\{a_n\}$ often have fairly simple limits which can then be used to approximate the actual a_n at hand. Table 1.1.1 provides an illustration (although here the sequence is fairly simple). It suggests that the limit value $a = 2.72$ shown in Table 1.1.1 provides a good approximation for $n \geq 30$ and gives a reasonable ballpark figure even for n as small as 5.

Contemplation of the table may raise a concern. There is no guarantee that the progress of the sequence toward its limit is as steady as the tabulated values suggest. The limit statement guarantees only that *eventually* the members of the sequence will be arbitrarily close to the limit value, not that each member will be closer than its predecessor. This is illustrated by the sequence (1.1.4). As another example, let

$$a_n = \begin{cases} 1/\sqrt{n} & \text{if } n \text{ is the square of an integer } (n = 1, 4, 9, \dots) \\ 1/n & \text{otherwise.} \end{cases} \tag{1.1.9}$$

Then $a_n \to 0$ (Problem 1.7) but does so in a somewhat irregular fashion. For example, for $n = 90, 91, \dots, 99$, we see a_n getting steadily closer to the limit value 0 only to again be substantially further away at $n = 100$. In sequences encountered in practice, such irregular behavior is rare. (For a statistical example in which it does occur, see Hodges (1957)). A table such as Table 1.1.1 provides a fairly reliable indication of smooth convergence to the limit.

Limits satisfy simple relationships such as: if $a_n \to a, b_n \to b$, then

$$a_n + b_n \to a + b \quad \text{and} \quad a_n - b_n \to a - b, \tag{1.1.10}$$

$$a_n \cdot b_n \to a \cdot b \tag{1.1.11}$$

and

$$a_n/b_n \to a/b \text{ provided } b \neq 0. \tag{1.1.12}$$

These results will not be proved here. Proofs and more detailed treatment of the material in this section and Section 1.3 are given, for example, in the classical texts (recently reissued) by Hardy (1992) and Courant (1988). For a slightly more abstract treatment, see Rudin (1976).

Using (1.1.12), it follows from (1.1.8), for example, that

$$\left(\frac{1+\frac{a}{n}}{1+\frac{b}{n}}\right)^n \to e^{a-b} \text{ as } n \to \infty. \tag{1.1.13}$$

An important special case not covered by Definition 1.1.1 arises when a sequence tends to ∞. We say that $a_n \to \infty$ if eventually (i.e., from some point on) the a's get larger than any given constant M. Proceeding as in Definition 1.1.1, this leads to

Definition 1.1.2 The sequence a_n tends to ∞; in symbols,

$$a_n \to \infty \quad \text{or} \quad \lim_{n\to\infty} a_n = \infty \tag{1.1.14}$$

if, given any M, no matter how large, there exists $n_0 = n_0(M)$ such that

$$a_n > M \text{ for all } n > n_0. \tag{1.1.15}$$

Some sequences tending to infinity are

$$a_n = n^\alpha \text{ for any } \alpha > 0 \tag{1.1.16}$$

(this covers sequences such as $\sqrt[3]{n} = n^{1/3}, \sqrt{n} = n^{1/2}, \ldots$ and $n^2, n^3, \ldots$);

$$a_n = e^{\alpha n} \text{ for any } \alpha > 0; \tag{1.1.17}$$

$$a_n = \log n,\ a_n = \sqrt{\log n},\ a_n = \log\log n. \tag{1.1.18}$$

To see, for example, that $\log n \to \infty$, we check (1.1.15) to find that $\log n > M$ provided $n > e^M$ (here we use the fact that $e^{\log n} = n$), so that we can take for n_0 the smallest integer that is $\geq e^M$.

Relations (1.1.10)–(1.1.12) remain valid even if a and/or b are $\pm\infty$ with the exceptions that $\infty - \infty, \infty \cdot 0$, and ∞/∞ are undefined.

The case $a_n \to -\infty$ is completely analogous (Problem 1.4) and requires the corresponding restrictions on (1.1.10)–(1.1.12).

Since throughout the book we shall be dealing with sequences, we shall in the remainder of the present section and in Section 4 consider relations between two sequences a_n and $b_n, n = 1, 2, \ldots$, which are rough analogs of the relations $a = b$ and $a < b$ between numbers.

Definition 1.1.3 Two sequences $\{a_n\}$ and $\{b_n\}$ are said to be (asymptotically) equivalent as $n \to \infty$; in symbols:

$$a_n \sim b_n \tag{1.1.19}$$

if

$$a_n/b_n \to 1. \tag{1.1.20}$$

This generalizes the concept of equality of two numbers a and b, to which it reduces for the sequences $a, a, a, \ldots$ and $b, b, b, \ldots$.

If b_n tends to a finite limit $b \neq 0$, (1.1.20) simply states that a_n tends to the same limit. However, if the limit b is 0 or $\pm\infty$, the statement $a_n \sim b_n$ contains important additional information. Consider, for example, the sequences $a_n = 1/n^2$ and $b_n = 1/n$, both of which tend to zero. Since their ratio a_n/b_n tends to zero, the two sequences are not equivalent. Here are two more examples:

$$a_n = n + n^2,\ b_n = n \tag{1.1.21}$$

and

$$a_n = n + n^2,\ b_n = n^2, \tag{1.1.22}$$

in both of which a_n and b_n tend to ∞. In the first, $a_n/b_n \to \infty$ so that a_n and b_n are not equivalent; in the second, $a_n/b_n \to 1$ so that they are equivalent.

A useful application of the idea of equivalence is illustrated by the sequences

$$a_n = \frac{1}{n} + \frac{3}{n^2} + \frac{1}{n^3},\ b_n = \frac{1}{n}. \tag{1.1.23}$$

Both a_n and b_n tend to zero. Since their ratio satisfies

$$\frac{a_n}{b_n} = 1 + \frac{3}{n} + \frac{1}{n^2} \to 1,$$

the two sequences are equivalent. The replacement of a complicated sequence such as a_n by a simpler asymptotically equivalent sequence b_n plays a central role in large-sample theory.

Replacing a true a_n by an approximating b_n of course results in an error. Consider, for example, the two equivalent sequences (1.1.22). When $n = 100$,

$$a_n = 10,100,\ b_n = 10,000,$$

and the error (or absolute error) is $|a_n - b_n| = 100$. On the other hand, the

(1.1.24) $$\text{relative error} = \left|\frac{a_n - b_n}{a_n}\right|$$

is $\frac{100}{10,100}$ which is less than .01. The small relative error corresponds to the fact that, despite the large absolute error of 100, b_n gives a pretty good idea of the size of a_n.

As the following result shows, asymptotic equivalence is closely related to relative error.

Lemma 1.1.1 *The sequences $\{a_n\}$ and $\{b_n\}$ are asymptotically equivalent if and only if the relative error tends to zero.*

Proof. The relative error

$$\left|\frac{a_n - b_n}{a_n}\right| = \left|1 - \frac{b_n}{a_n}\right| \to 0$$

if and only if $b_n/a_n \to 1$. ■

The following is a classical example of asymptotic equivalence which forms the basis of the application given in the next section.

Example 1.1.2 Stirling's formula. Consider the sequence

(1.1.25) $$a_n = n! = 1 \cdot 2 \cdots n.$$

Clearly, $a_n \to \infty$ as $n \to \infty$, but it is difficult from the defining formula to see how fast this sequence grows. We shall therefore try to replace it by a simpler equivalent sequence b_n. Since n^n is clearly too large, one might try, for example, $(n/2)^n$. This turns out to be still too large, but taking logarithms leads (not obviously) to the suggestion $b_n = (n/e)^n$. Now only a relatively minor further adjustment is required, and the final result (which we shall not prove) is Stirling's formula

(1.1.26) $$n! \sim \sqrt{2\pi n}\,(n/e)^n .$$

The following table adapted from Feller (Vol. 1) (1957), where there is also a proof of (1.1.26), shows the great accuracy of the approximation (1.1.26) even for small n.

It follows from Lemma 1.1.1 that the relative error tends to zero, and this is supported by the last line of the table. On the other hand, the absolute error tends to infinity and is already about 30,000 for $n = 10$. □

The following example provides another result, which will be used later.

TABLE 1.1.2. Stirling's approximation to $n!$

n	1	2	5	10	100
$n!$	1	2	120	3.6288×10^6	9.3326×10^{157}
(1.26)	.922	1.919	118.019	3.5987×10^6	9.3249×10^{157}
Error	.078	.081	1.981	$.0301 \times 10^6$	$.0077 \times 10^{157}$
Relative Error	.08	.04	.02	.008	.0008

Example 1.1.3 Sums of powers of integers. Let

$$S_n^{(k)} = 1^k + 2^k + \cdots + n^k \ (k \text{ a positive integer}) \tag{1.1.27}$$

so that, in particular,

$$S_n^{(0)} = n,\ S_n^{(1)} = \frac{n(n+1)}{2}, \text{ and } S_n^{(2)} = \frac{n(n+1)(2n+1)}{6}.$$

These formulas suggest that perhaps

$$S_n^{(k)} \sim \frac{n^{k+1}}{k+1} \text{ for all } k = 1, 2, \ldots, \tag{1.1.28}$$

and this is in fact the case. (For a proof, see Problem 1.14). □

Summary

1. A sequence of numbers a_n tends to the *limit* a if for all sufficiently large n the a's get arbitrarily close [i.e., within any preassigned distance ε] to a. If $a_n \to a$, then a can be used as an approximation for a_n when n is large.

2. Two sequences $\{a_n\}$ and $\{b_n\}$ are *asymptotically equivalent* if their ratio tends to 1. The members of a complicated sequence can often be approximated by those of a simpler sequence which is asymptotically equivalent. In such an approximation, the relative error tends to 0 as $n \to \infty$.

3. Stirling's formula provides a simple approximation for $n!$. The relative error in this approximation tends to 0 as $n \to \infty$ while the absolute error tends to ∞.

1.2 Embedding sequences

The principal aim of the present section is to introduce a concept which is central to large-sample theory: obtaining an approximation to a given situation by embedding it in a suitable sequence of situations. We shall illustrate this process by obtaining two different approximations for binomial probabilities corresponding to two different embeddings.

The probability of obtaining x successes in n binomial trials with success probability p is

$$P_n(x) = \binom{n}{x} p^x q^{n-x} \text{ where } q = 1 - p. \tag{1.2.1}$$

Suppose that n is even and that we are interested in the probability $P_n\left(\frac{n}{2}\right)$ of getting an even split between successes and failures. It seems reasonable to expect that this probability will tend to 0 as $n \to \infty$ and that it will be larger when $p = 1/2$ than when it is $\neq 1/2$.

To get a more precise idea of this behavior, let us apply Stirling's formula (1.1.26) to the three factorials in

$$P_n\left(\frac{n}{2}\right) = \frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}(pq)^{n/2}.$$

After some simplification, this leads to (Problem 2.1)

$$P_n\left(\frac{n}{2}\right) \sim \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{n}}\left(\sqrt{4pq}\right)^n. \tag{1.2.2}$$

We must now distinguish two cases.

Case 1. $p = 1/2$. Here we are asking for an even split between heads and tails in n tosses with a fair coin. The third factor in (1.2.2) is then 1, and we get the simple approximation

$$P_n\left(\frac{n}{2}\right) \sim \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{n}} \text{ when } p = 1/2. \tag{1.2.3}$$

This result confirms the conjecture that the probability tends to 0 as $n \to \infty$. The exact values of $P_n\left(\frac{n}{2}\right)$ and the approximation (1.2.3) are shown in Table 1.2.1 for varying n.

TABLE 1.2.1. $P_n\left(\frac{n}{2}\right)$ for $p = 1/2$

n	4	20	100	500	1,000	10,000
Exact	.375	.176	.0796	.0357	.0252	.00798
(1.2.3)	.399	.178	.0798	.0357	.0252	.00798

A surprising feature of the table is how slowly the probability decreases. Even for $n = 10,000$, the probability of an exactly even 5,000–5,000 split is not much below .01. Qualitatively, this could have been predicted from (1.2.3) because of the very slow increase of $\sqrt{n}$ as a function of n. The table indicates that the approximation is highly accurate for $n > 20$.

Case 2. $p \neq 1/2$. Since $\sqrt{4pq} < 1$ for all $p \neq 1/2$ (Problem 2.2), the approximate probabilities (1.2.3) for $p = 1/2$ are multiplied by the n^{th} power of a number between 0 and 1 when $p \neq 1/2$. They are therefore greatly reduced and tend to 0 at a much faster rate. The exact values of $P_n\left(\frac{n}{2}\right)$ and the approximation (1.2.2) are shown in Table 1.2.2 for the case $p = 1/3$. Again,

TABLE 1.2.2. $P_n\left(\frac{n}{2}\right)$ for $p = 1/3$

n	4	20	100	1,000	10,000
Exact	.296	.0543	.000220	6.692×10^{-28}	1.378×10^{-258}
(2.2)	.315	.0549	.000221	6.694×10^{-28}	1.378×10^{-258}

the approximation is seen to be highly accurate for $n > 20$.

A comparison of the two tables shows the radical difference in the speed with which $P_n\left(\frac{n}{2}\right)$ tends to 0 in the two cases.

So far we have restricted attention to the probability of an even split, that is, the case in which $\frac{x}{n} = \frac{1}{2}$. Let us now consider the more general case that x/n has any given fixed value $\alpha\,(0 < \alpha < 1)$, which, of course, requires that αn is an integer. Then

$$P_n(x) = \binom{n}{\alpha n}\left(p^\alpha q^{1-\alpha}\right)^n$$

and application of Stirling's formula shows in generalization of (1.2.2) that (Problem 2.3)

$$P_n(\alpha n) \sim \frac{1}{\sqrt{2\pi\alpha(1-\alpha)}} \cdot \frac{1}{\sqrt{n}}\gamma^n \tag{1.2.4}$$

with

$$\gamma = \left(\frac{p}{\alpha}\right)^\alpha \left(\frac{q}{1-\alpha}\right)^{1-\alpha}. \tag{1.2.5}$$

As before, there are two cases.

Case 1. $p = \alpha$. This is the case in which p is equal to the frequency of success, the probability of which is being evaluated. Here $\gamma = 1$ and (1.2.4) reduces to

$$P_n(\alpha n) \sim \frac{1}{\sqrt{2\pi\alpha(1-\alpha)}} \cdot \frac{1}{\sqrt{n}} \text{ when } p = \alpha. \tag{1.2.6}$$