Springer 大学数学图书——影印版

Basic Stochastic Processes

随机过程基础

Zdzisław Brzeźniak
Tomasz Zastawniak 著

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内容提要

随机过程在数学、科学和工程中有着越来越广泛的应用。本书包括随机过程一些基本而又重要的内容：条件期望，Markov 链，Poisson 过程和 Brown 运动；同时也包括 Ito 积分和随机微分方程等应用范围越来越广的内容。本书的习题是其基本内容的延伸，而且有十分完整的解答，非常适合高年级本科生和研究生自学使用或用作教学参考书。

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序　言

在学校教书多年，当学生（特别是本科生）问有什么好的参考书时，我们所能推荐的似乎除了教材还是教材，而且不同教材之间的差别并不明显、特色也不鲜明。所以多年前我们就开始酝酿，希望为本科学生引进一些好的参考书，为此清华大学数学科学系的许多教授与清华大学出版社共同付出了很多心血。

这里首批推出的十余本图书，是从 Springer 出版社的多个系列丛书中精心挑选出来的。在丛书的筹划过程中，我们挑选图书最重要的标准并不是完美，而是有特色并包容各个学派（有些书甚至有争议，比如从数学上看也许不够严格），其出发点是希望我们的学生能够吸纳百家之长；同时，在价格方面，我们也做了很多工作，以使得本系列丛书的价格能让更多学校和学生接受，使得更多学生能够从中受益。

本系列图书按其定位，大体有如下四种类型（一本书可以属于多类，但这里限于篇幅不能一一介绍）。

一、适用面比较广、有特色并可以用作教材或参考书的图书。例如：

● Lovász et al.: Discrete Mathematics. 2003

该书是离散数学的入门类型教材。与现有的教材（包括国外的教材）相比，它涵盖了离散数学新颖而又前沿的研究课题，同时还涉及信息科学方面既基本又有趣的应用；在着力打好数学基础的同时，也强调了数学与信息科学的关联。姚期智先生倡导和主持的清华大学计算机科学试验班，已经选择该书作为离散数学课程的教材。

二、在目前国内的数学教育中，课程主要以学科的纵向发展为主线，而对数学不同学科之间的联系讨论很少。学生缺乏把不同学科视为一个数学整体的训练，这方面的读物尤其欠缺。这是本丛书一个重要的着力点。最典型的是：

● Fine/Rosenberger: The Fundamental Theorem of Algebra. 1997

该书对数学中最重要的定理——代数基本定理给出了六种证明，方法涉及到分析、代数与拓扑；附录中还给出了 Gauss 的证明和 Cauchy 的证明。全书以一个数学问题为主线展开，纵横数学的核心领域；结构严谨、文笔流畅、浅显易懂、引人入胜，是一本少见的能够让读者入迷的好读物，用它来引导学生欣赏和领会“数学的美”绝对不会落于空谈。该书适于自学、讨论，也是极好的短学期课程教材。

● Baker: Matrix Groups. 2001

就内容而言，本书并不超出我国大学线性代数、抽象代数和一般拓扑学课程的内容，但是本书所讲的是李群和李代数的基础理论——这是现代数学和物理学非常重要的工具。各种矩阵群和矩阵代数是李群和李代数最典型和

最重要的例子，同时也能帮助学生建立数学不同学科之间的联系。从矩阵出发，既能把握李群和李代数的实质，又能学会计算和运用，所以这是一本不可多得的好书。

三、科学与技术的发展不断为数学提出新的研究课题，因此在数学学科的发展过程中，来自其他学科的推动力是不能忽视的。本系列中第三种类型的读物就是强调数学与其他学科的联系。例如：

- Woodhouse: Special Relativity. 2003

该书将物理与数学有机结合，体现了物理学家伽利略的名言："大自然是一部用数学语言写成的巨著。" 不仅如此，本书作者还通过对线性代数、微积分、场论等数学的运用进一步强调并贯穿这样的观点：数学的真谛和发展存在并产生于物理或自然规律及其发现中。精读此书有助于理解物理学和数学的整体关系。

- Britton: Essential Mathematical Biology. 2003

生命科学在本世纪一定会有很大发展，其对数学的需求和推动是可以预见的。因此生物数学在应用数学中占有日益重要的地位，数学系培养的学生至少一部分人应当对这个领域有所了解。随着生命科学的迅速发展，生物数学也发展很快。本书由浅入深，从经典的问题入手，最后走向学科前沿和近年的热点问题。该书至少可以消除学生对生物学的神秘感。

四、最后一类是适合本科学生的课外读物。这类图书对激发和引导学生学习数学的兴趣会非常有帮助，而且目前国内也急需这样的图书。例如：

- Daepp/Gorkin: Reading, Writing and Proving. 2003

该书对初学高等数学的读者来说特别有意义。它的基本出发点是引导读者以研究的心态去学习，让读者养成独立思考的习惯，并进而成为研究型的学习者。该书将一个学习数学的过程在某种意义下程序化，努力让学习者养成一个好的学习习惯，以及学会如何应对问题。该书特色鲜明，类似的图书确实很少。

- Brzezniak/Zastawniak: Basic Stochastic Processes. 1998

随机过程理论在数学、科学和工程中有越来越广泛的应用，本书适合国内的需要。其主要特点是：书中配有的习题是巩固和延伸学习内容的基本手段，而且有十分完整的解答，非常适合自学和作为教学参考书。这是一本难得的好书，它 1999 年出版，到 2000 年已经是第 3 次印刷，到 2003 年则第 6 次重印。

- Anglin/Lambek: The Heritage of Thales. 1995

该书的基本内容是数学的历史和数学的哲学。数学历史是该书的线索，数学是内容的主体，引申到的是数学哲学。它不是一本史论型的著作，而是采用专题式编写方式，每个专题相对独立，所以比较易读、易懂，是本科生学习数学过程中非常好的课外读物。

本系列丛书中的大部分图书还将翻译为中文出版，以适应更多读者的需要。丛书筹划过程中，冯克勤、郑志勇、卢旭光、郑建华、王殿军、杨利军、叶俊、扈志明等很多清华大学的教授都投入了大量精力。他们之中很多人也将是后面中文版的译者。此外，我们今后还将不断努力丰富引进丛书的种类，同时也会将选书的范围在可能情况下进一步扩大到其他高水平的出版机构。

教育是科学和技术发展的基石，数学教育更是基石的基础。因为是基础所以它重要；也因为是基础所以它显示度不高，容易不被重视。只有将人才培养放到更高的地位上，中国成为创新型国家的目标才会成为可能。

本系列丛书的正式推出，圆了一个我们多年的梦，但这无疑仅仅是开始。

白峰杉

2006 年 6 月于清华园

To our families

Preface

This book has been designed for a final year undergraduate course in stochastic processes. It will also be suitable for mathematics undergraduates and others with interest in probability and stochastic processes, who wish to study on their own.

The main prerequisite is probability theory: probability measures, random variables, expectation, independence, conditional probability, and the laws of large numbers. The only other prerequisite is calculus. This covers limits, series, the notion of continuity, differentiation and the Riemann integral. Familiarity with the Lebesgue integral would be a bonus. A certain level of fundamental mathematical experience, such as elementary set theory, is assumed implicitly.

Throughout the book the exposition is interlaced with numerous exercises, which form an integral part of the course. Complete solutions are provided at the end of each chapter. Also, each exercise is accompanied by a hint to guide the reader in an informal manner. This feature will be particularly useful for self-study and may be of help in tutorials. It also presents a challenge for the lecturer to involve the students as active participants in the course.

A brief introduction to probability is presented in the first chapter. This is mainly to fix terminology and notation, and to provide a survey of the results which will be required later on. However, conditional expectation is treated in detail in the second chapter, including exercises designed to develop the necessary skills and intuition. The reader is strongly encouraged to work through them prior to embarking on the rest of this course. This is because conditional expectation is a key tool for stochastic processes, which often presents some difficulty to the beginner.

Chapter 3 is about martingales in discrete time. We study the basic properties, but also some more advanced ones like stopping times and the Optional Stopping Theorem. In Chapter 4 we continue with martingales by presenting

Doob's inequalities and convergence results. Chapter 5 is devoted to time-homogenous Markov chains with emphasis on their ergodic properties. Some important results are presented without proof, but with a lot of applications. However, Markov chains with a finite state space are treated in full detail. Chapter 6 deals with stochastic processes in continuous time. Much emphasis is put on two important examples, the Poisson and Wiener processes. Various properties of these are presented, including the behaviour of sample paths and the Doob maximal inequality. The last chapter is devoted to the Itô stochastic integral. This is carefully introduced and explained. We prove a stochastic version of the chain rule known as the Itô formula, and conclude with examples and the theory of stochastic differential equations.

It is a pleasure to thank Andrew Carroll for his careful reading of the final draft of this book. His many comments and suggestions have been invaluable to us. We are also indebted to our students who took the Stochastic Analysis course at the University of Hull. Their feedback was instrumental in our choice of the topics covered and in adjusting the level of exercises to make them challenging yet accessible enough to final year undergraduates.

As this book is going into its 3rd printing, we would like to thank our students and readers for their support and feedback. In particular, we wish to express our gratitude to Iaonnis Emmanouil of the University of Athens and to Brett T. Reynolds and Chris N. Reynolds of the University of Wales in Swansea for their extensive and meticulous lists of remarks and valuable suggestions, which helped us to improve the current version of *Basic Stochastic Processes*.

We would greatly appreciate further feedback from our readers, who are invited to visit the Web Page `http://www.hull.ac.uk/php/mastz/bsp.html` for more information and to check the latest corrections in the book.

Zdzisław Brzeźniak and Tomasz Zastawniak
Kingston upon Hull, June 2000

Contents

1
Review of Probability

In this chapter we shall recall some basic notions and facts from probability theory. Here is a short list of what needs to be reviewed:

1) Probability spaces, σ-fields and measures;
2) Random variables and their distributions;
3) Expectation and variance;
4) The σ-field generated by a random variable;
5) Independence, conditional probability.

The reader is advised to consult a book on probability for more information.

1.1 Events and Probability

Definition 1.1

Let Ω be a non-empty set. A *σ-field* $\mathcal{F}$ on Ω is a family of subsets of Ω such that

1) the empty set $\emptyset$ belongs to $\mathcal{F}$;
2) if A belongs to $\mathcal{F}$, then so does the complement $\Omega \setminus A$;

3) if $A_1, A_2, \ldots$ is a sequence of sets in $\mathcal{F}$, then their union $A_1 \cup A_2 \cup \cdots$ also belongs to $\mathcal{F}$.

Example 1.1

Throughout this course $\mathbb{R}$ will denote the set of real numbers. The family of *Borel sets* $\mathcal{F} = \mathcal{B}(\mathbb{R})$ is a σ-field on $\mathbb{R}$. We recall that $\mathcal{B}(\mathbb{R})$ is the smallest σ-field containing all intervals in $\mathbb{R}$.

Definition 1.2

Let $\mathcal{F}$ be a σ-field on Ω. A *probability measure* P is a function

$$P : \mathcal{F} \to [0,1]$$

such that

1) $P(\Omega) = 1$;
2) if $A_1, A_2, \ldots$ are pairwise disjoint sets (that is, $A_i \cap A_j = \emptyset$ for $i \neq j$) belonging to $\mathcal{F}$, then

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots .$$

The triple $(\Omega, \mathcal{F}, P)$ is called a *probability space*. The sets belonging to $\mathcal{F}$ are called *events*. An event A is said to occur *almost surely* (a.s.) whenever $P(A) = 1$.

Example 1.2

We take the unit interval $\Omega = [0,1]$ with the σ-field $\mathcal{F} = \mathcal{B}([0,1])$ of Borel sets $B \subset [0,1]$, and *Lebesgue measure* $P = \text{Leb}$ on $[0,1]$. Then $(\Omega, \mathcal{F}, P)$ is a probability space. Recall that Leb is the unique measure defined on Borel sets such that

$$\text{Leb}[a,b] = b - a$$

for any interval $[a,b]$. (In fact Leb can be extended to a larger σ-field, but we shall need Borel sets only.)

Exercise 1.1

Show that if $A_1, A_2, \ldots$ is an *expanding* sequence of events, that is,

$$A_1 \subset A_2 \subset \cdots ,$$

then

$$P(A_1 \cup A_2 \cup \cdots) = \lim_{n\to\infty} P(A_n).$$

Similarly, if $A_1, A_2, \ldots$ is a *contracting* sequence of events, that is,

$$A_1 \supset A_2 \supset \cdots ,$$

then

$$P(A_1 \cap A_2 \cap \cdots) = \lim_{n\to\infty} P(A_n).$$

Hint Write $A_1 \cup A_2 \cup \cdots$ as the union of a sequence of disjoint events: start with A_1, then add a disjoint set to obtain $A_1 \cup A_2$, then add a disjoint set again to obtain $A_1 \cup A_2 \cup A_3$, and so on. Now that you have a sequence of disjoint sets, you can use the definition of a probability measure. To deal with the product $A_1 \cap A_2 \cap \cdots$ write it as a union of some events with the aid of De Morgan's law.

Lemma 1.1 (Borel–Cantelli)

Let $A_1, A_2, \ldots$ be a sequence of events such that $P(A_1) + P(A_2) + \cdots < \infty$ and let $B_n = A_n \cup A_{n+1} \cup \cdots$. Then $P(B_1 \cap B_2 \cap \cdots) = 0$.

Exercise 1.2

Prove the Borel–Cantelli lemma above.

Hint $B_1, B_2, \ldots$ is a contracting sequence of events.

1.2 Random Variables

Definition 1.3

If $\mathcal{F}$ is a σ-field on Ω, then a function $\xi : \Omega \to \mathbb{R}$ is said to be $\mathcal{F}$*-measurable* if

$$\{\xi \in B\} \in \mathcal{F}$$

for every Borel set $B \in \mathcal{B}(\mathbb{R})$. If $(\Omega, \mathcal{F}, P)$ is a probability space, then such a function ξ is called a *random variable.*

Remark 1.1

A short-hand notation for events such as $\{\xi \in B\}$ will be used to avoid clutter. To be precise, we should write

$$\{\omega \in \Omega : \xi(\omega) \in B\}$$

in place of $\{\xi \in B\}$. Incidentally, $\{\xi \in B\}$ is just a convenient way of writing the inverse image $\xi^{-1}(B)$ of a set.

Definition 1.4

The σ-field $\sigma(\xi)$ *generated* by a random variable $\xi : \Omega \to \mathbb{R}$ consists of all sets of the form $\{\xi \in B\}$, where B is a Borel set in $\mathbb{R}$.

Definition 1.5

The σ-field $\sigma\{\xi_i : i \in I\}$ generated by a family $\{\xi_i : i \in I\}$ of random variables is defined to be the smallest σ-field containing all events of the form $\{\xi_i \in B\}$, where B is a Borel set in $\mathbb{R}$ and $i \in I$.

Exercise 1.3

We call $f : \mathbb{R} \to \mathbb{R}$ a *Borel function* if the inverse image $f^{-1}(B)$ of any Borel set B in $\mathbb{R}$ is a Borel set. Show that if f is a Borel function and ξ is a random variable, then the composition $f(\xi)$ is $\sigma(\xi)$-measurable.

Hint Consider the event $\{f(\xi) \in B\}$, where B is an arbitrary Borel set. Can this event be written as $\{\xi \in A\}$ for some Borel set A?

Lemma 1.2 (Doob–Dynkin)

Let ξ be a random variable. Then each $\sigma(\xi)$-measurable random variable η can be written as

$$\eta = f(\xi)$$

for some Borel function $f : \mathbb{R} \to \mathbb{R}$.

The proof of this highly non-trivial result will be omitted.

Definition 1.6

Every random variable $\xi : \Omega \to \mathbb{R}$ gives rise to a probability measure

$$P_\xi(B) = P\{\xi \in B\}$$

on $\mathbb{R}$ defined on the σ-field of Borel sets $B \in \mathcal{B}(\mathbb{R})$. We call P_ξ the *distribution* of ξ. The function $F_\xi : \mathbb{R} \to [0,1]$ defined by

$$F_\xi(x) = P\{\xi \le x\}$$

is called the *distribution function* of ξ.