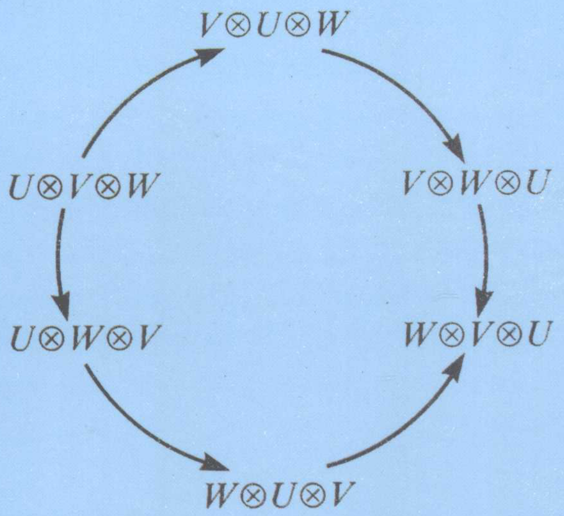
Vyjayanthi Chari and Andrew Pressley

# A GUIDE TO Quantum Groups

# 量子群入门

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# A GUIDE TO
# QUANTUM GROUPS

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# Contents

# Introduction

Quantum groups first arose in the physics literature, particularly in the work of L. D. Faddeev and the Leningrad school, from the 'inverse scattering method', which had been developed to construct and solve 'integrable' quantum systems. They have excited great interest in the past few years because of their unexpected connections with such, at first sight, unrelated parts of mathematics as the construction of knot invariants and the representation theory of algebraic groups in characteristic $p$.

In their original form, quantum groups are associative algebras whose defining relations are expressed in terms of a matrix of constants (depending on the integrable system under consideration) called a quantum R-matrix. It was realized independently by V. G. Drinfel'd and M. Jimbo around 1985 that these algebras are Hopf algebras, which, in many cases, are deformations of 'universal enveloping algebras' of Lie algebras. A little later, Yu. I. Manin and S. L. Woronowicz independently constructed non-commutative deformations of the algebra of functions on the groups $SL_2(\mathbb{C})$ and $SU_2$, respectively, and showed that many of the classical results about algebraic and topological groups admit analogues in the non-commutative case.

Thus, although many of the fundamental papers on quantum groups are written in the language of integrable systems, their properties are accessible by more conventional mathematical techniques, such as the theory of topological and algebraic groups and Lie algebras. Our aim in this book is to present the theory of quantum groups from this latter point of view. In fact, we shall concentrate on the study of the 'Lie algebras' of quantum groups, which seems to be the approach which has proved most powerful, particularly in applications, but we shall also discuss, in rather less detail, their relation with 'non-commutative algebraic geometry and topology'.

We shall now describe what a quantum group is, beginning by trying to explain the motivation for the use of the adjective 'quantum'.

In classical mechanics, the phase space $M$ of a dynamical system is a *Poisson manifold*. This means that the space $\mathcal{F}(M)$ of (differentiable) complex-valued functions on $M$ is equipped with a Lie bracket $\{\ ,\ \} : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M)$ (satisfying certain additional conditions), called the Poisson bracket. The dynamical equations defining the time evolution of the system are equivalent to the equations

$$\frac{d}{dt} f(m(t)) = \{\mathcal{H}_{\rm cl}, f\}(m(t))$$

for $f \in \mathcal{F}(M)$, where $\mathcal{H}_{\rm cl}$ is a fixed function on $M$ called the (classical) hamiltonian, and $m(t) \in M$ is the 'state' of the system at time $t$. For example, for a single particle moving along the real line, $M$ is the cotangent bundle $T^*(\mathbb{R})$, and if $q$ is the coordinate on $\mathbb{R}$ ('position') and $p$ the coordinate in the fibre direction ('momentum'), the Poisson bracket is

$$\{f_1, f_2\} = \frac{\partial f_1}{\partial p}\frac{\partial f_2}{\partial q} - \frac{\partial f_2}{\partial p}\frac{\partial f_1}{\partial q}.$$

In particular, the Poisson bracket of the coordinate functions is

$$\{p, q\} = 1. \tag{1}$$

In quantum mechanics, the space $M$ is replaced by the set of rays in a complex Hilbert space $V$, and the space $\mathcal{F}(M)$ of functions on $M$ by the algebra $\mathrm{Op}(V)$ of (not necessarily bounded) operators on $V$. The time evolution of an operator $A$ is given by

$$\frac{dA}{dt} = [\mathcal{H}_{\mathrm{qu}}, A]$$

for some operator $\mathcal{H}_{\mathrm{qu}} \in \mathrm{Op}(V)$, called the (quantum) hamiltonian. For example, in the case of a single particle moving along the real line, $V$ is the space $\mathrm{L}^2(\mathbb{R})$ of square-integrable functions of $q$, and the operators $P$ and $Q$ corresponding to the coordinate functions $p$ and $q$ are given by

$$P = -\sqrt{-1}h\frac{\partial}{\partial q}, \qquad Q = \text{multiplication by } q,$$

where $h$ is $1/2\pi$ times Planck's constant. Note that

$$[P, Q] = -\sqrt{-1}h\,\mathrm{id}_V. \tag{2}$$

The question is: how to pass from the classical to the quantum description of a system. This is the problem of *quantization*. Ideally, one would like a map $\mathcal{Q}$ which assigns to each function $f \in \mathcal{F}(M)$ an operator $\mathcal{Q}(f)$ on $V$. Moreover, since time evolution in the classical and quantum descriptions is given by taking the Poisson bracket and commutator with the hamiltonian, respectively, $\mathcal{Q}$ should satisfy the relation

$$\mathcal{Q}\{f_1, f_2\} = \frac{[\mathcal{Q}(f_1), \mathcal{Q}(f_2)]}{-\sqrt{-1}h}$$

(the normalization comes from (1) and (2)). Unfortunately, it is known that, even for the simplest case of a single particle moving along the real line, no such map $\mathcal{Q}$ exists.

There is, however, an alternative formulation of the quantization problem, introduced by J. E. Moyal in 1949. This begins by noting that the fundamental difference between the classical and quantum descriptions is that

$\mathcal{F}(M)$ is a commutative algebra, whereas $\mathrm{Op}(V)$ is non-commutative (when $\dim(V) > 1$). Moyal's idea is to try to reproduce the results of quantum mechanics by replacing the usual product on $\mathcal{F}(M)$ by a non-commutative product $*_h$, depending on a parameter $h$, such that $*_h$ becomes the usual product as $h \to 0$, just as 'quantum mechanics becomes classical mechanics as Planck's constant tends to zero', and such that

$$\lim_{h \to 0} \frac{f_1 *_h f_2 - f_2 *_h f_1}{h} = \{f_1\, ,\, f_2\}. \tag{3}$$

If we think of $\mathcal{F}(M)$ with the Moyal product $*_h$ as a non-commutative algebra of functions $\mathcal{F}_h(M)$, we find ourselves in the realm of non-commutative geometry in the sense of A. Connes. The philosophy here is that any 'space' is determined by the algebra of functions on it (with the usual product). For example, every affine algebraic variety over $\mathbb{C}$ is determined (up to isomorphism) by the commutative algebra of regular functions on it, whereas every compact topological space is determined by its commutative C*-algebra of complex-valued continuous functions. More precisely, the category of 'spaces' in these examples is dual to the category of the corresponding algebras. Thus, a non-commutative algebra should be viewed as the space of functions on a 'non-commutative space', and we can say that Moyal's construction gives a deformation of the classical phase space $M$ to a family of non-commutative (or 'quantum') spaces $M_h$ such that $\mathcal{F}_h(M)$ is the algebra of functions on $M_h$.

The category of quantum spaces, then, might be defined as the category dual to the category of associative, but not necessarily commutative, algebras. To define the notion of a quantum group, let us first return for a moment to the classical situation. If $G$ is a group, the multiplication $\mu : G \times G \to G$ of $G$ induces a homomorphism $\mu^* = \Delta : \mathcal{F}(G) \to \mathcal{F}(G \times G)$ of algebras of functions. Now, if we define the algebra $\mathcal{F}(G)$ and the tensor product appropriately, $\mathcal{F}(G \times G)$ will be isomorphic to $\mathcal{F}(G) \otimes \mathcal{F}(G)$ as an algebra. For example, if $G$ is an affine algebraic group over $\mathbb{C}$, and $\mathcal{F}(G)$ is the algebra of regular functions on $G$, the ordinary algebraic tensor product will do. Thus, we have a comultiplication $\Delta : \mathcal{F}(G) \to \mathcal{F}(G) \otimes \mathcal{F}(G)$. (The reason for this terminology is that the multiplication on $\mathcal{F}(G)$ can be viewed as a map $\mathcal{F}(G) \otimes \mathcal{F}(G) \to \mathcal{F}(G)$.) Similarly, the inverse map $\iota : G \to G$ induces a map $\iota^* = S : \mathcal{F}(G) \to \mathcal{F}(G)$, called the antipode, and evaluation at the identity element of $G$ is a homomorphism $\epsilon : \mathcal{F}(G) \to \mathbb{C}$, called the counit. The maps $\Delta$, $S$ and $\epsilon$ satisfy certain compatibility properties which reflect the defining properties of the inverse and the associativity of multiplication in $G$, and combine to give $\mathcal{F}(G)$ the structure of a *Hopf algebra.*

We might therefore *define the category of quantum groups to be the category dual to the category of (not necessarily commutative) Hopf algebras.* (We said 'might' here, and in our tentative definition of a quantum space, because, to ensure that the categories of quantum spaces and quantum groups have reasonable properties, it would be necessary to impose some restrictions on the class of algebras which are acceptable as 'quantized algebras of functions'. Manin suggests that one should work with 'Koszul algebras', but we shall not discuss this point here.) As is common practice in the literature, we shall often abuse terminology by referring to a Hopf algebra itself as a quantum group.

As the preceding discussion suggests, one way to try to construct non-classical examples of quantum groups is to look for deformations, in the category of Hopf algebras, of classical algebras of functions $\mathcal{F}(G)$. Just as the classical Poisson bracket can be recovered as the 'first order part' of Moyal's deformation (see (3)), so it turns out that the existence of a deformation $\mathcal{F}_h(G)$ of $\mathcal{F}(G)$ automatically endows the group $G$ itself with extra structure, namely that of a *Poisson–Lie group*. This is a Poisson structure on $G$ which is compatible with the group structure in a certain sense. Conversely, to construct deformations of $\mathcal{F}(G)$, it is natural to begin by describing the possible Poisson–Lie group structures on $G$ and then to attempt to extend these 'first order deformations' to full deformations. This is the approach taken in this book. Poisson–Lie groups are also of interest in their own right, for they form the natural setting for the study of classical integrable systems with symmetry.

There is another Hopf algebra associated to any Lie group $G$, namely the universal enveloping algebra $U(\mathfrak{g})$ of its Lie algebra $\mathfrak{g}$. This is essentially the *dual* of $\mathcal{F}(G)$ in the category of Hopf algebras. In general, the vector space dual $A^*$ of any finite-dimensional Hopf algebra $A$ is also a Hopf algebra: the multiplication $A^* \otimes A^* \to A^*$ is dual to the comultiplication $\Delta : A \to A \otimes A$ of $A$, and the comultiplication of $A^*$ is dual to the multiplication of $A$. Note that $A^*$ is commutative if and only if $A$ is cocommutative, i.e. if and only if $\Delta(A)$ is contained in the symmetric part of $A \otimes A$. If, as is usually the case in examples of interest, $A$ is infinite dimensional, this duality often continues to hold provided the dual and tensor product are defined appropriately. To a deformation $\mathcal{F}_h(G)$ of $\mathcal{F}(G)$ through (not necessarily commutative) Hopf algebras therefore corresponds a deformation $U_h(\mathfrak{g})$ of $U(\mathfrak{g})$ through (not necessarily cocommutative) Hopf algebras.

In fact, only non-cocommutative deformations of $U(\mathfrak{g})$ are of interest, since any deformation of $U(\mathfrak{g})$ through cocommutative Hopf algebras is necessarily of the form $U(\mathfrak{g}_h)$ for some deformation $\mathfrak{g}_h$ of $\mathfrak{g}$ through Lie algebras. However, many interesting Lie algebras have no non-trivial deformations. This is the case, for example, if $\mathfrak{g}$ is a (finite-dimensional) complex semisimple Lie algebra, such as the Lie algebra $sl_2(\mathbb{C})$ of $2 \times 2$ complex matrices of trace zero. This follows from the fact that the condition of semisimplicity is open, so that any small deformation of $\mathfrak{g}$ will still be semisimple, whereas the semisimple Lie algebras are discretely parametrized (by their Dynkin diagrams, for example).

The first example of a non-cocommutative deformation of this type was discovered by P. P. Kulish and E. K. Sklyanin in 1981 in the case $\mathfrak{g} = sl_2(\mathbb{C})$ (although the importance of its Hopf structure was not realized until later). Note that $sl_2(\mathbb{C})$ has a basis

$$(4) \qquad \bar{X}^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \bar{X}^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \bar{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

whose Lie brackets are given by

$$(5a) \qquad [\bar{X}^+, \bar{X}^-] = \bar{H}, \quad [\bar{H}, \bar{X}^\pm] = \pm 2\bar{X}^\pm.$$

The comultiplication is given on these basis elements by

$$(5b) \qquad \Delta(\bar{H}) = \bar{H} \otimes 1 + 1 \otimes \bar{H}, \quad \Delta(\bar{X}^\pm) = \bar{X}^\pm \otimes 1 + 1 \otimes \bar{X}^\pm,$$

an assignment which extends uniquely to an algebra homomorphism $\Delta : U(sl_2(\mathbb{C})) \to U(sl_2(\mathbb{C})) \otimes U(sl_2(\mathbb{C}))$. The deformation $U_h(sl_2(\mathbb{C}))$ is generated by elements $H$, $X^\pm$, which satisfy the relations

$$(6a) \qquad X^+X^- - X^-X^+ = \frac{e^{hH} - e^{-hH}}{e^h - e^{-h}}, \quad HX^\pm - X^\pm H = \pm 2X^\pm.$$

It has a non-cocommutative comultiplication given on generators by

$$(6b) \qquad \begin{gathered} \Delta(H) = H \otimes 1 + 1 \otimes H, \\ \Delta(X^+) = X^+ \otimes e^{hH} + 1 \otimes X^+, \quad \Delta(X^-) = X^- \otimes 1 + e^{-hH} \otimes X^-. \end{gathered}$$

Formally, at least, it is clear that (6a) and (6b) go over into (5a) and (5b) as $h \to 0$. The Hopf algebra defined in (6a,b) is called 'quantum $sl_2(\mathbb{C})$'. (See Chapter 6 for the formulas for the antipode and counit of $U_h(sl_2(\mathbb{C}))$, and for a way to make sense of expressions such as $e^{hH}$.)

The Hopf algebra dual to $U_h(sl_2(\mathbb{C}))$, the 'algebra $\mathcal{F}_h(SL_2(\mathbb{C}))$ of functions on quantum $SL_2(\mathbb{C})$', was discovered by L. D. Faddeev and L. A. Takhtajan in 1985. It is the associative algebra generated by elements $a$, $b$, $c$, $d$ with the following multiplicative relations:

$$(7) \qquad ab = e^{-h}ba, \quad ac = e^{-h}ca, \quad bd = e^{-h}db, \quad cd = e^{-h}dc,$$

$$(8) \qquad bc = cb, \quad ad - da + (e^h - e^{-h})bc = 0,$$

$$(9) \qquad ad - e^{-h}bc = 1,$$

and comultiplication

$$\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d,$$