DERIVATIVES IN FINANCIAL MARKETS WITH STOCHASTIC VOLATILITY

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Introduction

This book addresses problems in financial mathematics of pricing and hedging derivative securities in an environment of uncertain and changing market volatility. These problems are important to investors ranging from large trading institutions to pension funds. We introduce and systematically present mathematical and statistical tools that we have found to be very effective in this context. The material is suitable for a one-semester course for graduate students who have been exposed to methods of stochastic modeling and arbitrage pricing theory in finance. We have also aimed to make it easily accessible to derivatives practitioners in the financial engineering industry.

It is widely recognized that the simplicity of the popular Black–Scholes model, which relates derivative prices to current stock prices and quantifies risk through a *constant* volatility parameter, is no longer sufficient to capture modern market phenomena – especially since the 1987 crash. The natural extension of the Black–Scholes model that has been pursued in the literature and in practice is to modify the specification of volatility to make it a stochastic process. What makes this approach particularly challenging is first that volatility is a *hidden* process: it is driving prices and yet cannot be directly observed. Second, volatility tends to fluctuate at a high level for a while, then at a low level for a similar period, then high again, and so on. It "mean reverts" many times during the life of a derivative contract.

We describe here a method for modeling, analysis, and estimation that exploits this fast mean reversion of the volatility. It identifies the important groupings of market parameters, which otherwise are not obvious, and it turns out that estimation of these composites from market data is extremely efficient and stable. Furthermore, the methodology is robust in that it does not assume a specific volatility model.

Outline

In Chapter 1 we review the basic ideas and methods of the Black–Scholes theory as well as the stochastic calculus underpinning the models used. Chapter 2 motivates the stochastic volatility models and explains the difficulties induced by them. The main idea of volatility clustering, or *fast mean reversion,* is introduced in Chapter 3, where we describe – from scratch and with examples – the mathematics of this phenomenon. Tools for analyzing data to assess the rate of mean reversion of volatility are presented in Chapter 4. Our analysis shows clearly that the S&P 500 volatility is fast mean-reverting.

The problems we are interested in fall into two broad categories: pricing and hedging. In Chapters 5 and 6 we develop the asymptotic method that exploits volatility clustering for European derivatives. The theory identifies three group parameters that encode the effect of fast mean-reverting market volatility. We describe how these are estimated from the observed implied volatility skew and demonstrate their stability over time in the case of S&P 500 index options.

The extensions to exotic and American claims are described in Chapters 8 and 9, respectively. We outline in Chapter 11 how the tools are effective for fixed-income markets as well.

In Chapter 7 we describe hedging-type problems and how the asymptotic methodology is of use. These problems are somewhat different because, as we explain there, uncertain volatility gives rise to an incomplete market, where the effects of randomly changing volatility cannot be offset or hedged perfectly (in contrast to a known volatility, complete market environment).

Chapter 10 outlines various generalizations of the method: multidimensional situations, periodic effects, and non-Markovian models. We also show how it can be applied in other contexts such as the Merton portfolio optimization problem.

An overview of the problems presented in this book is shown in Figure 1.

Correcting Black–Scholes for Stochastic Volatility

The central idea of the book, presented in Chapters 3–6, is as follows. If volatility were running extremely fast then the market would behave as in a constant volatility Black–Scholes model. We will learn how this *effective volatility* arises. Then, since volatility is running fast but not extremely fast, we can treat the market as a perturbation of a constant volatility Black–Scholes market. We show how to compute the *correction,* which will reflect the effect of stochastic volatility on derivative prices. It involves the third derivative with respect to the price of the classical Black–Scholes price; we call this new "Greek" the *Epsilon*. What is

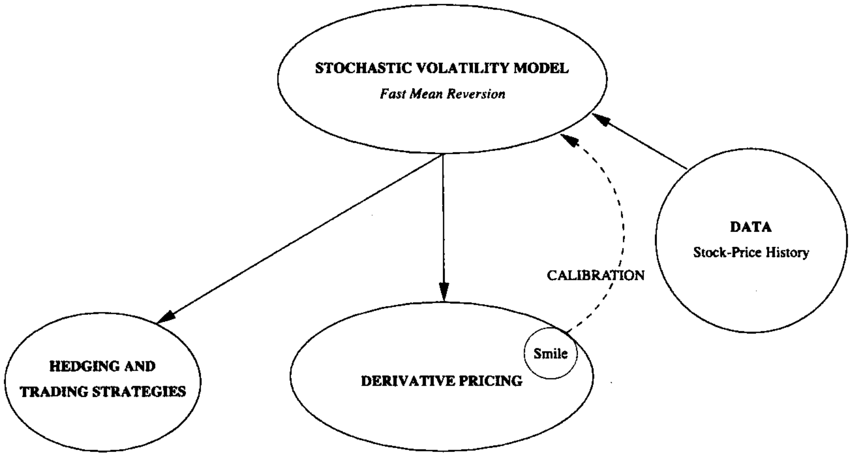

Figure 1. Overview of the problems discussed.

surprising is that all that is needed to compute this correction is the observed price and two quantities that are easily calibrated from the implied volatility surface. This calibration issue is discussed in Chapter 5.

A method for reducing hedging error is presented in Chapter 7. The method can be summarized in the case of European derivatives as follows.

(1) Fit an *affine function* of the log-moneyness-to-maturity ratio (LMMR) to the implied volatility surface across strikes and maturities for liquid European calls:

$$I = a\left(\frac{\log\left(\frac{\text{strike price}}{\text{asset price}}\right)}{\text{time to maturity}}\right) + b.$$

(2) From the estimated slope a and intercept b and the average volatility $\bar{\sigma}$ estimated from historical price data, deduce the two fundamental quantities

$$V_2 = \bar{\sigma}((\bar{\sigma} - b) - a(r + \tfrac{3}{2}\bar{\sigma}^2)),$$
$$V_3 = -a\bar{\sigma}^3,$$

which are small in the range of application of the method. We show that V_3 is directly related to the volatility *skew* and that V_2 contains the market price of volatility *risk*.

(3) Compute the Black–Scholes pricing function $P_0(t, x)$, which gives the price as a function of the present time t and underlying asset price x. For example, one could solve the Black–Scholes partial differential equation with the constant volatility $\bar{\sigma}$ and the appropriate terminal payoff.

(4) When the time to maturity is $T - t$, the *corrected Black–Scholes price* is given by

$$P_0(t, x) - (T - t)\left(V_2 x^2 \frac{\partial^2 P_0}{\partial x^2} + V_3 x^3 \frac{\partial^3 P_0}{\partial x^3}\right).$$

Observe that this computation does not require estimation of the present volatility and is *model-independent* in the sense that we are not calibrating any particular fully specified stochastic volatility model. The correction accounts for the market price of volatility risk.

(5) This approach leads to the following *hedging strategy*: Compute

$$Q = P_0 - (T - t)V_3\left(2x^2 \frac{\partial^2 P_0}{\partial x^2} + x^3 \frac{\partial^3 P_0}{\partial x^3}\right).$$

The proposed hedging strategy consists of holding

$$\frac{\partial Q}{\partial x}(t, x)$$

units of stock and

$$Q(t, x) - x\frac{\partial Q}{\partial x}(t, x)$$

worth of bonds, along the path of the stock-price process. This strategy *replicates* the claim at maturity. It is not self-financing, but we show that the correction compensates a systematic bias and reduces the variance of the cost. We then identify the remaining small nonhedgable part of that cost.

In the case of exotic derivatives treated in Chapter 8 and American derivatives in Chapter 9, we show that the same two quantities V_2 and V_3 are needed to implement this method. Analogous quantities for fixed-income markets are introduced in Chapter 11.

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Contents

1 The Black–Scholes Theory of Derivative Pricing

The aim of this first chapter is to review the basic objects, ideas, and results of the now classical Black–Scholes theory of derivative pricing. It is intended for readers who want to enter the subject or simply refresh their memory. This is not a complete treatment of this theory with detailed proofs but rather an intuitive but precise presentation that includes a few key calculations. Detailed presentations of the subject can be found in many books at various levels of mathematical rigor and generality, a few of which we list in the notes at the end of the chapter.

This book is about correcting the Black–Scholes theory in order to handle markets with stochastic volatility. The notation and many of the tools used in the constant volatility case will be used for the more complex markets throughout the book.

1.1 Market Model

In this simple model, suggested by Samuelson and used by Black and Scholes, there are two assets. One is a riskless asset (bond) with price β_t at time t, described by the ordinary differential equation

$$d\beta_t = r\beta_t\,dt, \tag{1.1}$$

where r, a nonnegative constant, is the instantaneous interest rate for lending or borrowing money. Setting $\beta_0 = 1$, we have $\beta_t = e^{rt}$ for $t \geq 0$. The price X_t of the other asset, the risky stock or stock index, evolves according to the stochastic differential equation

$$dX_t = \mu X_t\,dt + \sigma X_t\,dW_t, \tag{1.2}$$

where μ is a constant mean return rate, $\sigma > 0$ is a constant *volatility*, and $(W_t)_{t\geq 0}$ is a standard Brownian motion. This fundamental model and the intuitive content of equation (1.2) are presented in the following sections.

1.1.1 Brownian Motion

The Brownian motion is a stochastic process whose definition, existence, properties, and applications have been (and still are) the subject of numerous studies during the twentieth century. Our goal here is to give a very intuitive and practical presentation.

Brownian motion is a real-valued stochastic process with continuous trajectories that have independent and stationary increments. The trajectories are denoted by $t \to W_t$ and, for the standard Brownian motion, we have that:

- $W_0 = 0$;
- for any $0 < t_1 < \cdots < t_n$, the random variables $(W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}})$ are independent;
- for any $0 \leq s < t$, the increment $W_t - W_s$ is a centered (mean-zero) normal random variable with variance $\mathbb{E}\{(W_t - W_s)^2\} = t - s$. In particular, W_t is $\mathcal{N}(0, t)$-distributed.

We denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space where our Brownian motion is defined and the expectation $\mathbb{E}\{\cdot\}$ is computed. For instance, it could be $\Omega = C([0, +\infty) : \mathbb{R})$, the space of all continuous trajectories ω such that $W_t(\omega) = \omega(t)$. The σ-algebra $\mathcal{F}$ contains all sets of the form $\{\omega \in \Omega : |\omega(s)| < R, s \leq t\}$; $\mathbb{P}$ is the Wiener measure, which is the probability distribution of the standard Brownian motion.

The increasing family of σ-algebras $\mathcal{F}_t$ generated by $(W_s)_{s \leq t}$, the information on W up to time t, and all the sets of probability 0 in $\mathcal{F}$ is called the *natural filtration* of the Brownian motion. This *completion* by the null sets is important, in particular for the following reason. If two random variables X and Y are equal almost surely ($X = Y$ $\mathbb{P}$-a.s. means $\mathbb{P}\{X = Y\} = 1$) and if X is $\mathcal{F}_t$-measurable (meaning that any event $\{X_t \leq x\}$ belongs to $\mathcal{F}_t$), then Y is also $\mathcal{F}_t$-measurable.

A stochastic process $(X_t)_{t \geq 0}$ is *adapted* to the filtration $(\mathcal{F}_t)_{t \geq 0}$ if the random variable X_t is $\mathcal{F}_t$-measurable for every t. We say that (X_t) is $(\mathcal{F}_t)$*-adapted.* If another process (Y_t) is such that $X_t = Y_t$ $\mathbb{P}$-a.s. for every t, then it is also $(\mathcal{F}_t)$-adapted.

The independence of the increments of the Brownian motion and their normal distribution can be summarized using *conditional characteristic functions.* For $0 \leq s < t$ and $u \in \mathbb{R}$,

$$\mathbb{E}\{e^{iu(W_t - W_s)} \mid \mathcal{F}_s\} = e^{-u^2(t-s)/2}. \tag{1.3}$$

If W is a Brownian motion then, by independence of the increment $W_t - W_s$ from the past $\mathcal{F}_s$, the left-hand side of (1.3) is simply $\mathbb{E}\{e^{iu(W_t - W_s)}\}$, which is the characteristic function of a centered normal random variable with variance $t-s$, and is equal to the right-hand side. Conversely, if (1.3) holds then the continuous process (W_t) is a standard Brownian motion.

This independence of increments makes the Brownian motion an ideal candidate to define a complete family of independent infinitesimal increments dW_t, which are centered and normally distributed with variance dt and which will serve as a model of (Gaussian white) noise. The drawback is that the trajectories of (W_t) cannot be "nice" in the sense that they are not of bounded variation, as the following simple computation suggests. Let $t_0 = 0 < t_1 < \cdots < t_n = t$ be a subdivision of $[0, t]$, which we may suppose evenly spaced, so that $t_i - t_{i-1} = t/n$ for each interval. The quantity

$$\mathbb{E}\left\{\sum_{i=1}^{n} |W_{t_i} - W_{t_{i-1}}|\right\} = n\mathbb{E}\{|W_{t/n}|\} = n\sqrt{t/n}\,\mathbb{E}\{|W_1|\}$$

goes to $+\infty$ as $n \nearrow +\infty$, indicating that the integral with respect to dW_t cannot be defined in the usual way "trajectory by trajectory." We describe how such integrals can be defined in the next section.

1.1.2 Stochastic Integrals

For T a fixed finite time, let $(X_t)_{0\le t\le T}$ be a stochastic process adapted to $(\mathcal{F}_t)_{0\le t\le T}$, the filtration of the Brownian motion up to time T, such that

$$\mathbb{E}\left\{\int_0^T (X_t)^2\,dt\right\} < +\infty. \tag{1.4}$$

Using iterated conditional expectations and the independent increments property of Brownian motion, we note that

$$\mathbb{E}\left\{\left(\sum_{i=1}^{n} X_{t_{i-1}}(W_{t_i} - W_{t_{i-1}})\right)^2\right\} = \mathbb{E}\left\{\sum_{i=1}^{n}(X_{t_{i-1}})^2(t_i - t_{i-1})\right\}$$

for $t \le T$, which is a basic calculation in the construction of stochastic integrals. Note also that the Brownian increments on the left are forward in time and that the sum on the right converges to $\mathbb{E}\left\{\int_0^T (X_s)^2\,ds\right\}$, which, by (1.4), is finite.

The *stochastic integral* of (X_t) with respect to the Brownian motion (W_t) is defined as a limit in the mean-square sense ($L^2(\Omega)$),

$$\int_0^t X_s\, dW_s = \lim_{n \nearrow +\infty} \sum_{i=1}^{n} X_{t_{i-1}}(W_{t_i} - W_{t_{i-1}}), \tag{1.5}$$

as the mesh size of the subdivision goes to zero.

As a function of time t, this stochastic integral defines a continuous square integrable process such that

$$\mathbb{E}\left\{\left(\int_0^t X_s\, dW_s\right)^2\right\} = \mathbb{E}\left\{\int_0^t X_s^2\, ds\right\}. \tag{1.6}$$

It has the *martingale property*

$$\mathbb{E}\left\{\int_0^t X_u\, dW_u \mid \mathcal{F}_s\right\} = \int_0^s X_u\, dW_u \quad (\mathbb{P}\text{-a.s., } s \le t), \tag{1.7}$$

as can be easily deduced from the definition (1.5). The *quadratic variation* $\langle Y \rangle_t$ of the stochastic integral $Y_t = \int_0^t X_u\, dW_u$ is

$$\langle Y \rangle_t = \lim_{n \nearrow +\infty} \sum_{i=1}^{n} (Y_{t_i} - Y_{t_{i-1}})^2 = \int_0^t X_s^2\, ds \tag{1.8}$$

in the mean-square sense.

Stochastic integrals are mean-zero, continuous, and square integrable martingales. It is interesting to note that the converse is also true: every mean-zero, continuous, and square integrable martingale is a Brownian stochastic integral. This will be made precise and used in Section 1.4.

1.1.3 Risky Asset Price Model

The Black–Scholes model for the risky asset price corresponds to a continuous process (X_t) such that, in an infinitesimal amount of time dt, the infinitesimal return dX_t/X_t has mean $\mu\, dt$, proportional to dt, with a constant *rate of return* μ and centered random fluctuations that are independent of the past up to time t. These fluctuations are modeled by $\sigma\, dW_t$ where σ is a positive constant *volatility* and dW_t the infinitesimal increments of the Brownian motion. The corresponding formula for the infinitesimal return is

$$\frac{dX_t}{X_t} = \mu\, dt + \sigma\, dW_t, \tag{1.9}$$

which is the stochastic differential equation (1.2). The right side has the natural financial interpretation of a return term plus a risk term. We are also assuming that there are no dividends paid in the time interval that we are considering. It is easy to incorporate a continuous dividend rate in all that follows, but for simplicity we shall omit this.

In integral form, this equation is

$$X_t = X_0 + \mu \int_0^t X_s\, ds + \sigma \int_0^t X_s\, dW_s, \tag{1.10}$$

where the last integral is a stochastic integral as described in Section 1.1.2 and where X_0 is the initial value, which is assumed to be independent of the Brownian motion and square integrable.

Equation (1.10), or (1.2) in the differential form, is a particular case of a general class of stochastic differential equations driven by a Brownian motion:

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t \tag{1.11}$$

or, in integral form,

$$X_t = X_0 + \int_0^t \mu(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dW_s. \tag{1.12}$$

In the Black–Scholes model, $\mu(t, x) = \mu x$ and $\sigma(t, x) = \sigma x$; these are independent of t, differentiable in x, and linearly growing at infinity (since they are linear). This is enough to ensure existence and uniqueness of a continuous adapted and square integrable solution (X_t). The proof of this result is based on simple estimates like

$$\begin{aligned} \mathbb{E}\{X_t^2\} &= \mathbb{E}\left\{\left(X_0 + \mu \int_0^t X_s\, ds + \sigma \int_0^t X_s\, dW_s\right)^2\right\} \\ &\le 3\left(\mathbb{E}\{X_0^2\} + (\mu^2 T + \sigma^2) \int_0^t \mathbb{E}\{X_s^2\}\, ds\right), \end{aligned}$$

where we have used the inequality $(a + b + c)^2 \le 3(a^2 + b^2 + c^2)$, the Cauchy–Schwarz inequality

$$\mathbb{E}\left(\int_0^t X_s\, ds\right)^2 \le t \int_0^t \mathbb{E}\{X_s^2\}\, ds,$$

and (1.6). We deduce that

$$0 \le \mathbb{E}\{X_t^2\} \le c_1 + c_2 \int_0^t \mathbb{E}\{X_s^2\}\, ds$$

for $0 \le t \le T$ and constants $c_1, c_2 \ge 0$. By a direct application of Gronwall's lemma, we deduce that the solution is a priori square integrable. The construction of a solution and the proof of uniqueness can be obtained by similar and slightly more complicated estimates that use the Kolmogorov–Doob inequality for martingales.

Looking at equation (1.9), it is very tempting to write X_t/X_0 explicitly as the exponential of $(\mu t + \sigma W_t)$. However, this is not correct because the usual chain rule is