HEP Studies in Modern Mathematics

Introduction to Topology

Theory and Applications

拓扑学教程

——理论及应用

Min Yan

高等教育出版社
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Dedicated to my father, my mother and Paulina

Preface

For mathematicians, topology is a fundamental mathematical language widely used in many fields. For students, topology is an intellectually challenging and rewarding subject. This textbook aims to address the subject of topology from both angles. The development of the content is based on the following considerations.

First, the topology theory has the point set as well as the combinatorial (or algebraic) aspects. This book intends to give students a more comprehensive view of topology. So materials in point set topology and combinatorial topology are arranged in alternating chapters. Of course only the most basic topics can be covered in a semester. This means point set topology up to Hausdorff, connected, and compact properties, and combinatorial topology up to the Euler number and the classification of surfaces. A final chapter is added to cover the important and useful topics in point set topology. The topics in the final chapter are not covered in my lecture.

Second, the basic topological theory is a tool used for describing certain aspects of mathematics. So we should keep in mind how the topology is actually used in the other fields of mathematics. For example, the topologies are always introduced from topological basis or subbasis in practical applications. Therefore this book introduces the topological basis before the concept of topology, and emphasizes how to "compute" the topological concepts by making use of topological basis.

Third, the theory of point set topology can be very abstract, and the axiomatic approach can be daunting for students. This book starts with metric spaces, which is more concrete and familiar to students. The topological concepts are defined from the viewpoint of metrics but are quickly reinterpreted in terms of balls. Later on, by replacing the balls with the topological basis, students can easily understand the same concepts in the more abstract setting.

Fourth, the effective learning of abstract theory requires lots of practice. The book contains plenty of exercises. Moreover, the exercises immediately follow discussions, instead of being listed separately at the end of sections. Many exercises require the students to compute topological concepts in very specific and concrete topological spaces. There are also many exercises that ask the students to prove some basic results, some of which are used in the proofs.

The book was originally the lecture note for my topology course in The Hong Kong University of Science and Technology. I would like to thank the university and enthusiastic students for their support.

Min Yan

Contents

Chapter 1

Set and Map

1.1 Set

Sets and elements are the most basic concepts of mathematics. Given any *element* x and any *set* X, either x belongs to X (denoted $x \in X$), or x does not belong to X (denoted $x \notin X$). Elements are also figuratively called *points*.

Example 1.1.1. A set can be presented by listing all its elements.

1. $X_1 = \{1, 2, 3, \dots, n\}$ is the set of all integers between 1 and n.
2. $X_2 = \{2, -5\}$ is the set of all numbers satisfying the equation $x^2 + 3x - 10 = 0$.
3. $X_3 = \{a, b, c, \dots, x, y, z\}$ is the set of all latin alphabets.
4. $X_4 = \{\text{red, green, blue}\}$ is the set of basic colors that combine to form all the colors human can see.
5. $X_5 = \{\text{red, yellow}\}$ is the set of colors on the Chinese national flag.
6. The set X_6 of all registered students in the topology course is the list of names provided to me by the registration office.

Example 1.1.2. A set can also be presented by describing the properties satisfied by the elements.

1. Natural numbers $\mathbb{N} = \{n : n \text{ is obtained by repeatedly adding 1 to itself}\}$.
2. Prime numbers $\mathbb{P} = \{p : p \in \mathbb{N}, \text{ and the only integers dividing } p \text{ are } \pm 1 \text{ and } \pm p\}$.
3. Rational numbers $\mathbb{Q} = \{r : r \text{ is a quotient of two integers}\}$.
4. Open interval $(a, b) = \{x : x \in \mathbb{R}, a < x < b\}$.
5. Closed interval $[a, b] = \{x : x \in \mathbb{R}, a \leq x \leq b\}$.
6. Real polynomials $\mathbb{R}[t] = \{a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n : a_i \in \mathbb{R}\}$.
7. Continuous functions $C[0,1] = \{f : \lim_{t \to a} f(t) = f(a) \text{ for any } 0 \leq a \leq 1\}$ on $[0,1]$.
8. Unit sphere $S^2 = \{(x_1, x_2, x_3) : x_i \in \mathbb{R}, x_1^2 + x_2^2 + x_3^2 = 1\}$ in $\mathbb{R}^3$.
9. $X_1 = \{x : x \in \mathbb{N}, x \leq n\}$, the first set in Example 1.1.1.
10. $X_2 = \{x : x^2 + 3x - 10 = 0\}$, the second set in Example 1.1.1.

Exercise 1.1.1. Present the following sets: the set $\mathbb{Z}$ of integers, the unit sphere S^n in $\mathbb{R}^{n+1}$, the set $GL(n)$ of invertible $n \times n$ matrices, the set of latin alphabets in your name.

Exercise 1.1.2. Provide suitable names for the following subsets of $\mathbb{R}^2$.

1. $\{(x, y) : x = 0\}$.
2. $\{(x, y) : x = y\}$.
3. $\{(x, y) : x^2 + y^2 = 4\}$.
4. $\{(x, y) : x^2 + y^2 > 4\}$.
5. $\{(x, y) : y < 0\}$.
6. $\{(x, y) : x^2 + 4y^2 = 4\}$.
7. $\{(x, y) : |x| + |y| < 1\}$.
8. $\{(x, y) : |x| < 1, |y| < 1\}$.

A set X is a *subset* of Y if $x \in X$ implies $x \in Y$. In this case, we denote $X \subset Y$ and say "X is *contained in* Y", or denote $Y \supset X$ and say "Y *contains* X". The subset relation has the following properties:

- *reflexivity*: $X \subset X$.
- *antisymmetry*: $X \subset Y, Y \subset X \iff X = Y$.
- *transitivity*: $X \subset Y, Y \subset Z \implies X \subset Z$.

When the subsets are presented by properties satisfied by elements, the subset relation is the logical relation between the properties. For example, the subset relation $\{n\colon n^2 \text{ is even}\} \subset \{n\colon n \text{ is even}\}$ simply means "if n^2 is even, then n is even".

The special notation $\emptyset$ is reserved for the *empty set*, the set with no element. The empty set is a subset of any set.

The *power set* $\mathcal{P}(X)$ of a set X, also denoted as 2^X, is the collection of all subsets of X. For example, the power set of the set $\{1, 2, 3\}$ is

$$\mathcal{P}\{1,2,3\} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

The power set demonstrates that sets themselves can become elements of some other set, which we usually call a *collection* of sets. For example, the set

$$\{\emptyset, \{1,2\}, \{1,3\}, \{2,3\}\}$$

is the collection of subsets of $\{1, 2, 3\}$ with even number of elements.

Collections of sets appear quite often in everyday life and will play a very important role in the development of point set topology. For example, the set Parity $=$ {Even, Odd} is actually the collection of two sets

$$\text{Even} = \{\dots, -4, -2, 0, 2, 4, \dots\}, \quad \text{Odd} = \{\dots, -3, -1, 1, 3, 5, \dots\},$$

and the set Sign $= \{+, -, 0\}$ is the collection of three sets

$$+ = \{x\colon x > 0\}, \quad - = \{x\colon x < 0\}, \quad 0 = \{0\}.$$

Exercise 1.1.3. How many elements are in the power set of $\{1, 2, \dots, n\}$? (The answer suggests the reason for the notation 2^X.) How many of these contain even number of elements?

Exercise 1.1.4. List all elements in $\mathcal{P}(\mathcal{P}\{1, 2\})$, the power set of the power set of $\{1, 2\}$.

Exercise 1.1.5. Show that the set of numbers x satisfying $x^2 = 6x - 8$ is the same as the set of even integers between 1 and 5.

Exercise 1.1.6. For any real number $\epsilon > 0$, find a real number $\delta > 0$, such that

$$\{x\colon |x-1| < \delta\} \subset \{x\colon |x^2 - 1| < \epsilon\}.$$

Exercise 1.1.7. Find a number $n \in \mathbb{N}$, such that

$$\{m\colon m > n\} \subset \left\{m\colon \left|\frac{m}{m^2+1}\right| < 0.0001\right\}.$$

Exercise 1.1.8. For $r > 0$, let

$$D_r = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le r^2\},$$
$$S_r = \{(x,y) \in \mathbb{R}^2 : |x| \le r, |y| \le r\}.$$

What is the necessary and sufficient condition for $D_r \subset S_{r'}$? What is the necessary and sufficient condition for $S_{r'} \subset D_r$?

Exercise 1.1.9. A set X is a *proper subset* of Y if $X \subset Y$ and $X \ne Y$. Prove that if $X \subset Y \subset Z$, then the following are equivalent.

1. X is a proper subset of Z.
2. Either X is a proper subset of Y, or Y is a proper subset of Z.

New sets can be constructed from the existing ones by the following basic operations:

- *union*: $X \cup Y = \{x : x \in X \text{ or } x \in Y\}$.
- *intersection*: $X \cap Y = \{x : x \in X \text{ and } x \in Y\}$.
- *difference*: $X - Y = \{x : x \in X \text{ and } x \notin Y\}$.
- *product*: $X \times Y = \{(x,y) : x \in X, y \in Y\}$.

Two sets X and Y are *disjoint* if $X \cap Y = \emptyset$. In other words, X and Y share no common element. The union of disjoint sets is sometimes denoted as $X \sqcup Y$, called *disjoint union*. If $Y \subset X$, then $X - Y$ is also called the *complement* of Y in X. Moreover, X^n denotes the product of n copies of X.

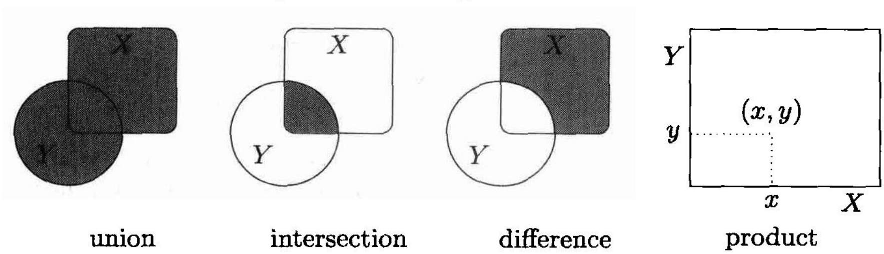

Figure 1.1.1. *set operations*

Some set operations can be extended to a collection $\{X_i : i \in I\}$ of sets:

- *union*: $\cup_i X_i = \{x : x \in X_i \text{ for some } i \in I\}$.
- *intersection*: $\cap_i X_i = \{x : x \in X_i \text{ for all } i \in I\}$.
- *product*: $\times_i X_i = \{(x_i)_{i \in I} : x_i \in X_i \text{ for each } i \in I\}$.

We also have the disjoint union $\sqcup_i X_i$ when the collection is pairwise disjoint: $X_i \cap X_j = \emptyset$ in case $i \ne j$.

The union and the intersection have the following properties:

- $X \cap Y \subset X \subset X \cup Y$, $X \cup \emptyset = X$, $X \cap \emptyset = \emptyset$.
- *commutativity*: $Y \cup X = X \cup Y$, $Y \cap X = X \cap Y$.
- *associativity*: $(X \cup Y) \cup Z = X \cup (Y \cup Z)$, $(X \cap Y) \cap Z = X \cap (Y \cap Z)$.
- *distributivity*: $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$, $(X \cap Y) \cup Z = (X \cup Z) \cap (Y \cup Z)$.

The properties are not hard to prove. For example, the proof of the associativity of the union involves the verification of $(X \cup Y) \cup Z \subset X \cup (Y \cup Z)$ and $(X \cup Y) \cup Z \supset X \cup (Y \cup Z)$. The first inclusion is verified below.

$$\begin{aligned} x \in (X \cup Y) \cup Z &\implies x \in X \cup Y \text{ or } x \in Z \\ &\implies x \in X \text{ or } x \in Y \text{ or } x \in Z \\ &\implies x \in X \text{ or } x \in Y \cup Z \\ &\implies x \in X \cup (Y \cup Z). \end{aligned}$$

The second inclusion can be verified similarly.

The properties for the union and the intersection can be extended to any (especially infinite) union and intersection.

The difference has the following properties:

- $X - Y = \emptyset \iff X \subset Y$.
- $X - Y = X \iff X \cap Y = \emptyset$.
- $Y \subset X \implies X - (X - Y) = Y$.
- *de Morgan's Law*[1]: $X - (Y \cup Z) = (X - Y) \cap (X - Z)$, $X - (Y \cap Z) = (X - Y) \cup (X - Z)$.

Note that de Morgan's law basically says that the complement operation exchanges the union and the intersection. The property can be extended to any union and intersection.

When several operations are mixed, we have a rule similar to the arithmetic operations. Usually the product $\times$ is taken first, then the union $\cup$ or the intersection $\cap$ is taken, and finally the difference $-$ is taken. For example, $x \in (X - Y \times Z) \cap W - U$ means $x \in X$, $x \notin Y \times Z$, $x \in W$, and $x \notin U$.

Exercise 1.1.10. What are the following sets?

1. $\{n \colon n \text{ is even}\} \cap \{n \colon n \text{ is divisible by } 5\}$.
2. $\{n \colon n \text{ is a positive integer}\} \cup \{n \colon n \text{ is a negative integer}\} \cup \{0\}$.
3. $\{n \colon n \text{ is even}\} \cup \{n \colon |n| < 10 \text{ and } n \text{ is an integer}\} - \{n \colon n \neq 2\} \cap \{n \colon n^2 \neq 6n - 8\}$.
4. $\{x \colon 1 < x \leq 10\} - \{x \colon 2x \text{ is an integer}\} \cap \{x \colon 6x^3 + 5x^2 - 33x + 18 = 0\}$.

[1] Augustus de Morgan, born June 27, 1806 in Madurai, India, died March 18, 1871 in London, England. De Morgan introduced the term "mathematical induction" to put the proving method on a rigorous basis. His most important contribution is to the subject of formal logic.

5. $\{x\colon x>0\}\times\{y\colon |y|<1\}-\{(x,y)\colon x+y<1\}-\{(x,y)\colon x>y\}$.
6. $\{(x,y)\colon x^2+y^2\leq 1\}-\{x\colon x\geq 0\}^2$.

Exercise 1.1.11. Prove the properties of set operations.

1. $(X\cap Y)\cap Z=X\cap(Y\cap Z)$.
2. $(X\cap Y)\cup Z=(X\cup Z)\cap(Y\cup Z)$.
3. $X-(Y\cap Z)=(X-Y)\cup(X-Z)$.
4. $X\cup Y-Z=(X-Z)\cup(Y-Z)$.
5. $(X-Y)\cup(Y-X)=X\cup Y-X\cap Y$.
6. $(X-Y)\cap(Y-X)=\emptyset$.
7. $X\times(Y\cap Z)=(X\times Y)\cap(X\times Z)$.
8. $X\times Y-X\times Z=X\times(Y-Z)$.

Exercise 1.1.12. Find all the unions and intersections among $A=\{\emptyset\}$, $B=\{\emptyset,A\}$, $C=\{\emptyset,A,B\}$.

Exercise 1.1.13. Express the following using sets X, Y, Z and operations $\cup$, $\cap$, $-$.

1. $A=\{x\colon x\in X \text{ and } (x\in Y \text{ or } x\in Z)\}$.
2. $B=\{x\colon (x\in X \text{ and } x\in Y) \text{ or } x\in Z\}$.
3. $C=\{x\colon x\in X,\ x\notin Y, \text{ and } x\in Z\}$.

Exercise 1.1.14. Let A and B be subsets of X. Prove that

$$A\subset B \iff X-A\supset X-B \iff A\cap(X-B)=\emptyset.$$

Exercise 1.1.15. Which statements are true?

1. $X\subset Z$ and $Y\subset Z \implies X\cup Y\subset Z$.
2. $X\subset Z$ and $Y\subset Z \implies X\cap Y\subset Z$.
3. $X\subset Z$ or $Y\subset Z \implies X\cup Y\subset Z$.
4. $Z\subset X$ and $Z\subset Y \implies Z\subset X\cap Y$.
5. $Z\subset X$ and $Z\subset Y \implies Z\subset X\cup Y$.
6. $Z\subset X\cap Y \implies Z\subset X$ and $Z\subset Y$.
7. $Z\subset X\cup Y \implies Z\subset X$ and $Z\subset Y$.
8. $X-(Y-Z)=(X-Y)\cup Z$.
9. $(X-Y)-Z=X-Y\cup Z$.
10. $X-(X-Z)=Z$.
11. $X\cap Y-Z=(X-Z)\cap(Y-Z)$.
12. $X\cap(Y-Z)=X\cap Y-X\cap Z$.
13. $X\cup(Y-Z)=X\cup Y-X\cup Z$.
14. $(X\cap Y)\cup(X-Y)=X$.
15. $X\subset U$ and $Y\subset V$ $\iff X\times Y\subset U\times V$.
16. $X\times(U\cup V)=X\times U\cup X\times V$.
17. $X\times(U-V)=X\times U-X\times V$.
18. $(X-Y)\times(U-V)=X\times U-Y\times V$.

Exercise 1.1.16. The union $X\cup Y$ of two sets X and Y is naturally divided into a union of three disjoint subsets $X-Y$, $Y-X$ and $X\cap Y$.

1. How many pieces can the union of three sets be naturally divided into?
2. Express the pieces in the first part in terms of the set operations. (One such piece is $X-(Y\cup Z)$, for example.)
3. How many pieces constitute the union of n sets?

1.2 Map

A *map* from a set X to a set Y is a process f that assigns, for each $x \in X$, a unique $y = f(x) \in Y$. The process should be *well-defined* in the following sense:

- *applicability*: The process can be applied to any input $x \in X$ to produce some output $f(x)$.

- *unambiguity*: For any input $x \in X$, the output $f(x)$ of the process is unique. In other words, the process f is *single-valued*.

In case Y is a set of numbers, the map f is also called a *function*.

A map f from X to Y is usually denoted as

$$f: X \to Y,\ x \mapsto f(x),$$

or

$$f(x) = y: X \to Y.$$

The sets X and Y are the *domain* and the *range* of the map. The point y is the *image* (or the *value*) of x.

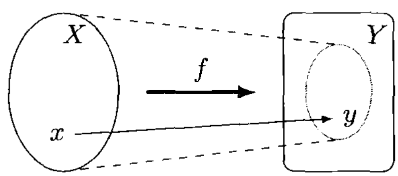

Figure 1.2.1. *domain, range, image*

Example 1.2.1. The map $f(x) = 2x^2 - 1: \mathbb{R} \to \mathbb{R}$ (equivalently, $f: \mathbb{R} \to \mathbb{R},\ x \mapsto 2x^2 - 1$) means the following process: Multiply x to itself to get x^2. Then multiply 2 to get $2x^2$. Finally subtract 1 to get $2x^2 - 1$. Since each step always works and gives unique outcome, the process is a map.

Example 1.2.2. The square root function $f(x) = \sqrt{x}: [0, \infty) \to \mathbb{R}$ is the following process: For any $x \geq 0$, find a *non-negative* number y, such that multiplying y to itself yields x. Then $f(x) = y$. Again, since y always exists and is unique, the process is a map.

If $[0, \infty)$ is changed to $\mathbb{R}$, then y does not exist for negative x. So the applicability condition is violated, and the process $\sqrt{x}: \mathbb{R} \to \mathbb{R}$ is not a map.

If the process is modified by no longer requiring y to be non-negative. Then the process can be applied to any $x \in [0, \infty)$, except there will be two outcomes (one positive, one negative) in general. So the unambiguity condition is violated, and the process is again not a map.

Example 1.2.3. The map $R_\theta(x_1, x_2) = (x_1 \cos\theta - x_2 \sin\theta, x_1 \sin\theta + x_2 \cos\theta): \mathbb{R}^2 \to \mathbb{R}^2$ is the process of rotating points on the plane around the origin by angle θ in the counterclockwise direction.

Example 1.2.4. The map Age: $X_6 \to \mathbb{N}$ is the process of subtracting the birth year from the current year. The map Instructor: Courses $\to$ Professors takes the course to the professor who teaches the course. For example, the map "Instructor" takes "topology" to "me".

Example 1.2.5. For any set X, the *identity map* is

$$id_X(x) = x \colon X \to X,$$

and the *diagonal map* is

$$\Delta_X(x) = (x, x) \colon X \to X^2.$$

For any sets X, Y and fixed $b \in Y$, the map

$$c(x) = b \colon X \to Y$$

is a *constant map*. Moreover, there are two *projection maps*

$$\pi_X(x, y) = x \colon X \times Y \to X, \quad \pi_Y(x, y) = y \colon X \times Y \to Y,$$

from the product of two sets.

Exercise 1.2.1. The following attempts to define a "square root" map. Which are actually maps?

1. For $x \in \mathbb{R}$, find $y \in \mathbb{R}$, such that $y^2 = x$. Then $f(x) = y$.
2. For $x \in [0, \infty)$, find $y \in \mathbb{R}$, such that $y^2 = x$. Then $f(x) = y$.
3. For $x \in [0, \infty)$, find $y \in [0, \infty)$, such that $y^2 = x$. Then $f(x) = y$.
4. For $x \in [1, \infty)$, find $y \in [1, \infty)$, such that $y^2 = x$. Then $f(x) = y$.
5. For $x \in [1, \infty)$, find $y \in (-\infty, -1]$, such that $y^2 = x$. Then $f(x) = y$.
6. For $x \in [0, 1)$, find $y \in [0, \infty)$, such that $y^2 = x$. Then $f(x) = y$.
7. For $x \in [1, \infty)$, find $y \in (-\infty, -4] \cup [1, 2)$, such that $y^2 = x$. Then $f(x) = y$.

Exercise 1.2.2. Describe the processes that define the maps.

1. $2^n \colon \mathbb{Z} \to \mathbb{R}$.
2. $\sin \colon \mathbb{R} \to \mathbb{R}$.
3. Angle: Ordered pairs of straight lines in $\mathbb{R}^2 \to [0, 2\pi)$.
4. Area_r : Rectangles $\to [0, \infty)$.
5. Area_t : Triangles $\to [0, \infty)$.
6. Absolute value: $\mathbb{R} \to \mathbb{R}$.

Exercise 1.2.3. Suppose we want to combine two maps $f \colon X \to Z$ and $g \colon Y \to Z$ to get a new map $h \colon X \cup Y \to Z$ by

$$h(x) = \begin{cases} f(x), & \text{if } x \in X, \\ g(x), & \text{if } x \in Y. \end{cases}$$

What is the condition for h to be a map?

The *image* of a subset $A \subset X$ under a map $f\colon X \to Y$ is

$$f(A) = \{f(a)\colon a \in A\} = \{y \in Y\colon y = f(a) \text{ for some } a \in A\}.$$

In the other direction, the *preimage* of a subset $B \subset Y$ is

$$f^{-1}(B) = \{x\colon f(x) \in B\}.$$

The image and the preimage have the following properties: The image and the preimage have the following properties:

- $f(A) \subset B \iff A \subset f^{-1}(B)$.
- $A \subset A' \implies f(A) \subset f(A')$.
- $B \subset B' \implies f^{-1}(B) \subset f^{-1}(B')$.
- $f(A \cup A') = f(A) \cup f(A')$.
- $f^{-1}(B \cup B') = f^{-1}(B) \cup f^{-1}(B')$.
- $f(A \cap A') \subset f(A) \cap f(A')$.
- $f^{-1}(B \cap B') = f^{-1}(B) \cap f^{-1}(B')$.
- $X - f^{-1}(B) = f^{-1}(Y - B)$.
- $A \subset f^{-1}(B) \iff f(A) \subset B$.

The properties can be extended to any union and intersection.

Example 1.2.6. Both the domain and the range of the map $f(x) = 2x^2 - 1\colon \mathbb{R} \to \mathbb{R}$ are $\mathbb{R}$. The image of the whole domain is $f(\mathbb{R}) = [-1, \infty)$. The image of $[0, \infty)$ is also $[-1, \infty)$, and the image of $[1, \infty)$ is $[1, \infty)$. The preimage of $[0, \infty)$ and 1 are respectively $\left(-\infty, -\frac{1}{\sqrt{2}}\right] \cup \left[\frac{1}{\sqrt{2}}, \infty\right)$ and $\{1, -1\}$.

Both the domain and the range of the rotation map $R_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ are $\mathbb{R}^2$. The image and the preimage of any circle centered at the origin are the circle itself. If the circle is not centered at the origin, then the image and the preimage are still circles, but at different locations.

The preimage $\text{Age}^{-1}(20)$ is all the 20-year-old students in the class. The preimage $\text{Instructor}^{-1}(\text{me})$ is all the courses I teach.

Exercise 1.2.4. Find the image and the preimage of a straight line under the rotation map R_θ. When is the image or the preimage the same as the original line?

Exercise 1.2.5. Show that in the property $f(A \cap A') \subset f(A) \cap f(A')$, the two sides are not necessarily the same.

Exercise 1.2.6. Which statements are true? If not, whether at least some directions or inclusions are true?

1. $A \cap A' = \emptyset \iff f(A) \cap f(A') = \emptyset$.
2. $B \cap B' = \emptyset \iff f^{-1}(B) \cap f^{-1}(B') = \emptyset$.
3. $f(A - A') = f(A) - f(A')$.
4. $f^{-1}(B - B') = f^{-1}(B) - f^{-1}(B')$.

Exercise 1.2.7. Prove that $f(f^{-1}(B)) = B \cap f(X)$. In particular, $f(f^{-1}(B)) = B$ if and only if $B \subset f(X)$.

Exercise 1.2.8. Prove that $f^{-1}(f(A)) = A$ if and only if $A = f^{-1}(B)$ for some $B \subset Y$.

Exercise 1.2.9. For a subset $A \subset X$, define the *characteristic function*

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases} : X \to \mathbb{R}.$$

1. Prove $\chi_{A \cap B} = \chi_A \chi_B$ and $\chi_{X-A} + \chi_A = 1$.
2. Express $\chi_{A \cup B}$ in terms of χ_A and χ_B.
3. Describe the image and the preimage under the map χ_A.

Exercise 1.2.10. Let $f\colon X \to Y$ be a map. Show that the "image map"

$$F\colon \mathcal{P}(X) \to \mathcal{P}(Y),\ A \mapsto f(A)$$

is indeed a map. Is the similar "preimage map" also a map?

A map $f\colon X \to Y$ is *onto* (or *surjective*) if $f(X) = Y$. This means that any $y \in Y$ is $f(x)$ for some $x \in X$. The map is *one-to-one* (or *injective*) if

$$f(x_1) = f(x_2) \implies x_1 = x_2.$$

This is equivalent to

$$x_1 \neq x_2 \implies f(x_1) \neq f(x_2).$$

In other words, different elements have different images. The map is a *one-to-one correspondence* (or *bijective*) if it is one-to-one and onto.

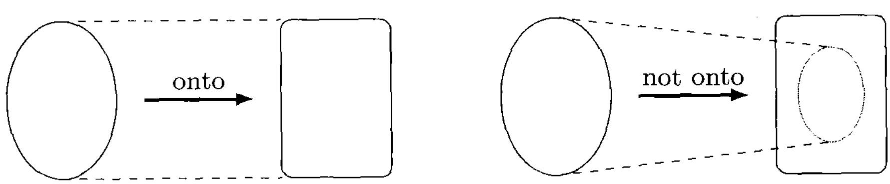

Figure 1.2.2. *onto and not onto*

For a map $f\colon X \to Y$, there are generally three possibilities at $y \in Y$:

1. *zero to one*: There is no $x \in X$ satisfying $f(x) = y$. This is the same as $y \notin f(X)$.