HZ BOOKS 华章教育

PEARSON

经 典 原 版 书 库

实分析

（英文版 · 第4版）

REAL ANALYSIS

H.L. Royden · P.M. Fitzpatrick

Fourth Edition

（美） H. L. Royden 斯坦福大学 P. M. Fitzpatrick 马里兰大学帕克分校 著

机械工业出版社
China Machine Press

经典原版书库

实分析

（英文版·第4版）

Real Analysis

(Fourth Edition)

H. L. Royden
斯坦福大学
（美）　　　　著
P. M. Fitzpatrick
马里兰大学帕克分校

机械工业出版社
China Machine Press

本书版权登记号：图字：01-2010-3826

图书在版编目（CIP）数据

实分析（英文版·第4版）/（美）罗伊登（Royden, H. L.），菲茨帕特里克（Fitzpatrick, P. M.）著. —北京：机械工业出版社，2010.7

（经典原版书库）

书名原文：Real Analysis, Fourth Edition

ISBN 978-7-111-31305-2

Ⅰ.实…　Ⅱ.①罗…　②菲…　Ⅲ.实分析－英文　Ⅳ.O174.1

中国版本图书馆CIP数据核字（2010）第131745号

机械工业出版社（北京市西城区百万庄大街22号　邮政编码　100037）
责任编辑：李俊竹
北京京师印务有限公司印刷
2010年8月第1版第1次印刷
150mm×214mm · 16.125印张
标准书号：ISBN 978-7-111-31305-2
定价：49.00元

凡购本书，如有缺页、倒页、脱页，由本社发行部调换
客服热线：(010) 88378991；88361066
购书热线：(010) 68326294；88379649；68995259
投稿热线：(010) 88379604
读者信箱：hzjsj@hzbook.com

Preface

The first three editions of H.L.Royden's *Real Analysis* have contributed to the education of generations of mathematical analysis students. This fourth edition of *Real Analysis* preserves the goal and general structure of its venerable predecessors—to present the measure theory, integration theory, and functional analysis that a modern analyst needs to know.

The book is divided the three parts: Part I treats Lebesgue measure and Lebesgue integration for functions of a single real variable; Part II treats abstract spaces—topological spaces, metric spaces, Banach spaces, and Hilbert spaces; Part III treats integration over general measure spaces, together with the enrichments possessed by the general theory in the presence of topological, algebraic, or dynamical structure.

The material in Parts II and III does not formally depend on Part I. However, a careful treatment of Part I provides the student with the opportunity to encounter new concepts in a familiar setting, which provides a foundation and motivation for the more abstract concepts developed in the second and third parts. Moreover, the Banach spaces created in Part I, the L^p spaces, are one of the most important classes of Banach spaces. The principal reason for establishing the completeness of the L^p spaces and the characterization of their dual spaces is to be able to apply the standard tools of functional analysis in the study of functionals and operators on these spaces. The creation of these tools is the goal of Part II.

NEW TO THE EDITION

- This edition contains 50% more exercises than the previous edition
- Fundamental results, including Egoroff's Theorem and Urysohn's Lemma are now proven in the text.
- The Borel-Cantelli Lemma, Chebychev's Inequality, rapidly Cauchy sequences, and the continuity properties possessed both by measure and the integral are now formally presented in the text along with several other concepts.

There are several changes to each part of the book that are also noteworthy:

Part I

- The concept of uniform integrability and the Vitali Convergence Theorem are now presented and make the centerpiece of the proof of the fundamental theorem of integral calculus for the Lebesgue integral
- A precise analysis of the properties of rapidly Cauchy sequences in the $L^p(E)$ spaces, $1 \leq p \leq \infty$, is now the basis of the proof of the completeness of these spaces
- Weak sequential compactness in the $L^p(E)$ spaces, $1 \leq p \leq \infty$, is now examined in detail and used to prove the existence of minimizers for continuous convex functionals.

Part II

- General structural properties of metric and topological spaces are now separated into two brief chapters in which the principal theorems are proven.
- In the treatment of Banach spaces, beyond the basic results on bounded linear operators, compactness for weak topologies induced by the duality between a Banach space and its dual is now examined in detail.
- There is a new chapter on operators in Hilbert spaces, in which weak sequential compactness is the basis of the proofs of the Hilbert-Schmidt theorem on the eigenvectors of a compact symmetric operator and the characterization by Riesz and Schuader of linear Fredholm operators of index zero acting in a Hilbert space.

Part III

- General measure theory and general integration theory are developed, including the completeness, and the representation of the dual spaces, of the $L^p(X, \mu)$ spaces for, $1 \leq p \leq \infty$. Weak sequential compactness is explored in these spaces, including the proof of the Dunford-Pettis theorem that characterizes weak sequential compactness in $L^1(X, \mu)$.
- The relationship between topology and measure is examined in order to characterize the dual of $C(X)$, for a compact Hausdorff space X. This leads, via compactness arguments, to (i) a proof of von Neumann's theorem on the existence of unique invariant measures on a compact group and (ii) a proof of the existence, for a mapping on a compact Hausdorf space, of a probability measure with respect to which the mapping is ergodic.

The general theory of measure and integration was born in the early twentieth century. It is now an indispensable ingredient in remarkably diverse areas of mathematics, including probability theory, partial differential equations, functional analysis, harmonic analysis, and dynamical systems. Indeed, it has become a unifying concept. Many different topics can agreeably accompany a treatment of this theory. The companionship between integration and functional analysis and, in particular, between integration and weak convergence, has been fostered here: this is important, for instance, in the analysis of nonlinear partial differential equations (see L.C. Evans' book *Weak Convergence Methods for Nonlinear Partial Differential Equations* [AMS, 1998]).

The bibliography lists a number of books that are not specifically referenced but should be consulted for supplementary material and different viewpoints. In particular, two books on the interesting history of mathematical analysis are listed.

SUGGESTIONS FOR COURSES: FIRST SEMESTER

In Chapter 1, all the background elementary analysis and topology of the real line needed for Part I is established. This initial chapter is meant to be a handy reference. Core material comprises Chapters 2, 3, and 4, the first five sections of Chapter 6, Chapter 7, and the first section of Chapter 8. Following this, selections can be made: Sections 8.2–8.4 are interesting for students who will continue to study duality and compactness for normed linear spaces, while Section 5.3 contains two jewels of classical analysis, the characterization of Lebesgue integrability and of Riemann integrability for bounded functions.

SUGGESTIONS FOR COURSES: SECOND SEMESTER

This course should be based on Part III. Initial core material comprises Section 17.1, Section 18.1–18.4, and Sections 19.1–19.3. The remaining sections in Chapter 17 may be covered at the beginning or as they are needed later: Sections 17.3–17.5 before Chapter 20, and Section 17.2 before Chapter 21. Chapter 20 can then be covered. None of this material depends on Part II. Then several selected topics can be chosen, dipping into Part II as needed.

- Suggestion 1: Prove the Baire Category Theorem and its corollary regarding the partial continuity of the pointwise limit of a sequence of continuous functions (Theorem 7 of Chapter 10), infer from the Riesz-Fischer Theorem that the Nikodym metric space is complete (Theorem 23 of Chapter 18), prove the Vitali-Hahn-Saks Theorem and then prove the Dunford-Pettis Theorem.
- Suggestion 2: Cover Chapter 21 (omitting Section 20.5) on Measure and Topology, with the option of assuming the topological spaces are metrizable, so 20.1 can be skipped.
- Suggestion 3: Prove Riesz's Theorem regarding the closed unit ball of an infinite dimensional normed linear space being noncompact with respect to the topology induced by the norm. Use this as a motivation for regaining sequential compactness with respect to weaker topologies, then use Helley's Theorem to obtain weak sequential compactness properties of the $L^p(X, \mu)$ spaces, $1 < p < \infty$, if $L^q(X, \mu)$ is separable and, if Chapter 21 has already been covered, weak-$*$ sequential compactness results for Radon measures on the Borel σ-algebra of a compact metric space.

SUGGESTIONS FOR COURSES: THIRD SEMESTER

I have used Part II, with some supplemental material, for a course on functional analysis, for students who had taken the first two semesters; the material is tailored, of course, to that chosen for the second semester. Chapter 16 on bounded linear operators on a Hilbert space may be covered right after Chapter 13 on bounded linear operators on a Banach space, since the results regarding weak sequential compactness are obtained directly from the existence of an orthogonal complement for each closed subspace of a Hilbert space. Part II should be interlaced with selections from Part III to provide applications of the abstract space theory to integration. For instance, reflexivity and weak compactness can be considered in general $L^p(X, \mu)$ spaces, using material from Chapter 19. The above suggestion 1 for the second semester course can be taken in the third semester rather than the second, providing a truly striking application of the Baire Category Theorem. The establishment, in Chapter 21, of the representation of the dual of $C(X)$, where X is a compact Hausdorff space, provides another collection of spaces, spaces of signed Radon measures, to which the theorems of Helley, Alaoglu, and Krein-Milman apply. By covering Chapter 22 on Invariant Measures, the student will encounter applications of Alaoglu's Theorem and the Krein-Milman Theorem to prove the existence of Haar measure on a compact group and the existence of measures with respect to which a mapping is ergodic (Theorem 14 of Chapter 22), and an application of Helley's Theorem to establish the existence of invariant measures (the Bogoliubov-Krilov Theorem).

I welcome comments at pmf@math.umd.edu. A list of errata and remarks will be placed on www.math.umd.edu/~pmf/RealAnalysis.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge my indebtedness to teachers, colleagues, and students. A penultimate draft of the entire manuscript was read by Diogo Arsénio, whom I warmly thank for his observations and suggestions, which materially improved that draft. Here in my mathematical home, the University of Maryland, I have written notes for various analysis courses, which have been incorporated into the present edition. A number of students in my graduate analysis classes worked through parts of drafts of this edition, and their comments and suggestions have been most helpful: I thank Avner Halevy, Kevin McGoff, and Himanshu Tiagi. I am pleased to acknowledge a particular debt to Brendan Berg who created the index, proofread the final manuscript, and kindly refined my tex skills. I have benefited from conversations with many friends and colleagues; in particular, with Diogo Arsénio, Michael Boyle, Michael Brin, Craig Evans, Manos Grillakis, Brian Hunt, Jacobo Pejsachowicz, Eric Slud, Robert Warner, and Jim Yorke. Publisher and reviewers: J. Thomas Beale, Duke University; Richard Carmichael, Wake Forest University; Michael Goldberg, Johns Hopkins University; Paul Joyce, University of Idaho; Dmitry Kaliuzhnyi-Verbovetskyi, Drexel University; Giovanni Leoni, Carnegie Mellon University; Bruce Mericle, Mankato State University; Stephen Robinson, Wake Forest University; Martin Schechter, University of California-Irvine; James Stephen White, Jacksonville State University; and Shanshuang Yang, Emory University.

Patrick M. Fitzpatrick
College Park, MD
November, 2009

Contents

PART ONE

LEBESGUE INTEGRATION FOR FUNCTIONS OF A SINGLE REAL VARIABLE

Preliminaries on Sets, Mappings, and Relations

Contents

In these preliminaries we describe some notions regarding sets, mappings, and relations that will be used throughout the book. Our purpose is descriptive and the arguments given are directed toward plausibility and understanding rather than rigorous proof based on an axiomatic basis for set theory. There is a system of axioms called the Zermelo-Frankel Axioms for Sets upon which it is possible to formally establish properties of sets and thereby properties of relations and functions. The interested reader may consult the introduction and appendix to John Kelley's book, *General Topology* [Kel75], Paul Halmos's book, *Naive Set Theory* [Hal98], and Thomas Jech's book, *Set Theory* [Jec06].

UNIONS AND INTERSECTIONS OF SETS

For a set A,[1] the membership of the element x in A is denoted by $x \in A$ and the nonmembership of x in A is denoted by $x \notin A$. We often say a member of A belongs to A and call a member of A a *point* in A. Frequently sets are denoted by braces, so that $\{x \mid \text{statement about } x\}$ is the set of all elements x for which the statement about x is true.

Two sets are the same provided they have the same members. Let A and B be sets. We call A a **subset** of B provided each member of A is a member of B; we denote this by $A \subseteq B$ and also say that A is contained in B or B contains A. A subset A of B is called a **proper subset** of B provided $A \neq B$. The **union** of A and B, denoted by $A \cup B$, is the set of all points that belong either to A or to B; that is, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$. The word *or* is used here in the nonexclusive sense, so that points which belong to both A and B belong to $A \cup B$. The **intersection** of A and B, denoted by $A \cap B$, is the set of all points that belong to both A and B; that is, $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$. The **complement** of A in B, denoted by $B \sim A$, is the set of all points in B that are not in A; that is, $B \sim A = \{x \mid x \in B, x \notin A\}$. If, in a particular discussion, all of the sets are subsets of a reference set X, we often refer to $X \sim A$ simply as the complement of A.

The set that has no members is called the **empty-set** and denoted by $\emptyset$. A set that is not equal to the empty-set is called nonempty. We refer to a set that has a single member as a **singleton set**. Given a set X, the set of all subsets of X is denoted by $\mathcal{P}(X)$ or 2^X; it is called the **power set** of X.

In order to avoid the confusion that might arise when considering sets of sets, we often use the words "collection" and "family" as synonyms for the word "set." Let $\mathcal{F}$ be a collection of sets. We define the union of $\mathcal{F}$, denoted by $\bigcup_{F \in \mathcal{F}} F$, to be the set of points

[1]The *Oxford English Dictionary* devotes several hundred pages to the definition of the word "set."

that belong to at least one of the sets in $\mathcal{F}$. We define the intersection of $\mathcal{F}$, denoted by $\bigcap_{F\in\mathcal{F}} F$, to be the set of points that belong to every set in $\mathcal{F}$. The collection of sets $\mathcal{F}$ is said to be **disjoint** provided the intersection of any two sets in $\mathcal{F}$ is empty. For a family $\mathcal{F}$ of sets, the following identities are established by checking set inclusions.

De Morgan's identities

$$X\sim\left[\bigcup_{F\in\mathcal{F}} F\right]=\bigcap_{F\in\mathcal{F}}[X\sim F] \quad\text{and}\quad X\sim\left[\bigcap_{F\in\mathcal{F}} F\right]=\bigcup_{F\in\mathcal{F}}[X\sim F],$$

that is, the complement of the union is the intersection of the complements, and the complement of the intersection is the union of the complements.

For a set Λ, suppose that for each $\lambda\in\Lambda$, there is defined a set E_λ. Let $\mathcal{F}$ be the collection of sets $\{E_\lambda \mid \lambda\in\Lambda\}$. We write $\mathcal{F}=\{E_\lambda\}_{\lambda\in\Lambda}$ and refer to this as an **indexing** (or **parametrization**) of $\mathcal{F}$ by the **index set** (or **parameter set**) Λ.

Mappings between sets

Given two sets A and B, by a **mapping** or **function** from A into B we mean a correspondence that assigns to each member of A a member of B. In the case B is the set of real numbers we always use the word "function." Frequently we denote such a mapping by $f\colon A\to B$, and for each member x of A, we denote by $f(x)$ the member of B to which x is assigned. For a subset A' of A, we define $f(A')=\{b \mid b=f(a)$ for some member a of $A'\}$: $f(A')$ is called the image of A' under f. We call the set A the **domain** of the function f and $f(A)$ the **image** or **range** of f. If $f(A)=B$, the function f is said to be **onto**. If for each member b of $f(A)$ there is exactly one member a of A for which $b=f(a)$, the function f is said to be **one-to-one**. A mapping $f\colon A\to B$ that is both one-to-one and onto is said to be **invertible**; we say that this mapping establishes a **one-to-one correspondence** between the sets A and B. Given an invertible mapping $f\colon A\to B$, for each point b in B, there is exactly one member a of A for which $f(a)=b$ and it is denoted by $f^{-1}(b)$. This assignment defines the mapping $f^{-1}\colon B\to A$, which is called the **inverse** of f. Two sets A and B are said to be **equipotent** provided there is an invertible mapping from A onto B. Two sets which are equipotent are, from the set-theoretic point of view, indistinguishable.

Given two mappings $f\colon A\to B$ and $g\colon C\to D$ for which $f(A)\subseteq C$ then the composition $g\circ f\colon A\to D$ is defined by $[g\circ f](x)=g(f(x))$ for each $x\in A$. It is not difficult to see that the composition of invertible mappings is invertible. For a set D, define the identity mapping $id_D\colon D\to D$ is defined by $id_D(x)=x$ for all $x\in D$. A mapping $f\colon A\to B$ is invertible if and only if there is a mapping $g\colon B\to A$ for which

$$g\circ f=id_A \text{ and } f\circ g=id_B.$$

Even if the mapping $f\colon A\to B$ is not invertible, for a set E, we define $f^{-1}(E)$ to be the set $\{a\in A \mid f(a)\in E\}$; it is called the **inverse image** of E under f. We have the following useful properties: for any two sets E_1 and E_2,

$$f^{-1}(E_1\cup E_2)=f^{-1}(E_1)\cup f^{-1}(E_2),\ f^{-1}(E_1\cap E_2)=f^{-1}(E_1)\cap f^{-1}(E_2)$$

and

$$f^{-1}(E_1\sim E_2)=f^{-1}(E_1)\sim f^{-1}(E_2).$$

Finally, for a mapping $f: A \to B$ and a subset A' of its domain A, the **restriction** of f to A', denoted by $f|_{A'}$, is the mapping from A' to B which assigns $f(x)$ to each $x \in A'$.

EQUIVALENCE RELATIONS, THE AXIOM OF CHOICE, AND ZORN'S LEMMA

Given two nonempty sets A and B, the **Cartesian product** of A with B, denoted by $A \times B$, is defined to be the collection of all ordered pairs (a, b) where $a \in A$ and $b \in B$ and we consider $(a, b) = (a', b')$ if and only if $a = a'$ and $b = b'$.[2] For a nonempty set X, we call a subset R of $X \times X$ a **relation** on X and write $x R x'$ provided (x, x') belongs to R. The relation R is said to be **reflexive** provided $x R x$, for all $x \in X$; the relation R is said to be **symmetric** provided $x R x'$ if $x' R x$; the relation R is said to be **transitive** provided whenever $x R x'$ and $x' R x''$, then $x R x''$.

Definition *A relation R on a set X is called an* **equivalence relation** *provided it is reflexive, symmetric, and transitive.*

Given an equivalence relation R on a set X, for each $x \in X$, the set $R_x = \{x' \mid x' \in X,\ x R x'\}$ is called the **equivalence class** of x (with respect to R). The collection of equivalence classes is denoted by X/R. For example, given a set X, the relation of equipotence is an equivalence relation on the collection 2^X of all subsets of X. The equivalence class of a set with respect to the relation equipotence is called the **cardinality** of the set.

Let R be an equivalence relation on a set X. Since R is symmetric and transitive, $R_x = R_{x'}$ if and only if $x R x'$ and therefore the collection of equivalence classes is disjoint. Since the relation R is reflexive, X is the union of the equivalence classes. Therefore X/R is a disjoint collection of nonempty subsets of X whose union is X. Conversely, given a disjoint collection $\mathcal{F}$ of nonempty subsets of X whose union is X, the relation of belonging to the same set in $\mathcal{F}$ is an equivalence relation R on X for which $\mathcal{F} = X/R$.

Given an equivalence relation on a set X, it is often necessary to choose a subset C of X which consists of exactly one member from each equivalence class. Is it obvious that there is such a set? Ernst Zermelo called attention to this question regarding the choice of elements from collections of sets. Suppose, for instance, we define two real numbers to be rationally equivalent provided their difference is a rational number. It is easy to check that this is an equivalence relation on the set of real numbers. But it is not easy to identify a set of real numbers that consists of exactly one member from each rational equivalence class.

Definition *Let $\mathcal{F}$ be a nonempty family of nonempty sets. A* **choice function** *f on $\mathcal{F}$ is a function f from $\mathcal{F}$ to $\bigcup_{F \in \mathcal{F}} F$ with the property that for each set F in $\mathcal{F}$, $f(F)$ is a member of F.*

Zermelo's Axiom of Choice *Let $\mathcal{F}$ be a nonempty collection of nonempty sets. Then there is a choice function on $\mathcal{F}$.*

---

[2] In a formal treatment of set theory based on the Zermelo-Frankel Axioms, an ordered pair (a, b) is defined to be the set $\{\{a\}, \{a, b\}\}$ and a function with domain in A and image in B is defined to be a nonempty collection of ordered pairs in $A \times B$ with the property that if the ordered pairs (a, b) and (a, b') belong to the function, then $b = b'$.

Very roughly speaking, a choice function on a family of nonempty sets "chooses" a member from each set in the family. We have adopted an informal, descriptive approach to set theory and accordingly we will freely employ, without further ado, the Axiom of Choice.

Definition *A relation R on a set nonempty X is called a* **partial ordering** *provided it is reflexive, transitive, and, for x, x' in X,*

$$\text{if } x\,R\,x' \text{ and } x'\,R\,x, \text{ then } x = x'.$$

A subset E of X is said to be **totally ordered** *provided for x, x' in E, either $x\,R\,x'$ or $x'\,R\,x$. A member x of X is said to be an* **upper bound** *for a subset E of X provided $x' R x$ for all $x' \in E$, and said to be* **maximal** *provided the only member x' of X for which $x\,R\,x'$ is $x' = x$.*

For a family $\mathcal{F}$ of sets and $A, B \in \mathcal{F}$, define $A\,R\,B$ provided $A \subseteq B$. This relation of **set inclusion** is a partial ordering of $\mathcal{F}$. Observe that a set F in $\mathcal{F}$ is an upper bound for a subfamily $\mathcal{F}'$ of $\mathcal{F}$ provided every set in $\mathcal{F}'$ is a subset of F and a set F in $\mathcal{F}$ is maximal provided it is not a proper subset of any set in $\mathcal{F}$. Similarly, given a family $\mathcal{F}$ of sets and $A, B \in \mathcal{F}$ define $A\,R\,B$ provided $B \subseteq A$. This relation of **set containment** is a partial ordering of $\mathcal{F}$. Observe that a set F in $\mathcal{F}$ is an upper bound for a subfamily $\mathcal{F}'$ of $\mathcal{F}$ provided every set in $\mathcal{F}'$ contains F and a set F in $\mathcal{F}$ is maximal provided it does not properly contain any set in $\mathcal{F}$.

Zorn's Lemma *Let X be a partially ordered set for which every totally ordered subset has an upper bound. Then X has a maximal member.*

We will use Zorn's Lemma to prove some of our most important results, including the Hahn-Banach Theorem, the Tychonoff Product Theorem, and the Krein-Milman Theorem. Zorn's Lemma is equivalent to Zermelo's Axiom of Choice. For a proof of this equivalence and related equivalences, see Kelley [Kel75], pp. 31–36.

We have defined the Cartesian product of two sets. It is useful to define the Cartesian product of a general parametrized collection of sets. For a collecton of sets $\{E_\lambda\}_{\lambda \in \Lambda}$ parametrized by the set Λ, the Cartesian product of $\{E_\lambda\}_{\lambda \in \Lambda}$, which we denote by $\Pi_{\lambda \in \Lambda} E_\lambda$, is defined to be the set of functions f from Λ to $\bigcup_{\lambda \in \Lambda} E_\lambda$ such that for each $\lambda \in \Lambda$, $f(\lambda)$ belongs to E_λ. It is clear that the Axiom of Choice is equivalent to the assertion that the Cartesian product of a nonempty family of nonempty sets is nonempty. Note that the Cartesian product is defined for a parametrized family of sets and that two different parametrizations of the same family will have different Cartesian products. This general definition of Cartesian product is consistent with the definition given for two sets. Indeed, consider two nonempty sets A and B. Define $\Lambda = \{\lambda_1, \lambda_2\}$ where $\lambda_1 \neq \lambda_2$ and then define $E_{\lambda_1} = A$ and $E_{\lambda_2} = B$. The mapping that assigns to the function $f \in \Pi_{\lambda \in \Lambda} E_\lambda$ the ordered pair $(f(\lambda_1), f(\lambda_2))$ is an invertible mapping of the Cartesian product $\Pi_{\lambda \in \Lambda} E_\lambda$ onto the collection of ordered pairs $A \times B$ and therefore these two sets are equipotent. For two sets E and Λ, define $E_\lambda = E$ for all $\lambda \in \Lambda$. Then the Cartesian product $\Pi_{\lambda \in \Lambda} E_\lambda$ is equal to the set of all mappings from Λ to E and is denoted by E^Λ.