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Jianhua Zheng

亚纯函数值分布理论

Value Distribution
of Meromorphic
Functions



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内 容 简 介

本书共 7 章, 研究在复平面上或在以原点为顶点的角域上亚纯的函数的值分布, 即通过某些值点来刻画亚纯函数。前两章研究各类特征函数及这样的实函数的性质。第 3、4 章放在新引入的奇异方向—— T 方向, 包括存在性、分布, 与其他方向的关系上, T 方向与分布值和亏值总数的关系。射线分布值确定亚纯函数的增长性的问题在第 5 章详细研究。第 6 章研究亚纯函数对应的 Riemann 曲面, 逆函数的奇异性及其与不动点的关系。最后一章介绍具有重要地位的 F.Nevanlinna 猜想的 Eremenko 应用位势论的证明。

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Preface

This book is devoted to the study of value distribution of functions which are meromorphic on the complex plane or in an angular domain with vertex at the origin. We characterize such meromorphic functions in terms of distribution of some of their value points. The study, together with certain related topics, is known as theory of value distribution of meromorphic functions. The theory is too vast to be justified within a single work. Therefore we selected and organized the content based on their significant importance to our understanding and interests in this book. I gladly acknowledge my indebtedness in particular to the books of M. Tsuji, A. A. Goldberg and I. V. Ostrovskii, Yang L. and the papers of A. Eremenko.

An outline of the book is provided below. The introduction of the Nevanlinna characteristic to the study of meromorphic functions is a new starting symbol of the theory of value distribution. The Nevanlinna characteristic is powerful, and its thought has been used to produce various characteristics such as the Nevanlinna characteristic and Tsuji characteristic for an angular domain. And from geometric point of view, namely the Ahlfors theory of covering surfaces, the Ahlfors-Shimizu characteristic have also been introduced. These characteristics are real-valued functions defined on the positive real axis. Therefore, in the first chapter, we collect the basic results about positive real functions that are often used in the study of meromorphic function theory. Some of these results are distributed in other books, some in published papers, and some have been newly established in order to serve our specific objectives in the book.

In the present book, we discuss value distribution not only in the complex plane, but also in an angular domain. Therefore, we introduce, in the second chapter, various characteristics of a meromorphic function: The Nevanlinna characteristic for a disk, the Nevanlinna characteristic for an angle, the Tsuji characteristic and the Ahlfors-Shimizu characteristic for an angle. Although they were distributed in other books, we collected all of them, and more importantly, we carefully compared them with one another to reveal their relations that enabled us to produce new results and applications. We establish the first and second fundamental theorems for the various characteristics and the corresponding integrated counting functions, and provide an estimate of the error term related to the Nevanlinna characteristic for an

angle in terms of the Nevanlinna characteristic in a larger angle. We discuss in an angle the growth order of a meromorphic function and exponent of convergence of its a -points by means of the Ahlfors-Shimizu characteristic. We establish unique theorems in an angular domain with the help of the Tsuji characteristic, which is a new topic, because this has never been touched before while only the case of the whole complex plane was discussed.

After providing a brief overview of the characteristics in Chapter 2, we carefully investigate, in the third chapter, a new singular direction of a meromorphic function called T direction, which is different from the Julia, Borel and Nevanlinna directions. A singular direction is characterized essentially with the help of a property that in any angle containing it, the function assumes abundantly any value possibly except at most two values. The word “abundantly” is expressed by “infinitely often” for the Julia directions and by the growth order of the function for the Borel directions. The definition of T directions is to compare the integrated counting function in an angle to the characteristic and so it does not depend on the growth order, which is different from the Borel directions. So we can naturally consider T directions of meromorphic functions with zero order or infinite order. The second fundamental theorem of Nevanlinna is considered as the background of T directions. The following inequality

$$\limsup_{r \rightarrow \infty} \frac{N(r, \mathbb{C}, f = a)}{T(r, f)} > 0$$

always holds for all but at most two values of a . For a T direction, we consider the above inequality in any angle containing it instead of the whole complex plane. First we discuss the existence of T directions including the case of small functions in our consideration, next do relationship with the Borel directions, then common T directions of the function and its derivatives including the Hayman T directions. The singular directions of meromorphic solutions of linear differential equations possess some special properties, which are carefully studied and finally, we survey the results on the uniqueness and singular directions of an algebroid function.

The book includes discussion of argument distribution as well as modulo distribution and their relations. In the fourth chapter, we reveal relations between the numbers of deficient values and T directions. The results established there are new and unpublished elsewhere. The essential idea for discussion of this topic comes from observation that if the function assumes two values a and b at few points and is in close proximity to a complex number $c \neq a, b$ at enough points in a bounded domain, then it is close to c in the whole domain possibly outside a small set and that if the function is analytic, in view of the two constant theorem for the harmonic measure, we can use the modulo of the function on some part of the boundary of the domain to control the function modulo inside the domain. In the final section, we make a survey on this topic.

In the fifth chapter, we discuss the growth of the meromorphic functions that have two radially distributed values and a Nevanlinna deficient value. We first consider the growth of the meromorphic functions without any restriction imposed on their order and then those with the finite lower order. We attain our purpose in terms of the Nevanlinna characteristic for an angle, as Goldberg and Ostrovskii did, but our

starting point is to establish an estimate of the Nevanlinna characteristic for a disk centered at the origin in terms of $B_{\alpha,\beta}(r, f)$ under an observation of the Nevanlinna deficient value, and then $B_{\alpha,\beta}(r, f)$ is estimated by two $C_{\alpha,\beta}(r, *)$ which may deal with the derivatives with help of fundamental inequalities for the Nevanlinna characteristic for an angle, and finally, $C_{\alpha,\beta}(r, *)$ are replaced by the integrated counting functions $N(r, \Omega, *)$ in terms of the relations between them. Thus the Nevanlinna characteristic for a disk can be estimated by two $N(r, \Omega, *)$ and we reduce the study of this subject to estimation of $B_{\alpha,\beta}$ in terms of $C_{\alpha,\beta}$. However, this comes from the study of fundamental inequality for the Nevanlinna characteristic for an angle. As we know, most of the fundamental inequalities for a disk can be validly and easily transferred to the case of an angle and therefore, we give out a simple and elementary approach to the discussions of this subject. When the function is of the finite lower order, we use the Baernstein spread relation to discuss the estimation of the Nevanlinna characteristic for a disk in terms of $B_{\alpha,\beta}(r, f)$ and hence we can attain deeper results for this subject.

In the sixth chapter, we collect and develop results about singularities of the inverse of a meromorphic function. A transcendental meromorphic function is equipped with a parabolic simply connected Riemann surface. The boundary points of the Riemann surface correspond to transcendental singularities of the inverse of the function, that is, asymptotic values of the function, and vice versa. We discuss relationships between the number of direct singularities and the growth (lower) order. The isolated transcendental singularity is logarithmic, and hence we observe that an asymptotic value over which the singularity is not logarithmic is a limit of other singular values. For a meromorphic function of finite order, such an asymptotic value is a limit point of critical values, which is the Bergweiler-Eremenko's result. We show Eremenko's construction of a transcendental meromorphic function with the finite given order which has every value on the extended complex plane as its asymptotic value, and next discuss the fixed points of bounded-type meromorphic functions, that is, meromorphic functions whose singular value set are bounded, from which we observe that meromorphic functions possess special characters if their singular values are suitably restricted.

The final chapter is mainly devoted to the Eremenko's proof of the famous F. Nevanlinna conjecture on meromorphic functions with maximum total sum of Nevanlinna deficiencies. The conjecture was proved first by David Drasin, but his proof is very complicated. A. Eremenko used the potential theory to give a simple proof to the conjecture, from which we see the power of the potential theory in the study of value distribution of meromorphic functions. The theory to study subharmonic functions is the potential theory. The defence of two subharmonic functions is called δ -subharmonic. The logarithm of modulo of a meromorphic function is a δ -subharmonic function. Therefore, some problems about value distribution of meromorphic functions can be transferred to those about subharmonic functions. And the limit functions of a sequence of subharmonic functions produced by the subharmonic function in question are easier to be characterized than the meromorphic functions. The property or behavior of the limit functions can be used to describe the

meromorphic functions. This is one of the approaches in which the potential theory are used to discuss problems about meromorphic functions.

For the benefit of readers, and for our intent to introduce and develop the potential theory in value distributions, we introduce and gather the basic knowledge about the potential theory including the normality of subharmonic function family in the sense of \mathcal{L}_{loc} and the Nevanlinna theory of subharmonic functions which consist of works of Anderson, Baernstein, Eremenko, Sodin, and others. The works of these mathematicians are very special and very important, and in our opinion, represent one aspect of value distribution theory which is worth further investigating and developing.

The first draft of this book was finished at the end of 2006, and main content of the book, except the seventh chapter was lectured in the summer course for post-graduated students held at Jiang Xi Normal University in the summer of 2007. I am indebted to Professor Yi Caifeng for her organizing the summer school, to Professor He Yuzhan for his comments and offering me some important materials, and to Professor Ye Zhuang for his support of this book. I would like to send many thanks to others including my students who pointed out some mistakes or some tough statements in the original draft when they read. This book has been partially supported by the National Natural Science Foundation of China.

Jianhua Zheng
Beijing,
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Contents

1	Preliminaries of Real Functions	1
1.1	Functions of a Real Variable	1
1.1.1	The Order and Lower Order of a Real Function	1
1.1.2	The Pólya Peak Sequence of a Real Function	4
1.1.3	The Regularity of a Real Function	8
1.1.4	Quasi-invariance of Inequalities	14
1.2	Integral Formula and Integral Inequalities	19
1.2.1	The Green Formula for Functions with Two Real Variables	19
1.2.2	Several Integral Inequalities	20
	References	23
2	Characteristics of a Meromorphic Function	25
2.1	Nevanlinna's Characteristic in a Domain	26
2.2	Nevanlinna's Characteristic in an Angle	47
2.3	Tsuji's Characteristic	58
2.4	Ahlfors-Shimizu's Characteristic	66
2.5	Estimates of the Error Terms	77
2.6	Characteristic of Derivative of a Meromorphic Function	86
2.7	Meromorphic Functions in an Angular Domain	93
2.8	Deficiency and Deficient Values	103
2.9	Uniqueness of Meromorphic Functions Related to Some Angular Domains	110
	References	120
3	T Directions of a Meromorphic Function	123
3.1	Notation and Existence of T Directions	124
3.2	T Directions Dealing with Small Functions	130
3.3	Connection Among T Directions and Other Directions	134
3.4	Singular Directions Dealing with Derivatives	146
3.5	The Common T Directions of a Meromorphic Function and Its Derivatives	151

3.6	Distribution of the Julia, Borel Directions and T Directions	163
3.7	Singular Directions of Meromorphic Solutions of Some Equations	166
3.8	Value Distribution of Algebroid Functions	178
	References	181
4	Argument Distribution and Deficient Values	185
4.1	Deficient Values and T Directions	186
4.2	Retrospection	202
	References	205
5	Meromorphic Functions with Radially Distributed Values	207
5.1	Growth of Such Meromorphic Functions	208
5.2	Growth of Such Meromorphic Functions with Finite Lower Order	215
5.3	Retrospection	222
	References	228
6	Singular Values of Meromorphic Functions	229
6.1	Riemann Surfaces and Singularities	230
6.2	Density of Singularities	241
6.3	Meromorphic Functions of Bounded Type	252
	References	265
7	The Potential Theory in Value Distribution	267
7.1	Signed Measure and Distributions	268
7.2	δ -Subharmonic Functions	270
7.2.1	Basic Results Concerning δ -Subharmonic Functions	271
7.2.2	Normality of Family of δ -Subharmonic Functions	275
7.2.3	The Nevanlinna Theory of δ -Subharmonic Functions	285
7.3	Eremenko's Proof of the Nevanlinna Conjecture	292
	References	305
	Index	307

Chapter 1

Preliminaries of Real Functions

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Abstract: The various characteristics of meromorphic functions are main tool in the study of value distribution of meromorphic functions this book will introduce. They are real-valued functions defined on the positive real axis. In this chapter, we discuss certain properties of such real functions for application in later chapters. We begin with the order and the lower order of such functions which include the proximate order and the type function. We discuss the existence of the Pólya peak sequence. Also, we identify a sequence of positive numbers with some of the Pólya peak properties. We mainly introduce a result of Edrei and Fuchs for the regularity, thereby, improving the lemma of Borel and quasi-invariance of inequalities of two real functions under differentiation and integration. Finally, we exhibit the Green formula and collect several integral inequalities.

Key words: Real functions, Proximate order, Pólya peak, Regularity, Quasi-invariance

1.1 Functions of a Real Variable

In investigation of theory of meromorphic functions, we often meet the study of some properties of functions of a real variable, because various characteristics of meromorphic functions are such functions. Therefore, in this section, we collect the main properties of such functions which will be frequently used in the sequel.

1.1.1 The Order and Lower Order of a Real Function

Let $T(r)$ be a non-negative continuous function on $[r_0, \infty)$ for some $r_0 \geq 0$ and define $\log^+ x = \log \max\{1, x\}$. For $T(r)$, we define its lower order μ and order λ in turn as follows:

$$\mu = \mu(T) = \liminf_{r \rightarrow \infty} \frac{\log^+ T(r)}{\log r}$$

and

$$\lambda = \lambda(T) = \limsup_{r \rightarrow \infty} \frac{\log^+ T(r)}{\log r}.$$

We concentrate mainly on the function $T(r)$ which tends to infinity as r does. The order of a positive increasing continuous function can be characterized in term of an integral value.

Lemma 1.1.1. *Let $T(r)$ be a continuous, non-decreasing and positive function on $[r_0, \infty)$. Then for each $\rho < \lambda(T)$, we have*

$$\int_r^\infty \frac{T(t)}{t^{\rho+1}} dt = \infty;$$

Conversely, if the above equation holds for certain ρ , then $\lambda(T) \geq \rho$.

Proof. Suppose that the integral is finite, and then for all $r \geq r_0$,

$$K > \int_r^{2r} \frac{T(t)}{t^{\rho+1}} dt \geq \frac{T(r)}{(2r)^{\rho+1}} r = 2^{-\rho-1} T(r) r^{-\rho},$$

where $K = \int_{r_0}^\infty \frac{T(t)}{t^{\rho+1}} dt$. This immediately deduces $\lambda(T) \leq \rho$ and the former half part of the lemma follows.

If $\lambda(T) < \rho$, then for each s with $\lambda(T) < s < \rho$, we have $T(r) < r^s$ for all sufficiently large r . Thus $T(r)r^{-\rho-1} < r^{-(\rho-s)-1}$, which yields the integral $\int_{r_0}^\infty \frac{T(t)}{t^{\rho+1}} dt$ is convergent.

This completes the proof of Lemma 1.1.1. □

A continuous function may be too complicated to grasp, and thus sometime it is necessary to modify it by preserving, roughly speaking, only the values of r at which $T(r)$ can be approximately written into r^λ . The precise statement is as under

Theorem 1.1.1. *Let $T(r)$ be a continuous and positive function for $r \geq r_0 > 0$ and tend to infinity as $r \rightarrow \infty$ with $\lambda = \lambda(T) < \infty$. Then, there exists a function $\lambda(r)$ with the following properties:*

- (1) $\lambda(r)$ is a monotone and piecewise continuous differentiable function for $r \geq r_0$ with $\lim_{r \rightarrow \infty} \lambda(r) = \lambda$;
- (2) $\lim_{r \rightarrow \infty} \lambda'(r) r \log r = 0$;
- (3) $\limsup_{r \rightarrow \infty} \frac{T(r)}{r^{\lambda(r)}} = 1$;
- (4) for each positive number d ,

$$\lim_{r \rightarrow \infty} \frac{U(dr)}{U(r)} = d^\lambda, \quad U(r) = r^{\lambda(r)}. \quad (1.1.1)$$

We shall call the function $\lambda(r)$ the proximate order of $T(r)$ and the function $U(r)$ the type function of $T(r)$. It is obvious that the proximate order and the type function of a real function are not unique. As $\lambda > 0$, $U(r) = e^{\lambda(r) \log r}$ is increasing for all larger r . A simple calculation implies that a monotone increasing function $T(r)$ satisfying (1.1.1) must have $\mu(T) = \lambda(T) = \lambda$. The formula (1.1.1) is the key point of Theorem 1.1.1 and it makes sense essentially for the limit being finite. This explains the necessity for the condition that a function $T(r)$ in question is of finite order. However, in the case of infinite order, we have the following

Theorem 1.1.2. *Let $T(r)$ be a continuous and positive function for $r \geq r_0 > 0$ and tend to infinity as $r \rightarrow \infty$ with $\lambda = \lambda(T) = \infty$. Assume that $\omega(r)$ is a positive, continuous and non-increasing function with $\int_1^\infty \frac{\omega(t)}{t} dt < +\infty$.*

Then, there exists a function $\lambda(r)$ with the following properties

- (1) $\lambda(r)$ is non-decreasing and continuous and tends to infinity as $r \rightarrow \infty$;
- (2) $\limsup_{r \rightarrow \infty} \frac{T(r)}{r^{\lambda(r)}} = 1$;
- (3) Set $U(r) = r^{\lambda(r)}$ and

$$\lim_{r \rightarrow \infty} \frac{U(r + \omega(U(r)))}{U(r)} = 1. \quad (1.1.2)$$

The proofs of Theorem 1.1.1 and Theorem 1.1.2 can be found in Chuang [2].

The following result will be used often in the next chapters.

Lemma 1.1.2. *Let $T(r)$ be a non-negative and non-decreasing function in $0 < r < \infty$. If*

$$\liminf_{r \rightarrow \infty} \frac{T(dr)}{T(r)} \geq c > 1$$

for some $d > 1$, then

$$\int_1^r \frac{T(t)}{t} dt \leq \frac{2c \log d}{c-1} T(r) + O(1);$$

If

$$\liminf_{r \rightarrow \infty} \frac{T(dr)}{T(r)} > d^\omega$$

for some $d > 1$ and $\omega > 0$, then

$$\int_1^r \frac{T(t)}{t^{\omega+1}} dt \leq K \frac{T(r)}{r^\omega} + O(1),$$

where K is a positive constant.

Proof. Write $s = \frac{c+1}{2}$ and we can find a natural number N such that for $r \geq r_0 = d^N$, we have $T(d^{-1}r) < s^{-1}T(r)$. Then for each $r \geq r_0 = d^N$, we have $n \geq N$ such that $d^n \leq r < d^{n+1}$, and let us estimate the following integral

$$\begin{aligned}
\int_{r_0}^r \frac{T(t)}{t} dt &= \sum_{k=N}^{n-1} \int_{d^k}^{d^{k+1}} \frac{T(t)}{t} dt + \int_{d^n}^r \frac{T(t)}{t} dt \\
&\leq \sum_{k=N}^{n-1} T(d^{k+1}) \log d + T(r) \log d \\
&= T(d^n) \log d \sum_{k=N}^{n-1} \frac{T(d^{k+1})}{T(d^n)} + T(r) \log d \\
&< T(d^n) \log d \sum_{k=0}^{\infty} s^{-k} + T(r) \log d \\
&\leq \frac{2c \log d}{c-1} T(r).
\end{aligned}$$

This yields the first desired inequality.

Now, we come to the proof of the second part of Lemma 1.1.2. Under the given assumption, for $r \geq r_0 = d^N$ and some $\varepsilon > 0$ we have $T(d^{-1}r) < (d + \varepsilon)^{-\omega} T(r)$. Thus, it follows that

$$\begin{aligned}
\int_{r_0}^r \frac{T(t)}{t^{\omega+1}} dt &= \sum_{k=N}^{n-1} \int_{d^k}^{d^{k+1}} \frac{T(t)}{t^{\omega+1}} dt + \int_{d^n}^r \frac{T(t)}{t^{\omega+1}} dt \\
&\leq \sum_{k=N}^{n-1} T(d^{k+1}) \frac{1}{\omega} \left(\frac{1}{d^{k\omega}} - \frac{1}{d^{(k+1)\omega}} \right) + T(r) \frac{1}{\omega} \left(\frac{1}{d^{n\omega}} - \frac{1}{r^\omega} \right) \\
&< \frac{1}{\omega} T(d^n) \sum_{k=N}^{n-1} (d + \varepsilon)^{-\omega(n-k-1)} \left(\frac{1}{d^{k\omega}} - \frac{1}{d^{(k+1)\omega}} \right) + \frac{1}{\omega} \frac{T(r)}{d^{n\omega}} \\
&< \frac{d^\omega - 1}{\omega} \frac{T(d^n)}{(d + \varepsilon)^{n\omega}} \frac{\left(\frac{d+\varepsilon}{d}\right)^{(n+1)\omega} - 1}{\left(\frac{d+\varepsilon}{d}\right)^\omega - 1} + \frac{1}{\omega} \frac{T(r)}{d^{n\omega}} \\
&\leq K_0 \frac{T(r)}{d^{n\omega}} < K_0 d^\omega \frac{T(r)}{r^\omega},
\end{aligned}$$

where $K_0 = \frac{d^\omega - 1}{\omega} \frac{(d+\varepsilon)^\omega}{(d+\varepsilon)^\omega - d^\omega} + \frac{1}{\omega}$.

This completes the proof of Lemma 1.1.2. □

1.1.2 The Pólya Peak Sequence of a Real Function

In this subsection, we consider the Pólya peak for a $T(r)$, which was first introduced by Edrei [6].

Definition 1.1.1. A sequence of positive numbers $\{r_n\}$ is called a sequence of Pólya peaks of order β for $T(r)$ (outside a set E) provided that there exist four sequences $\{r'_n\}$, $\{r''_n\}$, $\{\varepsilon_n\}$ and $\{\varepsilon'_n\}$ such that

$$(1) \quad r_n \notin E, \quad r'_n \rightarrow \infty, \quad \frac{r_n}{r'_n} \rightarrow \infty, \quad \frac{r''_n}{r'_n} \rightarrow \infty, \quad \varepsilon_n \rightarrow 0, \quad \varepsilon'_n \rightarrow 0 \quad (n \rightarrow \infty);$$

- (2) $\liminf_{n \rightarrow \infty} \frac{\log T(r_n)}{\log r_n} \geq \beta$;
- (3) $T(t) < (1 + \varepsilon_n) \left(\frac{t}{r_n}\right)^\beta T(r_n)$, $t \in [r'_n, r''_n]$;
- (4) $T(t)/t^{\beta-\varepsilon'_n} \leq KT(r_n)/r_n^{\beta-\varepsilon'_n}$, $1 \leq t \leq r''_n$ and for a positive constant K .

Actually, it is easy to see that (2) follows from (4). It is obvious that any subsequence of a Pólya peak sequence is still a sequence of the Pólya peak. Please note that the above definition of the Pólya peaks has some differences from that in other literatures where a sequence of Pólya peak is only required to satisfy (1) and (3) listed in Definition 1.1.1. The sequence $\{r_n\}$ is called a sequence of relaxed Pólya peaks of order β for a constant $C > 1$, provided that (1), (2) and (4) in Definition 1.1.1 hold and (3) does for C in place of “ $(1 + \varepsilon_n)$ ”. It is easily seen that for a sequence $\{r_n\}$ of Pólya peak and $d \geq 1$, $\{dr_n\}$ must be a sequence of the relaxed Pólya peak.

The following is a modifying version of well-known result which can be found in Section 8.1 of Yang [12].

Theorem 1.1.3. *Let $T(r)$ be a non-negative and non-decreasing continuous function in $0 < r < \infty$ with $0 \leq \mu(T) < \infty$ and $0 < \lambda(T) \leq \infty$. Then for arbitrary finite and positive number β satisfying $\mu \leq \beta \leq \lambda$ and a set F with finite logarithmic measure, i.e., $\int_F t^{-1} dt < \infty$, there exists a sequence of the Pólya peaks of order β for $T(r)$ outside F .*

Proof. We choose a sequence of positive numbers $\{\varepsilon_n\}$ with $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. By induction, we seek the desired Pólya peak sequence $\{r_n\}$. Suppose we have r_{n-1} and want to find r_n .

First of all consider the case when $\beta = \lambda(T) < \infty$. It is easy to see that for n ,

$$\limsup_{t \rightarrow \infty} \frac{T(t)}{t^{\beta-\varepsilon_n}} = \infty \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{T(t)}{t^{\beta+\varepsilon_n}} = 0.$$

Therefore, we can find a real number $u > \max\{n\varepsilon_n^{-1}, r_{n-1}\}$ such that

$$T(u)u^{-\beta+\varepsilon_n} = \max_{1 \leq t \leq u} \{T(t)t^{-\beta+\varepsilon_n}\}$$

and a $v \geq u$ such that

$$T(v)v^{-\beta-\varepsilon_n} = \max_{t \geq u} \{T(t)t^{-\beta-\varepsilon_n}\}.$$

We choose r_n with $u \leq r_n \leq v$ such that

$$T(r_n)r_n^{-\beta+\varepsilon_n} = \max_{u \leq t \leq v} \{T(t)t^{-\beta+\varepsilon_n}\} \geq T(u)u^{-\beta+\varepsilon_n}.$$

Thus for $t \leq v$, we have

$$T(r_n)r_n^{-\beta+\varepsilon_n} \geq T(t)t^{-\beta+\varepsilon_n} \quad (1.1.3)$$

and for $t \geq r_n$

$$T(t)t^{-\beta-\varepsilon_n} \leq T(v)v^{-\beta-\varepsilon_n} \leq T(r_n)r_n^{-\beta+\varepsilon_n}v^{-2\varepsilon_n} \leq T(r_n)r_n^{-\beta-\varepsilon_n}$$

and, therefore, for $r_n \leq t \leq r_n/\varepsilon_n$,

$$\begin{aligned} T(t)t^{-\beta+\varepsilon_n} &= T(t)t^{-\beta-\varepsilon_n}t^{2\varepsilon_n} \leq T(r_n)r_n^{-\beta-\varepsilon_n}t^{2\varepsilon_n} \\ &= T(r_n)r_n^{-\beta+\varepsilon_n} \left(\frac{t}{r_n} \right)^{2\varepsilon_n} \\ &\leq \left(\frac{1}{\varepsilon_n} \right)^{2\varepsilon_n} T(r_n)r_n^{-\beta+\varepsilon_n}. \end{aligned} \quad (1.1.4)$$

Combining (1.1.3) and (1.1.4) deduces that r_n satisfies (4) for $r_n'' = r_n/\varepsilon_n$. This also immediately yields

$$T(t) \leq e^{-2\varepsilon_n \log \varepsilon_n} \left(\frac{t}{r_n} \right)^\beta T(r_n) \text{ for } \varepsilon_n r_n \leq t \leq \varepsilon_n^{-1} r_n. \quad (1.1.5)$$

Now let us consider the case when $\mu \leq \beta < \lambda$. Assume without any loss of generalities that $\varepsilon_n < \lambda - \beta$. Then

$$\limsup_{t \rightarrow \infty} \frac{T(t)}{t^{\beta+\varepsilon_n}} = \infty \text{ and } \liminf_{t \rightarrow \infty} \frac{T(t)}{t^{\beta+\varepsilon_n/2}} = 0.$$

Application of a theorem of Edrei [6] deduces the existence of r_n with $r_n > \max\{r_{n-1}, \varepsilon_n^{-\frac{2\beta+\varepsilon_n}{\varepsilon_n}}\}$ such that

$$T(t) \leq \left(\frac{t}{r_n} \right)^{\beta+\varepsilon_n} T(r_n)$$

for $1 \leq t \leq r_n^{\frac{\beta+\varepsilon_n}{\beta+\varepsilon_n/2}}$. This immediately implies (1.1.5) and r_n satisfies (4), because for $1 \leq t \leq \varepsilon_n^{-1} r_n (< r_n^{\frac{\beta+\varepsilon_n}{\beta+\varepsilon_n/2}})$,

$$\left(\frac{t}{r_n} \right)^{2\varepsilon_n} \leq e^{2\varepsilon_n |\log \varepsilon_n|}$$

and the quantity on the right side is bounded and tends to 1.

Thus, we have gotten a sequence $\{r_n\}$ satisfying (1.1.5) and (4) in Definition 1.1.1.

Put $d_n = 1 + 1/n$ and $V = \cup_{n=1}^{\infty} [r_n, d_n r_n]$. V has the infinite logarithmic measure and, therefore, there exist a subsequence of $\{[r_n, d_n r_n]\}$, each member of which contains at least a point outside F . Without any loss of generalities we can assume for each n a $\hat{r}_n \in [r_n, d_n r_n] \setminus F$. Then for $\hat{\varepsilon}_n \hat{r}_n \leq t \leq \hat{r}_n/\hat{\varepsilon}_n$ with $\hat{\varepsilon}_n = d_n \varepsilon_n$, we have

$$\begin{aligned}
T(t) &\leq \left(\frac{t}{r_n}\right)^{\beta+\varepsilon_n} T(r_n) \leq (d_n)^{\beta+\varepsilon_n} \left(\frac{t}{\hat{r}_n}\right)^{\beta+\varepsilon_n} T(\hat{r}_n) \\
&\leq (d_n)^{\beta+\varepsilon_n} \left(\frac{1}{\hat{\varepsilon}_n}\right)^{2\varepsilon_n} \left(\frac{t}{\hat{r}_n}\right)^{\beta} T(\hat{r}_n),
\end{aligned}$$

this implies that $\{\hat{r}_n\}$ satisfies (3) in Definition 1.1.1. It is easy to show $\{\hat{r}_n\}$ satisfies other conditions of the Pólya peak.

This completes the proof of Theorem 1.1.3. \square

Chuang considered in [4] the type function and in [3] the Pólya peak sequence of a continuous real function and revealed some relations between the type function and the Pólya peak sequence by demonstrating their existence simultaneously starting from a basic theorem, that is, Theorem 1 of [3] or Lemma 4.4 of [4]. In fact, we easily obtain a sequence of the Pólya peak of order $\lambda(T)$ from the type function, for an example, a careful calculation implies that a sequence of positive real numbers $\{r_n\}$ with $U(r_n) = (1 + o(1))T(r_n)$ must be a Pólya peak sequence of $T(r)$ of order $\lambda(T)$. Drasin and Shea [5] obtained a necessary and sufficient condition for existence of a sequence of Pólya peaks of order β which satisfies only (1) and (3) listed in Definition 1.1.1. Set

$$\lambda^*(T) = \sup \left\{ \tau : \limsup_{x, A \rightarrow \infty} \frac{T(Ax)}{A^\tau T(x)} = \infty \right\}$$

and

$$\mu_*(T) = \inf \left\{ \tau : \liminf_{x, A \rightarrow \infty} \frac{T(Ax)}{A^\tau T(x)} = 0 \right\}.$$

It is proved in [5] that $\mu_*(T) \leq \mu(T) \leq \lambda(T) \leq \lambda^*(T)$ and if $\mu_* < \infty$, then a sequence of Pólya peaks of order β satisfying only (1) and (3) listed in Definition 1.1.1 exists if and only if $\mu_* \leq \beta \leq \lambda^*$ and $\beta < \infty$. However, we do not know if this condition is sufficient to the existence of our Pólya peak sequence. Usually, we call λ^* and μ_* respectively the Pólya order and Pólya lower order of $T(r)$.

Generally, there exists no Pólya peak sequence of $T(r)$ whose lower order is of infinite order. However, we have the following, which will be often used in the sequel.

Lemma 1.1.3. *Let $T(r)$ be an increasing and non-negative continuous function with the infinite order and F a set of positive real numbers having finite logarithmic measure. Then given a sequence $\{s_n\}$ of positive real numbers, there exists an unbounded sequence $\{r_n\}$ of positive real numbers outside F such that*

$$\frac{T(t)}{t^{s_n}} \leq e \frac{T(r_n)}{r_n^{s_n}}, \quad 1 \leq t \leq r_n.$$

Proof. Since $T(r)$ is of infinite order, for a fixed s_n we have

$$\limsup_{t \rightarrow \infty} \frac{T(t)}{t^{s_n}} = \infty$$

and it is easy to see that we can find a sequence $\{\hat{r}_m\}$ such that $\hat{r}_m > 2^{nm}$ and $\hat{r}_{m+1} > e^{1/s_n} \hat{r}_m$ and

$$\frac{T(t)}{t^{s_n}} \leq \frac{T(\hat{r}_m)}{\hat{r}_m^{s_n}}, \quad 1 \leq t \leq \hat{r}_m.$$

Set

$$F_n = \bigcup_{m=1}^{\infty} [\hat{r}_m, e^{1/s_n} \hat{r}_m].$$

Then

$$\int_{F_n} \frac{dt}{t} = \sum_{m=1}^{\infty} \int_{\hat{r}_m}^{e^{1/s_n} \hat{r}_m} \frac{dt}{t} = \sum_{m=1}^{\infty} \frac{1}{s_n} = \infty$$

so that $F_n \setminus F$ has the infinite logarithmic measure. We can find a $r_n \in F_n \setminus F$ such that for some m , $\hat{r}_m \leq r_n \leq e^{1/s_n} \hat{r}_m$ and choose a r'_n in $[\hat{r}_m, r_n]$ such that

$$\frac{T(r'_n)}{r_n'^{s_n}} = \max \left\{ \frac{T(t)}{t^{s_n}} : \hat{r}_m \leq t \leq r_n \right\}.$$

Thus for $1 \leq t \leq r_n$, we have

$$\frac{T(t)}{t^{s_n}} \leq \frac{T(r'_n)}{r_n'^{s_n}} \leq \left(\frac{r_n}{r'_n} \right)^{s_n} \frac{T(r_n)}{r_n^{s_n}} \leq e \frac{T(r_n)}{r_n^{s_n}}.$$

The desired sequence $\{r_n\}$ has been attained. □

1.1.3 The Regularity of a Real Function

We first of all consider the density and the logarithmic density of a Lebesgue measurable set on the positive real axis. However, we begin with a general case, which will bring us some benefits.

An absolutely continuous function $\psi(r)$ on an interval $[a, b]$ has finite derivative almost everywhere in the sense of Lebesgue and $\psi'(r) \in L^1([a, b])$ and for each $r \in [a, b]$

$$\psi(r) = \psi(a) + \int_a^r \psi'(t) dt$$

and an indefinite integral of a function in $L^1([a, b])$ is absolutely continuous. A convex function is absolutely continuous and its right (left) derivative is non-decreasing. We say that an increasing function $\psi(r)$ is a convex function of another increasing $\varphi(r)$ if the right (left) derivative $d\psi(t)/d\varphi(t)$ exists and is non-decreasing.

We denote by m the Lebesgue measure on the positive real axis. Let E be a Lebesgue measurable subset of the positive real axis and $\psi(r)$ a positive and ab-