Graduate Texts in Mathematics

David Eisenbud
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The Geometry of Schemes

概型的几何

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Introduction

What schemes are

The theory of schemes is the foundation for algebraic geometry formulated by Alexandre Grothendieck and his many coworkers. It is the basis for a grand unification of number theory and algebraic geometry, dreamt of by number theorists and geometers for over a century. It has strengthened classical algebraic geometry by allowing flexible geometric arguments about infinitesimals and limits in a way that the classic theory could not handle. In both these ways it has made possible astonishing solutions of many concrete problems. On the number-theoretic side one may cite the proof of the Weil conjectures, Grothendieck's original goal (Deligne [1974]) and the proof of the Mordell Conjecture (Faltings [1984]). In classical algebraic geometry one has the development of the theory of moduli of curves, including the resolution of the Brill–Noether–Petri problems, by Deligne, Mumford, Griffiths, and their coworkers (see Harris and Morrison [1998] for an account), leading to new insights even in such basic areas as the theory of plane curves; the firm footing given to the classification of algebraic surfaces in all characteristics (see Bombieri and Mumford [1976]); and the development of higher-dimensional classification theory by Mori and his coworkers (see Kollár [1987]).

No one can doubt the success and potency of the scheme-theoretic methods. Unfortunately, the average mathematician, and indeed many a beginner in algebraic geometry, would consider our title, "The Geometry of Schemes", an oxymoron akin to "civil war". The theory of schemes is widely regarded as a horribly abstract algebraic tool that hides the appeal of geometry to promote an overwhelming and often unnecessary generality.

By contrast, experts know that schemes make things simpler. The ideas behind the theory — often not told to the beginner — are directly related to those from the other great geometric theories, such as differential geometry, algebraic topology, and complex analysis. Understood from this perspective, the basic definitions of scheme theory appear as natural and necessary ways of dealing with a range of ordinary geometric phenomena, and the constructions in the theory take on an intuitive geometric content which makes them much easier to learn and work with.

It is the goal of this book to share this "secret" geometry of schemes. Chapters I and II, with the beginning of Chapter III, form a rapid introduction to basic definitions, with plenty of concrete instances worked out to give readers experience and confidence with important families of examples. The reader who goes further in our book will be rewarded with a variety of specific topics that show some of the power of the scheme-theoretic approach in a geometric setting, such as blow-ups, flexes of plane curves, dual curves, resultants, discriminants, universal hypersurfaces and the Hilbert scheme.

What's in this book?

Here is a more detailed look at the contents:

Chapter I lays out the basic definitions of schemes, sheaves, and morphisms of schemes, explaining in each case why the definitions are made the way they are. The chapter culminates with an explanation of fibered products, a fundamental technical tool, and of the language of the "functor of points" associated with a scheme, which in many cases enables one to characterize a scheme by its geometric properties.

Chapter II explains, by example, what various kinds of schemes look like. We focus on affine schemes because virtually all of the differences between the theory of schemes and the theory of abstract varieties are encountered in the affine case — the general theory is really just the direct product of the theory of abstract varieties à la Serre and the theory of affine schemes. We begin with the schemes that come from varieties over an algebraically closed field (II.1). Then we drop various hypotheses in turn and look successively at cases where the ground field is not algebraically closed (II.2), the scheme is not reduced (II.3), and where the scheme is "arithmetic" — not defined over a field at all (II.4).

In Chapter II we also introduce the notion of *families* of schemes. Families of varieties, parametrized by other varieties, are central and characteristic aspects of algebraic geometry. Indeed, one of the great triumphs of scheme theory — and a reason for much of its success — is that it incorporates this aspect of algebraic geometry so effectively. The central concepts of *limits*, and *flatness* make their first appearance in section II.3 and are discussed in detail, with a number of examples. We see in particular how to take flat limits of families of subschemes, and how nonreduced schemes occur naturally as limits in flat families.

In all geometric theories the compact objects play a central role. In many theories (such as differential geometry) the compact objects can be embedded in affine space, but this is not so in algebraic geometry. This is the reason for the importance of projective schemes, which are *proper*— this is the property corresponding to compactness. Projective schemes form the most important family of nonaffine schemes, indeed the most important family of schemes altogether, and we devote Chapter III to them. After a discussion of properness we give the construction of Proj and describe in some detail the examples corresponding to projective space over the integers and to double lines in three-dimensional projective space (in affine space all double lines are equivalent, as we show in Chapter II, but this is not so in projective space). We also discuss the important geometric constructions of tangent spaces and tangent cones, the universal hypersurface and intersection multiplicities.

We devote the remainder of Chapter III to some invariants of projective schemes. We define free resolutions, graded Betti numbers and Hilbert functions, and we study a number of examples to see what these invariants yield in simple cases. We also return to flatness and describe its relation to the Hilbert polynomial.

In Chapters IV and V we exhibit a number of classical constructions whose geometry is enriched and clarified by the theory of schemes. We begin Chapter IV with a discussion of one of the most classical of subjects in algebraic geometry, the flexes of a plane curve. We then turn to blow-ups, a tool that recurs throughout algebraic geometry, from resolutions of singularities to the classification theory of varieties. We see (among other things) that this very geometric construction makes sense and is useful for such apparently non-geometric objects as arithmetic schemes. Next, we study the *Fano schemes* of projective varieties — that is, the schemes parametrizing the lines and other linear spaces contained in projective varieties — focusing in particular on the Fano schemes of lines on quadric and cubic surfaces. Finally, we introduce the reader to the *forms* of an algebraic variety — that is, varieties that become isomorphic to a given variety when the field is extended.

In Chapter V we treat various constructions that are defined locally. For example, Fitting ideals give one way to define the *image* of a morphism of schemes. This kind of image is behind Sylvester's classical construction of resultants and discriminants, and we work out this connection explicitly. As an application we discuss the set of all tangent lines to a plane curve (suitably interpreted for singular curves) called the *dual* curve. Finally, we discuss the double point locus of a morphism.

In Chapter VI we return to the functor of points of a scheme, and give some of its varied applications: to group schemes, to tangent spaces, and to describing moduli schemes. We also give a taste of the way in which geometric definitions such as that of tangent space or of openness can be extended from schemes to certain functors. This extension represents the beginning of the program of enlarging the category of schemes to a more flexible one, which is akin to the idea of adding distributions to the ordinary theory of functions.

Since we believe in learning by doing we have included a large number of exercises, spread through the text. Their level of difficulty and the background they assume vary considerably.

Didn't you guys already write a book on schemes?

This book represents a major revision and extension of our book *Schemes: The Language of Modern Algebraic Geometry*, published by Wadsworth in 1992. About two-thirds of the material in this volume is new. The introductory sections have been improved and extended, but the main difference is the addition of the material in Chapters IV and V, and related material elsewhere in the book. These additions are intended to show schemes at work in a number of topics in classical geometry. Thus for example we define blowups and study the blowup of the plane at various nonreduced points; and we define duals of plane curves, and study how the dual degenerates as the curve does.

What to do with this book

Our goal in writing this manuscript has been simply to communicate to the reader our sense of what schemes are and why they have become the fundamental objects in algebraic geometry. This has governed both our choice of material and the way we have chosen to present it. For the first, we have chosen topics that illustrate the geometry of schemes, rather than developing more refined tools for working with schemes, such as cohomology and differentials. For the second, we have placed more emphasis on instructive examples and applications, rather than trying to develop a comprehensive logical framework for the subject.

Accordingly, this book can be used in several different ways. It could be the basis of a second semester course in algebraic geometry, following a course on classical algebraic geometry. Alternatively, after reading the first two chapters and the first half of Chapter III of this book, the reader may wish to pass to a more technical treatment of the subject; we would recommend Hartshorne [1977] to our students. Thirdly, one could use this book selectively to complement a course on algebraic geometry from a book such as Hartshorne's. Many topics are treated independently, as illustrations, so that they can easily be disengaged from the rest of the text.

We expect that the reader of this book will already have some familiarity with algebraic varieties. Good sources for this include Harris [1995], Hartshorne [1977, Chapter 1], Mumford [1976], Reid [1988], or Shafarevich [1974, Part 1], although all these sources contain more than is strictly necessary.

Beginners do not stay beginners forever, and those who want to apply schemes to their own areas will want to go on to a more technically oriented treatise fairly soon. For this we recommend to our students Hartshorne's book *Algebraic Geometry* [1977]. Chapters 2 and 3 of that book contain many fundamental topics not treated here but essential to the modern uses of the theory. Another classic source, from which we both learned a great deal, is David Mumford's *The Red Book of Varieties and Schemes* [1988]. The pioneering work of Grothendieck [Grothendieck 1960; 1961a; 1961b; 1963; 1964; 1965; 1966; 1967] and Dieudonné remains an important reference.

Who helped fix it

We are grateful to many readers who pointed out errors in earlier versions of this book. They include Leo Alonso, Joe Buhler, Herbert Clemens, Vesselin Gashorov, Andreas Gathmann, Tom Graber, Benedict Gross, Brendan Hassett, Ana Jeremias, Alex Lee, Silvio Levy, Kurt Mederer, Mircea Mustata, Arthur Ogus, Keith Pardue, Irena Peeva, Gregory Smith, Jason Starr, and Ravi Vakil.

Silvio Levy helped us enormously with his patience and skill. He transformed a crude document into the book you see before you, providing a level of editing that could only come from a professional mathematician devoted to publishing.

How we learned it

Our teacher for most of the matters presented here was David Mumford. The expert will easily perceive his influence; and a few of his drawings, such as that of the projective space over the integers, remain almost intact. It was from a project originally with him that this book eventually emerged. We are glad to express our gratitude and appreciation for what he taught us.

David Eisenbud
Joe Harris

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I
Basic Definitions

Just as topological or differentiable manifolds are made by gluing together open balls from Euclidean space, schemes are made by gluing together open sets of a simple kind, called *affine schemes.* There is one major difference: in a manifold one point looks locally just like another, and open balls are the only open sets necessary for the construction; they are all the same and very simple. By contrast, schemes admit much more local variation; the smallest open sets in a scheme are so large that a lot of interesting and nontrivial geometry happens within each one. Indeed, in many schemes no two points have isomorphic open neighborhoods (other than the whole scheme). We will thus spend a large portion of our time describing affine schemes.

We will lay out basic definitions in this chapter. We have provided a series of easy exercises embodying and applying the definitions. The examples given here are mostly of the simplest possible kind and are not necessarily typical of interesting geometric examples. The next chapter will be devoted to examples of a more representative sort, intended to indicate the ways in which the notion of a scheme differs from that of a variety and to give a sense of the unifying power of the scheme-theoretic point of view.

I.1 Affine Schemes

An *affine scheme* is an object made from a commutative ring. The relationship is modeled on and generalizes the relationship between an affine variety and its coordinate ring. In fact, one can be led to the definition of scheme in the following way. The basic correspondence of classical algebraic geometry is the bijection

$$\{\text{affine varieties}\} \longleftrightarrow \left\{ \begin{array}{c} \text{finitely generated, nilpotent-free rings} \\ \text{over an algebraically closed field } K \end{array} \right\}$$

Here the left-hand side corresponds to the geometric objects we are naively interested in studying: the zero loci of polynomials. If we start by saying that these are the objects of interest, we arrive at the restricted category of rings on the right. Scheme theory arises if we adopt the opposite point of view: if we do not accept the restrictions "finitely generated," "nilpotent-free" or "K-algebra" and insist that the right-hand side include all commutative rings, what sort of geometric object should we put on the left? The answer is "affine schemes"; and in this section we will show how to extend the preceding correspondence to a diagram

$$\begin{array}{ccc} \{\text{affine varieties}\} & \longleftrightarrow & \left\{ \begin{array}{c} \text{finitely generated, nilpotent-free rings} \\ \text{over an algebraically closed field } K \end{array} \right\} \\ \downarrow & & \downarrow \\ \{\text{affine schemes}\} & \longleftrightarrow & \{\text{commutative rings with identity}\} \end{array}$$

We shall see that in fact the ring and the corresponding affine scheme are equivalent objects. The scheme is, however, a more natural setting for many geometric arguments; speaking in terms of schemes will also allow us to globalize our constructions in succeeding sections.

Looking ahead, the case of differentiable manifolds provides a paradigm for our approach to the definition of schemes. A differentiable manifold M was originally defined to be something obtained by gluing together open balls — that is, a topological space with an atlas of coordinate charts. However, specifying the manifold structure on M is equivalent to specifying which of the continuous functions on any open subset of M are differentiable. The property of differentiability is defined locally, so the differentiable functions form a subsheaf $\mathscr{C}^\infty(M)$ of the sheaf $\mathscr{C}(M)$ of continuous functions on M (the definition of sheaves is given below). Thus we may give an alternative definition of a differentiable manifold: it is a topological space M together with a subsheaf $\mathscr{C}^\infty(M) \subset \mathscr{C}(M)$ such that the pair $(M, \mathscr{C}^\infty(M))$ is locally isomorphic to an open subset of $\mathbb{R}^n$ with its sheaf of differentiable functions. Sheaves of functions can also be used to define many other kinds of geometric structure — for example, real analytic manifolds, complex analytic manifolds, and Nash manifolds may all be defined in this way. We will adopt an analogous approach in defining schemes: a scheme will be a topological space X with a sheaf $\mathcal{O}$, locally isomorphic to an affine scheme as defined below.

Let R be a commutative ring. The affine scheme defined from R will be called $\operatorname{Spec} R$, the *spectrum* of R. As indicated, it (like any scheme) consists of a set of points, a topology on it called the *Zariski topology*, and a sheaf $\mathcal{O}_{\operatorname{Spec} R}$ on this topological space, called the *sheaf of regular functions*, or *structure sheaf* of the scheme. Where there is a possibility of confusion we will use the notation $|\operatorname{Spec} R|$ to refer to the underlying set or topological space, without the sheaf; though if it is clear from context what we mean ("an open subset of $\operatorname{Spec} R$," for example), we may omit the vertical bars.

We will give the definition of the affine scheme $\operatorname{Spec} R$ in three stages, specifying first the underlying set, then the topological structure, and finally the sheaf.

I.1.1 Schemes as Sets

We define a *point* of $\operatorname{Spec} R$ to be a prime—that is, a prime ideal—of R. To avoid confusion, we will sometimes write $[\mathfrak{p}]$ for the point of $\operatorname{Spec} R$ corresponding to the prime $\mathfrak{p}$ of R. We will adopt the usual convention that R itself is not a prime ideal. Of course, the zero ideal (0) is a prime if R is a domain.

If R is the coordinate ring of an ordinary affine variety V over an algebraically closed field, $\operatorname{Spec} R$ will have points corresponding to the points of the affine variety—the maximal ideals of R—and also a point corresponding to each irreducible subvariety of V. The new points, corresponding to subvarieties of positive dimension, are at first rather unsettling but turn out to be quite convenient. They play the role of the "generic points" of classical algebraic geometry.

Exercise I-1. Find $\operatorname{Spec} R$ when R is (a) $\mathbb{Z}$; (b) $\mathbb{Z}/(3)$; (c) $\mathbb{Z}/(6)$; (d) $\mathbb{Z}_{(3)}$; (e) $\mathbb{C}[x]$; (f) $\mathbb{C}[x]/(x^2)$.

Each element $f \in R$ defines a "function", which we also write as f, on the space $\operatorname{Spec} R$: if $x = [\mathfrak{p}] \in \operatorname{Spec} R$, we denote by $\kappa(x)$ or $\kappa(\mathfrak{p})$ the quotient field of the integral domain $R/\mathfrak{p}$, called the *residue field* of X at x, and we define $f(x) \in \kappa(x)$ to be the image of f via the canonical maps

$$R \to R/\mathfrak{p} \to \kappa(x).$$

Exercise I-2. What is the value of the "function" 15 at the point $(7) \in \operatorname{Spec} \mathbb{Z}$? At the point (5)?

Exercise I-3. (a) Consider the ring of polynomials $\mathbb{C}[x]$, and let $p(x)$ be a polynomial. Show that if $\alpha \in \mathbb{C}$ is a number, then $(x - \alpha)$ is a prime of $\mathbb{C}[x]$, and there is a natural identification of $\kappa((x - \alpha))$ with $\mathbb{C}$ such that the value of $p(x)$ at the point $(x - \alpha) \in \operatorname{Spec} \mathbb{C}[x]$ is the number $p(\alpha)$.

(b) More generally, if R is the coordinate ring of an affine variety V over an algebraically closed field K and $\mathfrak{p}$ is the maximal ideal corresponding to a point $x \in V$ in the usual sense, then $\kappa(x) = K$ and $f(x)$ is the value of f at x in the usual sense.

In general, the "function" f has values in fields that vary from point to point. Moreover, f is not necessarily determined by the values of this "function". For example, if K is a field, the ring $R = K[x]/(x^2)$ has only one prime ideal, which is (x); and thus the element $x \in R$, albeit nonzero, induces a "function" whose value is 0 at every point of $\operatorname{Spec} R$.

We define a *regular function* on $\operatorname{Spec} R$ to be simply an element of R. So a regular function gives rise to a "function" on $\operatorname{Spec} R$, but is not itself determined by the values of this "function".

I.1.2 Schemes as Topological Spaces

By using regular functions, we make $\operatorname{Spec} R$ into a topological space; the topology is called the *Zariski topology.* The closed sets are defined as follows. For each subset $S \subset R$, let

$$V(S) = \{x \in \operatorname{Spec} R \mid f(x) = 0 \text{ for all } f \in S\} = \{[\mathfrak{p}] \in \operatorname{Spec} R \mid \mathfrak{p} \supset S\}.$$

The impulse behind this definition is to make each $f \in R$ behave as much like a continuous function as possible. Of course the fields $\kappa(x)$ have no topology, and since they vary with x the usual notion of continuity makes no sense. But at least they all contain an element called zero, so one can speak of the locus of points in $\operatorname{Spec} R$ on which f is zero; and if f is to be like a continuous function, this locus should be closed. Since intersections of closed sets must be closed, we are led immediately to the definition above: $V(S)$ is just the intersection of the loci where the elements of S vanish.

For the family of sets $V(S)$ to be the closed sets of a topology it is necessary that it be closed under arbitrary intersections; from the description above it is clear that for any family of sets S_a we have $\bigcap_a V(S_a) = V\bigl(\bigcup_a S_a\bigr)$, as required. It is worth noting also that, if I is the ideal generated by S, then $V(I) = V(S)$.

An open set in the Zariski topology is simply the complement of one of the sets $V(S)$. The open sets corresponding to sets S with just one element will play a special role, essentially because they are again spectra of rings; for this reason they get a special name and notation. If $f \in R$, we define the *distinguished* (or *basic*) open subset of $X = \operatorname{Spec} R$ associated with f to be

$$X_f = |\operatorname{Spec} R| \setminus V(f).$$

The points of X_f — that is, the prime ideals of R that do not contain f — are in one-to-one correspondence with the prime ideals of the localization