普通高等教育“十一五”国家级规划教材

Calculus（Ⅰ）

马继刚　邹云志　P.W.Aitchison

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内容提要

本书是英文版大学数学微积分教材，分为上、下两册。上册为单变量微积分学，包括函数、极限和连续、导数、中值定理及导数的应用以及一元函数积分学等内容；下册为多变量微积分学，包括空间解析几何及向量代数、多元函数微分学、重积分、线积分与面积分、级数及微分方程初步等内容。

本书由两位国内作者和一位外籍教授共同完成，在内容体系安排上与国内主要微积分教材一致，同时也充分参考和借鉴了国外尤其是北美一些大学微积分教材的诸多特点，内容深入浅出，语言简洁通俗。

本书适合作为大学本科生一学年微积分教学的教材，也可作为非双语教学的参考书。

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前　言

“教学的艺术就是帮助学生发现问题的艺术”

近年来，国内外在教育教学方面的交流和互动日益加强，很多大学在引进国外优质教学资源、推动高等教育国际合作、提升教育模式的创新能力以及培养大学生的国际视野方面做了很多积极有益的尝试。

四川大学早在2000年就组建了四川大学－华盛顿大学联合创新班。从那时起，我们就开始配合学校的拔尖创新人才培养计划，在这类学生群体中开展了数学课的教育教学探索与实践，其中一个实践就是在微积分、线性代数这些公共基础课上结合国内外优秀教材，产生一些适合中国学生的英文教材。2004年我们同国外作者开始合作编写英文微积分讲义 *Calculus*。在讲义试用的基础上，本书经过了多次丰富和完善，并在2006年被列入普通高等教育“十一五”国家级规划教材。

本书（上、下册）的宗旨就是以浅显易懂的语言去帮助学生发现和体验微积分的思想、方法和知识。其内容体系的安排和国内常见的微积分教材基本一致，同时借鉴了国外教材注重从图像、分析和数值等多方面来阐述微积分知识的特点。上册为单变量微积分：第一章为函数和极限，第二、三章为微分学，积分学在第四章。下册为多变量微积分及级数理论和微分方程初步：第五章为向量代数和空间解析几何，第六章为多元函数微分学，第七、八章为多元函数积分学，第九章为级数，第十章为微分方程初步。本书适合大学本科理工、经管等非数学类专业微积分双语教学使用，也可供其他学习微积分知识的师生参考。

本书在创作的过程中得到了四川大学数学学院、四川大学教务处、四川大学吴玉章学院的领导和很多同事对本书的大力支持和鼓励。我们特别感谢数学学院分管副院长王宝富教授，吴玉章学院刘黎研究员、李娟老师对本书的关心、鼓励和支持；我们也感谢四川大学03～06级联合班同学及07～09级吴玉章学院的同学为本书提的诸多好建议；感谢四川大学数学学院徐小湛教授及吴玉章学院戴旭光、叶军军、阮英波、杨达、刘梦、徐惠、张诚、林聪、周义博、万宏，李凯、谭理、杨勇、何斯美琪、胡清泉以及数学学院温超、研究生张卫兵，丁可伟等；另外还要感谢美国 Wellesley High School 的 Steven Bu 同学为本书做的校对工作；感谢杨志蕊和 Universite Pierre Mendes France 的罗昱同

学为本书配的插图；还特别感谢国内外很多微积分教材的作者：James Stewart，Ross L. Finney，Robert A Adams，Chirstoper George B. Thomas，Maurice D. Weir，Joel Hass，Frank R. Giordano，以及四川大学和同济大学等兄弟院校，他们优秀的著作一直影响和激励着我们。

由于时间和水平的关系，书中难免存在不妥甚至错误的地方，恳请读者批评指正，以便我们及时改进和更正。

编 者

2009年8月

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CHAPTER 1

FUNCTIONS, LIMITS AND CONTINUITY

Calculus is based on many mathematical concepts, notations and theorems that were developed and refined by mathematicians during hundreds of years of research. These basic concepts are described in this chapter. The most fundamental mathematical concept is the *function* that is used to describe, in a mathematical way, the relationship between two or more changing quantities or variables. In this chapter we study functions that determine the value of one variable (the dependent variable) from the value of the other variable (the independent variable), using a specific rule or formula relating the variables. Such a function is called a function of one variable, because there is only one independent variable.

In order to use functions in calculus, the concept of the *limit of a function* and the associated concept of *continuous function* are essential. The limit of a function at a specific value of the independent variable is simply a specific number that the function is trending towards. This seems, at first sight, to be an unlikely basis for the development of mathematics of such fantastic importance as calculus, but in fact it brings about the key developments of both types of calculus.

This chapter starts with a summary of mathematical notations and concepts that provide the building blocks for the development of calculus, including sets, numbers, inequalities, summations, absolute value, logical symbols, binomial coefficients and the binomial theorem.

1.1 Mathematical Sign Language

Mathematical formulations should be precise and concise. To achieve this aim, mathematical language employs precisely-defined symbols and formulas, and we will review them next.

1.1.1 Sets

A *set* is a collection of *elements*, *or members*, that are often numbers but may be other mathematical or non-mathematical objects. Sets are denoted by letters such as S or T, and can be defined simply by listing all elements, such as

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

Sets can also be defined by giving the characterizing properties of the elements. For example, the set S defined above can also be defined by any of the following characterizing properties:

$$S = \{\text{natural numbers from 1 to 10}\}$$

$$S = \{x \in \mathbb{N} \mid 1 \leqslant x \leqslant 10\}$$

In the example above, $\mathbb{N}$ is the symbol for the set of all non-negative integers 0, 1, 2, ⋯, called the *natural number*. The curly brackets indicate that it is a set, the "$x \in \mathbb{N}$" means that it is a set of numbers x that are in $\mathbb{N}$, and the "$1 \leqslant x \leqslant 10$" means that x is further restricted to be between 1 and 10. Another example of a set defined by a property is

$$T = \{\text{real numbers that are zeros of } \sin x\}$$

$$= \{x \mid x = n\pi \text{ for some integer } n\}$$

We write

$x \in S$ if x is an element of S (or shorter: "x is in S")

$x \notin S$ if x is not an element of S (or shorter: "x is not in S")

For example, $0 \in \mathbb{N}$ means that zero is a member of the set of natural numbers (or shorter, 0 is a natural number), and $\pi \notin \mathbb{N}$ means that $\pi = 3.141\,592\,653\,5\cdots$ is not a member of the set of natural numbers (or shorter, π is not a natural number).

If A, B are two sets, then they can be compared in various ways:

$A \subseteq B$ or in words: "A *is a subset of* B",

means: every element from A is also an element of B.

$A = B$ or in words: "A equals B",

means: A and B have precisely the same elements.

$A \subset B$ or in words: "A is a *proper subset* of B",

means: A is a subset of B, but A is not equal to B.

If A, B are sets, then new sets can be created in various ways:

$A \cup B$ or in words: "A *union* B",

means: the set of all elements that are either in A or in B.

$A \cap B$ or in words: "*A intersect B*",

means: the set of all elements that are both in A and in B.

$A \setminus B$ or in words: "*A minus B*",

means: the set of all elements that are in A but not in B.

$A \times B$ or in words "the *product set* of A and B" (also called the *direct product* and the *Cartesian product*),

means: the set of all *pairs* (a, b) where $a \in A$ and $b \in B$.

In particular, $\mathbb{R}$ is the symbol representing the set of all real numbers and $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the set of all real number pairs $(x, y) \in \mathbb{R}^2$ (also referred to as *points* of the *real plane* with respect to a two-dimensional Cartesian coordinate system).

It is useful in some circumstances to be able to refer to a special set that has no members at all. Hence

$\varnothing$ denotes the *empty* set, the only set that contains no elements.

1.1.2 Numbers

In the following, the use of the three dots "$\cdots$" means that the pattern established by the adjacent numbers is followed for ever (an infinite number of times). Sets can have a finite or infinite number of members but all of the number sets described next have an infinite number of members.

$\mathbb{N}$ denotes the set of all *natural numbers*, $\{0, 1, 2, 3, 4, \cdots\}$.

$\mathbb{Z}$ denotes the set of all *integers*: $\{\cdots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \cdots\}$. Note that $\mathbb{N} \subset \mathbb{Z}$ (the natural numbers is a proper subset of the integers).

$\mathbb{Q}$ denotes the set of all *rational numbers*, $\mathbb{Q} = \left\{ \frac{p}{q} \,\middle|\, p \in \mathbb{Z}, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$.

Note that $\mathbb{Z} \subset \mathbb{Q}$ because an integer p is considered to be the same as the fraction $\frac{p}{1}$.

$\mathbb{R}$ denotes the set of all *real numbers*. Note that $\mathbb{Q} \subset \mathbb{R}$. The definition of the real number $\mathbb{R}$ is very complicated and requires the use of limits, discussed later in Section 1.5.

$\mathbb{C}$ denotes the set of all *complex numbers*: $\mathbb{C} = \{a + ib \mid a \in \mathbb{R} \text{ and } b \in \mathbb{R}\}$ where i is a special number (not in $\mathbb{R}$) satisfying $i^2 = -1$. Note that $\mathbb{R} \subset \mathbb{C}$ because a real number a is considered to be the same as the complex number $a + i0$, where "0" means the real number zero).

Even though the different number sets above are defined in totally different ways, they all have operations of addition and multiplication that obey the same set of basic properties. If a, b, c are any numbers in one of the above number sets, then a very brief summary is as follow (where the notation ab means multiplication: $a \times b$):

(1) $a + b$ and ab are both in the set (closed under addition and multiplication);

(2) $a + b = b + a$ and $ab = ba$ (Commutative Laws);

(3) $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$ (Associative Laws);

(4) $a + 0 = a$ (0 is the additive identity);

(5) $a \times 0 = 0$;

(6) $1 \times a = a$ (1 is the multiplicative identity);

(7) $a(b + c) = ab + ac$ (Distributive Law);

(8) For any number a in any of these sets (except $\mathbb{N}$), there is another number written as $-a$ (the negative of a) such that $a + (-a) = 0$;

(9) For any number $b \neq 0$ in any of these sets (except $\mathbb{N}$ and $\mathbb{Z}$), there is another number written $\frac{1}{b}$ (the reciprocal) such that $b \times \frac{1}{b} = 1$.

Note

Subtraction is a special kind of addition in which "a minus b" is written as $a - b$ and defined by $a - b = a + (-b)$. *Division* is a special kind of multiplication in which "a divided by b" is written as $\frac{a}{b}$ and defined by $\frac{a}{b} = a \times \left(\frac{1}{b}\right)$. However, the operations of subtraction and division do not obey the Commutative and Associative Laws.

The real numbers and its subsets, have a unifying geometric property that any straight line that is infinitely long in both directions can be made into a *number line*. This means that we can create a one-to-one correspondence between the real numbers and points of the number line. A number line preserves our intuitive ideas of ordering and size of numbers. In particular the integers $\cdots$, -4, -3, -2, -1, 0, 1, 2, 3, 4, $\cdots$ are at equally spaced points on the number line, and "$a < b$" is exactly equivalent to "a is to the left of b on the number line" (see below for more details about inequalities). Number lines are used as axes in Cartesian coordinate system, and in other important applications.

1.1.3 Intervals

Intervals are special and important subsets of real numbers. They often appear as solution sets to inequalities and are important in the definition of many functions. Intervals come in various forms, summarized below. In the following it is assumed that a and b are real numbers satisfying $a < b$, except in the definition of the closed interval $[a, b]$ where we allow the possibility that $a = b$. The symbol ∞, used below, is **not** a real number, but is a notational convenience to indicate that the interval is unbounded to the right and $-\infty$ is used to indicate that the interval is unbounded to the left.

The *open* interval (a, b) is the set of real numbers $\{x \in \mathbb{R} \mid a < x < b\}$.

The *closed* interval $[a, b]$ is the set of real numbers $\{x \in \mathbb{R} \mid a \leqslant x \leqslant b\}$.

The *half-open* interval $(a, b]$ is the set of real numbers $\{x \in \mathbb{R} \mid a < x \leqslant b\}$.

The *half-open* interval $[a, b)$ is the set of real numbers $\{x \in \mathbb{R} \mid a \leqslant x < b\}$

The above intervals are intervals with finite lengths $b - a$. The following intervals have infinite length.

The *infinite* interval $[a, \infty)$ is the set of real numbers $\{x \in \mathbb{R} \mid a \leqslant x\}$.

The (*open*) *infinite* interval (a, ∞) is the set of real numbers $\{x \in \mathbb{R} \mid a < x\}$.

The *infinite* interval $(-\infty, b]$ is the set of real number $\{x \in \mathbb{R} \mid x \leqslant b\}$.

The *infinite* interval $(-\infty, \infty)$ is the same as the set $\mathbb{R}$ of all real numbers.

Note

(1) The closed interval $[a, a]$ is just the single value a.

(2) The symbol (a, b), where a and b are real numbers, has many uses in mathematics and in particular it is also used to denote a point in the plane defined by a two-dimensional Cartesian coordinate system. Thus sometimes it is necessary to make this clear using words like "the interval (a, b)" or "the point is (a, b)".

(3) $\{a, b\}$ is notation for the set with members a and b, and **not** for an interval or a point.

1.1.4 Implication and Equivalence

If S and T are two related mathematical or non-mathematical *statements* that can be true or false, then more complicated statements involving S and T also may be true or false. We write:

$S \Rightarrow T$ or in words: "*S implies T*", or "*T only if S*",

means: if S is true, then T is also true. Many mathematical theorems have this structure.

$\tilde{}T \Rightarrow \tilde{}S$ or in words: "not T implies not S",

means: if T is not true, then S is not true. This statement is called the *contra-*

positive of the first statement $S \Rightarrow T$, and the two statements, $S \Rightarrow T$ and $\tilde{T} \Rightarrow \tilde{S}$, mean exactly the same thing: if either one is true, then so is the other, and if either one is false, then so is the other.

$S \Leftrightarrow T$ or in words: "S is equivalent to T", or "S *if and only if* T", or, "S *iff* T"),

means: $S \Rightarrow T$ and $T \Rightarrow S$, or equivalently S and T are both true, or S and T are both false.

Note

The notation $S \Leftrightarrow T$ is often written in theorems in the form "S if and only if T", which is equivalent to the two statements "S if T" and "S only if T". The statement "S if T" should be read as "S is true **if** T is true" and is exactly the same as: $T \Rightarrow S$. The statement "S only if T" should be read as "S is true **only if** T is true" and is exactly the same as: $S \Rightarrow T$.

1.1.5 Inequalities and Numbers

Inequalities describe relations between the sizes of real numbers. Inequalities cannot be applied to complex numbers.

A real number x is *positive*, written as $x > 0$, iff (if and only if) it is the square of another non-zero real number. In symbols:

$$x > 0 \Leftrightarrow \text{there is a } y \in \mathbb{R} \text{ such that } y \neq 0 \text{ and } x = y^2$$

In our usual notation for real numbers, the positive numbers are those that can be writ ten without a negative sign (1, $\frac{1}{2}$, $\sqrt{3}$, π, 3.142, etc.), whereas *negative* numbers (all numbers that are not positive or zero) can only be written with a negative sign in front $\left(-1, -\frac{1}{2}, -\sqrt{3}, -\pi, -3.142, \text{etc.}\right)$. Since $-(-b) = b$ for any real number b, the negative of a negative number is a positive number.

We say that real numbers a, $b \in \mathbb{R}$ satisfy the following relationships:

$a < b$ or in words: "a is less than b" (or alternatively: "b is larger than a"),

means: $b - a > 0$ ($b - a$ is a positive number).

$a \leqslant b$ or in words: "a is less than or equal to b" (or alternatively: "a is not larger than b", or: "b is greater than or equal to a"),

means: either $a < b$ or $a = b$.

Note

The two inequalities: $a < b$ and $b > a$ have identical meaning. Similarly the two inequalities: $a \leqslant b$ and $b \geqslant a$ have identical meaning.

Some important properties of inequalities, for any real numbers a, b, c:

(1) $(a < b$ and $b < c) \Rightarrow a < c$ (Transitivity);

(2) $(a < b$ and c arbitrary$) \Rightarrow (a + c < b + c)$;

(3) $(a < b$ and $c > 0) \Rightarrow ac < bc$;

(4) $(a < b$ and $c < 0) \Rightarrow ac > bc$ (notice that the direction of the inequality changes when an inequality is multiplied by a negative number);

(5) $a < b \Leftrightarrow -b < -a$ (this follows from point (4) above when both sides of $a < b$ are multiplied by -1);

(6) $0 < a < b \Rightarrow 0 < \frac{1}{b} < \frac{1}{a}$ (this follows from point (3) above);

(7) $a \in \mathbb{R} \Rightarrow a^2 \geqslant 0$.

Similar results hold for the "$\leqslant$" or "$\geqslant$" signs. Some examples are:

(8) $(a \leqslant b$ and $b < c) \Rightarrow a < c$;

(9) $(a \leqslant b$ and $b \leqslant c) \Rightarrow a \leqslant c$;

(10) $(a \leqslant b$ and c arbitrary$) \Rightarrow (a + c \leqslant b + c)$;

(11) $(a \leqslant b$ and $c > 0) \Rightarrow ac \leqslant bc$;

(12) $(a < b$ and $c \geqslant 0) \Rightarrow ac \leqslant bc$;

(13) $(a \leqslant b$ and $c < 0) \Rightarrow ac \geqslant bc$;

(14) $a \leqslant b \Leftrightarrow -b \leqslant -a$;

(15) $0 < a \leqslant b \Rightarrow 0 < \frac{1}{b} \leqslant \frac{1}{a}$;

(16) If $a \geqslant b \geqslant 0$ then $\sqrt{a} \geqslant \sqrt{b}$ and $-\sqrt{a} \leqslant -\sqrt{b}$ (taking the negative square root reverses the direction of the inequality).

Note

All of the above can be written with the alternative notation for the inequality using "$>$" instead of "$<$", such as:

(1) $(c > b$ and $b > a) \Rightarrow c > a$ (Transitivity);

(2) $(b > a$ and c arbitrary$) \Rightarrow (b + c > a + c)$;

(3) $(b > a$ and $c > 0) \Rightarrow bc > ac$,

and so on.

1.1.6 Absolute Value of a Number

The *absolute value* of a real number a, $|a|$, is defined as:

$$|a| = \begin{cases} a, & \text{if } a \geqslant 0 \\ -a, & \text{if } a < 0 \end{cases}$$

That is, $|a|$ is the same as a if a is positive or zero, and $|a|$ changes a to the positive number $-a$ if a is negative. For any real numbers x, y, some properties of absolute value are:

(1) $|x| \geqslant 0$, and $|x| = 0$ only if $x = 0$;

(2) $|xy| = |x||y|$, and $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, provided that $y \neq 0$;

(3) $|-x| = |x|$ and so $|x-y| = |y-x|$;

(4) $|x|^2 = x^2$;

(5) $\sqrt{x^2} = |x|$ (This is very important. Some commercial software package have this wrong), that is, if x represents a negative number then $\sqrt{x^2} = -x$;

(6) If $a > 0$ then $|x| < a$ is equivalent to $-a < x < a$ or $x \in (-a, a)$;

(7) If $a > 0$ then $|x| \leqslant a$ is equivalent to $-a \leqslant x \leqslant a$ or $x \in [-a, a]$;

(8) If $a > 0$ then $|x-b| < a$ is equivalent to $b-a < x < b+a$ or x is in the open interval $(b-a, b+a)$;

(9) If $a > 0$ then $|x-b| \leqslant a$ is equivalent to $b-a \leqslant x \leqslant b+a$ or $x \in [b-a, b+a]$;

(10) $|x| \leqslant 1 \Rightarrow |x| \geqslant |x|^n$ if n is any positive integer;

(11) $1 \leqslant |x| \Rightarrow |x| \leqslant |x|^n$ if n is any positive integer;

(12) $|a-b|$ is the distance between a and b on the number line;

(13) $|x+y| \leqslant |x| + |y|$;

(14) $||x| - |y|| \leqslant |x-y|$.

Note

Once again $\sqrt{x^2} = |x|$. That is, $\sqrt{x^2} = \begin{cases} x, & \text{if } x \geqslant 0, \\ -x, & \text{if } x < 0. \end{cases}$ The reason for this is that the square root function $\sqrt{z}$ is always defined to be the positive square root of the number z.

A *neighborhood* of $x = a$ means an open interval (b, c) containing a, so that $b < a < c$. A δ-neighborhood $(\delta > 0)$ of $x = a$, denoted by $U(a, \delta)$, is the open interval $(a-\delta, a+\delta)$, or equivalently, all x satisfying $|x-a| < \delta$. That is,

$$U(a, \delta) = \{x \mid |x-a| < \delta\}$$

Since $|x-a|$ stands for the distance between x and a, so $U(a, \delta)$ represents all the x whose distance from a is less than δ, and consequently a is called the center of the