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Oscar Zariski

Algebraic Surfaces

代数曲面

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Preface to the First Edition

The aim of the present monograph is to give a systematic exposition of the theory of algebraic surfaces emphasizing the interrelations between the various aspects of the theory: algebro-geometric, topological and transcendental. To achieve this aim, and still remain inside the limits of the allotted space, it was necessary to confine the exposition to topics which are absolutely fundamental. The present work therefore makes no claim to completeness, but it does, however, cover most of the central points of the theory.

A presentation of the theory of surfaces, to be effective at all, must above all give the typical methods of proof used in the theory and their underlying ideas. It is especially true of algebraic geometry that in this domain the methods employed are at least as important as the results. The author has therefore avoided, as much as possible, purely formal accounts of results. The proofs given are of necessity condensed, for reasons of space, but no attempt has been made to condense them beyond the point of intelligibility. In many instances, due to exigencies of simplicity and rigor, the proofs given in the text differ, to a greater or less extent, from the proofs given in the original papers.

The author regrets that he has not been able, for the reasons outlined above, to include in his work two interesting and important developments of the theory: (I) the classification of surfaces by means of their invariants, due chiefly to ENRIQUES; (II) the theory of real algebraic surfaces, due to COMESSATTI. Fortunately, excellent and quite recent accounts of these two developments are available (I. GEPPERT, a; II. COMESSATTI, b; see "Bibliography").

Thanks are due to Dr. S. F. BARBER, National Research Fellow, and to Dr. R. J. WALKER of Princeton University, for careful reading of the manuscript and for many valuable suggestions.

Baltimore, June 12, 1934 O. ZARISKI

Preface to the Appendices

The many changes in mathematical taste and terminology and our limited knowledge of the literature have made all but impossible our task of satisfactorily updating ZARISKI's definitive account of the classical theory of algebraic surfaces. When, as the chief author of the appendices to the present edition, I sent the manuscripts to ZARISKI for his inspection, I felt acutely the deficiencies of our contributions. Is any potential reader skilled enough to be familiar with all the diverse foundations and abstract tools referred to in these appendices, patient enough to unwind the tangled relationships between old and new lines of argument, indulgent enough to forgive the gaps and gross oversimplifications caused by our parochial point of view and interested enough to want to read our hodge-podge that jumps back and forth between references and brief allusions?

The original edition of this book came at a very opportune moment. The Italian school, judged by its own standard, had completed a mature theory of algebraic surfaces. ZARISKI brought together the techniques of topology, analysis and algebraic geometry proper and put together a coherent and essentially complete account of this theory. It seems to us that the original text of the book is an excellent place for a student to learn the methods of classical geometry and to find the old results – some of them familiar results in the modern theory but still stated here clearly without the trappings of any abstract machinery. To help such a student one of the aims of our appendices is to clarify the connection between the modern and the Italian terminology and between essentially equivalent modern and Italian theorems. Chapters 2 and 3 particularly are hard for a modern reader to follow. The reason for this is that to the Italians, a surface was essentially a birational equivalence class and the models used were almost always (non-normal) surfaces in $\mathbf{P}^3$; whereas today 2 surfaces are thought to be "the same" only if they are biregularly equivalent (i.e. isomorphic as schemes), and since the models used must usually lie in at least $\mathbf{P}^5$, the basic definitions are made by appealing to methods familiar in differential and analytic geometry (charts, tangent vectors and differentials, line bundles) rather than by projective methods. Thus for instance the text uses extensively the concept of a linear system *with assigned base points*. This is a complicated notion which is forced on you if you want to set up a birationally invariant theory of linear systems.

However it seems to me that it has relatively few applications – the main one in this book is ENRIQUES' proof in Chapter 4 of the birational invariance of the arithmetic genus – and the reader should be advised that for most of Chapter 4 and Chapter 5 through 8, the concept of an ordinary linear system will suffice. On the other hand, as the reader of LIPMAN's appendix to Chapter 2 will see, it can be used together with the "ZARISKI RIEMANN surface" of the function field to give a very clear idea of the birationally-invariant geometry of the surface.

The largest change in the scope of the modern theory (as opposed to changes in foundation or in technique or generalizations of results on surfaces to varieties of arbitrary dimension) is probably the extension to characteristic p and the consequent tie-up with arithmetic problems over finite fields. In almost every appendix, we have come back repeatedly to the question: which results are still true in char. p and where and how are they proven. For instance the extension to characteristic p of the theory of PICARD varieties with its application to the theory of correspondences lead WEIL to initiate the abstract theory of abelian varieties and to seek methods in char. p of constructing auxiliary spaces which could be defined in char. 0 by analytic methods. In the case of the topology of a surface, the extension to char. p has gone hand-in-hand with a whole new technique for defining cohomology groups – the theory of étale cohomology – and we have sketched a few of its parallels with the simplicial homology of a surface in the appendix to Chapter 6. Another major advance since this book first appeared is HODGE's theory of KÄHLER manifolds and the (p, q)-decomposition of their cohomology. This is briefly outlined in the appendix to Chapter 7. Finally the theory of the resolution and structure of singularities, in the hands of ZARISKI, ABHYANKAR, HIRONAKA and others, has grown tremendously. In fact, it has grown way beyond the scope of this book and it seemed impossible in a short appendix to do justice to the results as well as the many new geometric ideas that have been introduced here. Therefore, with regret, we abandoned the project of adding an Appendix to Chapter 1.

Like the original text, we have tried to point out clearly the main gaps in the classical theory – e.g. the problem of the completeness of the characteristic systems of various algebraic families (cf. Chapter 5 and its appendix), and the "strong LEFSCHETZ theorem" (cf. the appendix to Chapter 6). The first of these is now known to be sometimes true, sometimes false, but much work remains to be done before we have a clear idea which of these is the exception and which the rule. The second has been proven in char. 0 by trancendental means but is still unknown in char. p. In the case of char. p, I would like to mention what seem to me to be the 2 most important outstanding problems in the theory of algebraic surfaces: a) to find a theory of "integrals" in char. p, leading to

DeRham cohomology groups over a suitable p-adic coefficient field; b) to prove or disprove Tate's conjecture on the existence of divisors on an algebraic surface in terms of the eigenvalues of the Frobenius acting on H^2.

Warwick, December 1970 David Mumford

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Chapter I

Theory and Reduction of Singularities

1. Algebraic varieties and birational transformations (KRONECKER, 1; MACAULAY, a; VAN DER WAERDEN, a_2, 4, 5; BERTINI, a; SEVERI, c, 26). Let $x_1, x_2, \ldots, x_{r+1}$ be homogeneous point coördinates in a complex projective r-dimensional space S_r. An *algebraic variety* V in S_r is the locus of points (x) satisfying a system of algebraic equations,

$$(1) \qquad f_1(x_1, \ldots, x_{r+1}) = 0, \ldots, f_n(x_1, \ldots, x_{r+1}) = 0,$$

where $f_1, f_2, \ldots, f_n$ are homogeneous polynomials. If φ is a homogeneous polynomial in the x's which vanishes at all the common zeros of $f_1, \ldots, f_n$, i. e. at every point of V, we say briefly that φ *vanishes* ($\varphi = 0$) *on* V. The variety V is *irreducible*, if from $\varphi\psi = 0$ on V it follows necessarily that one at least of the polynomials φ, ψ vanishes on V. In the language of the theory of ideals this definition can be formulated as follows: *V is irreducible if the homogeneous polynomial ideal* $(f_1, \ldots, f_n)$ (H-ideal) *is a primary ideal*[1] (MACAULAY, a, p. 33; VAN DER WAERDEN, a_2, p. 54). From the theorem of HILBERT-NETTO (MACAULAY, a, p. 48; VAN DER WAERDEN, a_2, p. 11) it follows then that either φ^ϱ or ψ^ϱ is a member of $(f_1, \ldots, f_n)$, where ϱ is a convenient integer.

The set of all homogeneous polynomials which vanish on V constitutes an ideal, and, if V is irreducible, this ideal is *prime*, i. e. if $\varphi\psi$ is a member of the ideal, then at least one of the polynomials φ and ψ is a member of the ideal. There is only one prime polynomial ideal associated with a given irreducible algebraic variety (VAN DER WAERDEN, a_2, p. 53).

The application of KRONECKER's method of elimination (KRONECKER, 1) toward the actual determination of the solutions of the system of equations (1) leads to important conclusions concerning the *parametric representation of an irreducible variety* V and allows us to give a rigorous definition of *the dimension* of V (MACAULAY, a, p. 27).

[1] If we were dealing with non-homogeneous coördinates $x_1, x_2, \ldots, x_r$ and with a system of non-homogeneous equations $\psi_1 = 0, \ldots, \psi_n = 0$, then from the hypothesis that $(\psi_1, \ldots, \psi_n)$ is a primary ideal it also would follow that V is irreducible, provided that we include in V only such points at infinity which are limit points of points at finite distance on V.

It is then found that for a generic choice of the coördinate system in S_r an irreducible variety V admits a representation of the type:

$$(2) \quad \begin{cases} \varrho y_i = t_i \varphi', & i = 1, 2, \ldots, k+2, k \leqq r, \\ \varrho y_{k+2+j} = \psi_j(t_1, \ldots, t_{k+2}), & j = 1, 2, \ldots, r-k-1, \\ \varphi(t_1, t_2, \ldots, t_{k+2}) = 0, \end{cases}$$

where φ and the ψ_j's are homogeneous polynomials of like degree, ϱ is a factor of proportionality, φ is irreducible and $\varphi' = \frac{\partial \varphi}{\partial t_1}$. This representation gives *all* the points of V provided that we include all the limit points, which arise when φ' and the polynomials ψ_j vanish simultaneously at a zero of φ. This statement follows from the following lemma, due to RITT: *If a polynomial* g *does not vanish on an irreducible variety* V, *then any point of* V, *at which* $g = 0$, *is a limit point of such points of* V, *at which* $g \neq 0$. In the present case the polynomial g is φ'. A simple proof of the above lemma was given by VAN DER WAERDEN (4). The integer k which occurs in (2) is called the *dimension* of V. To indicate that V is of dimension k we denote the variety by V_k. The form of the equations (2) permits us to prove without difficulties the following theorem: *An arbitrary linear subspace* S_{r-k} *of* S_r *has always points in common with* V_k *(and for a generic* S_{r-k} *the number of common points is finite), while there exist subspaces* S_{r-k-1} *which have no points in common with* V_k. This property of an irreducible V_k can be used for the definition of k.

The dimension k can also be defined as follows: $k+1$ is the maximum number of variables x_i, say $x_1, x_2, \ldots, x_{k+1}$, such that no homogeneous polynomial in these $k+1$ variables is a member of the prime H-ideal associated with V.

A *neighborhood* on V of a point $P_0(x_1^0, \ldots, x_{r+1}^0)$ of V is the set of all points $P(x_1, \ldots, x_{r+1})$ on V such that $|x_i - x_i^0| < \varepsilon$, where ε is a small real positive number. An irreducible V_k is also characterized by the property that if P_0 is *any* point of V_k, any algebraic variety which contains a neighborhood of P_0 contains the whole V_k (MACAULAY, a, p. 28).

In agreement with the above definitions it can be proved that: (1) the locus of points common to two or more algebraic varieties in S_r, or, more generally, satisfying a system of algebraic equations involving the point coördinates x_i and possibly certain parameters, consists of a finite number of irreducible algebraic varieties, possibly of different dimensions; (2) a V_{r-1} in S_r, i. e., an *hypersurface* in S_r, can be represented by *one* equation between the point coördinates x_i.

The parametric representation of an irreducible V_k is frequently used as the point of departure for the definition of a V_k (see, for instance, SEVERI, b, Introduction; d, p. 14). An irreducible V_k in S_r is then defined as the locus of points admitting a parametric representation

$$\varrho x_i = \varphi_i(t_1, t_2, \ldots, t_{k+2}), \qquad i = 1, 2, \ldots, r+1, \tag{3}$$

where the $k+2$ parameters t_j satisfy an *irreducible* homogeneous algebraic equation

$$f(t_1, t_2, \ldots, t_{k+2}) = 0, \tag{4}$$

and where the φ_i's are forms of like degree such that the Jacobian matrix of the $r+2$ polynomials φ_i and f is of rank $k+2$ on (4). This last condition implies that there exists a $(k+2)$-rowed determinant, which is not divisible by the irreducible polynomial f, and is necessary in order to insure that the variety be exactly of dimension k and not less. The points (x) represented by (3) and (4), together with their limit points, which arise when the polynomials φ_i vanish simultaneously at a zero of f, constitute an irreducible V_k also in the sense of the first definition. The associated prime polynomial ideal is the set of all homogeneous polynomials $\psi(x_1, \ldots, x_{r+1})$ such that $\psi(\varphi_1, \ldots, \varphi_{r+1})$ is divisible by f.

Another definition of the dimension of an irreducible variety, independent of the elimination theory, has been given by VAN DER WAERDEN (a_2, pp. 61–64).

Let V_k and V'_l be two irreducible algebraic varieties in $S_r(x_1, x_2, \ldots, x_{r+1})$ and $S'_\varrho(y_1, y_2, \ldots, y_{\varrho+1})$ respectively. An *algebraic correspondence* T between the two varieties is a correspondence between their points such that the coördinates of corresponding points (x) and (y) $(= T(x))$ satisfy a system of algebraic equations

$$\begin{cases} \psi_1(x_1, \ldots, x_{r+1}; y_1, \ldots, y_{\varrho+1}) = 0, \\ \psi_2(x_1, \ldots, x_{r+1}; y_1, \ldots, y_{\varrho+1}) = 0, \ldots, \end{cases} \tag{5}$$

where $\psi_1, \psi_2, \ldots$ are polynomials homogeneous in each set of variables: $x_1, \ldots, x_{r+1}; y_1, \ldots, y_{\varrho+1}$. The possibility of the correspondence not being defined for all points of V_k is not excluded, in consequence of the fact that for generic points (x) on V_k the equations (5) may be inconsistent on V'_l. In this case the points (x) of V_k for which homologous points $(y) = T(x)$ on V'_l exist, lie on a finite number of algebraic varieties on V_k of dimension $< k$. Similar considerations apply to V'_l and to the inverse correspondence T^{-1}.

In various questions concerning algebraic correspondences between two varieties V_k and V'_l it is found useful to consider the *variety of pairs of points* of V_k and V'_l. We put

$$\sigma X_{ij} = x_i y_j, \qquad i = 1, 2, \ldots, r+1; \; j = 1, 2, \ldots, \varrho+1, \tag{5'}$$