

CLASSICS IN MATHEMATICS

Oscar Zariski

Algebraic Surfaces

代数曲面

Springer

世界图书出版公司
www.wpcbj.com.cn

Oscar Zariski

Algebraic Surfaces

Reprint of the 1971 Edition



Springer

Oscar Zariski

Algebraic Surfaces

Second Supplemented Edition

With Appendices by

S. S. Abhyankar, J. Lipman, and D. Mumford



Springer-Verlag Berlin Heidelberg New York
1971

图书在版编目 (CIP) 数据

代数曲面: 英文/ (波兰) 扎里斯基 (Zariski, O.)

著. —影印本. —北京: 世界图书出版公司北京公司, 2010. 2

ISBN 978-7-5100-0516-9

I. ①代… II. ①扎… III. ①代数曲面—英文 IV. 0187. 1

中国版本图书馆 CIP 数据核字 (2010) 第 010692 号

书 名: Algebraic Surfaces

作 者: Oscar Zariski

中 译 名: 代数曲面

责任编辑: 高蓉 刘慧

出 版 者: 世界图书出版公司北京公司

印 刷 者: 三河国英印务有限公司

发 行: 世界图书出版公司北京公司 (北京朝内大街 137 号 100010)

联系电话: 010-64021602, 010-64015659

电子信箱: kjb@wpcbj.com.cn

开 本: 24 开

印 张: 12

版 次: 2010 年 01 月

版权登记: 图字: 01-2009-1059

书 号: 978-7-5100-0516-9/0 · 732

定 价: 35.00 元

The first edition was published 1935 as Band III, Heft 5
of the series *Ergebnisse der Mathematik und ihrer Grenzgebiete*

AMS Subject Classifications (1970):

Primary 14 J XX, 14 C XX, 14 E XX. Secondary 32-XX

ISBN 3-540-05335-2 Second edition Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-05335-2 Second edition Springer-Verlag New York Heidelberg Berlin

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically those of translation, reprinting, re-use of illustrations, broadcasting, reproduction by photocopying machine or similar means, and storage in data banks. Under § 54 of the German Copyright Law where copies are made for other than private use, a fee is payable to the publisher, the amount of the fee to be determined by agreement with the publisher.

© by Springer-Verlag Berlin Heidelberg 1935, 1971. Library of Congress Catalog Card Number 70-148144.
Printed in Germany. Printing and binding: Brühlsche Universitätsdruckerei, Gießen.

Oscar Zariski

Originally published as Vol. 61 of the
Ergebnisse der Mathematik und ihrer Grenzgebiete, 2nd sequence

Mathematics Subject Classification (1991):

Primary 14JXX, 14CXX, 14EXX

Secondary 32-XX

ISBN 3-540-58658-X Springer-Verlag Berlin Heidelberg New York

Photograph by kind permission of George Bergman

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustration, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provision of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer-Verlag. Violations are liable for prosecution under the German Copyright Law.

© Springer-Verlag Berlin Heidelberg 1995

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

This reprint has been authorized by Springer-Verlag (Berlin/Heidelberg/New York) for sale in the Mainland China only and not for export therefrom.

Preface to the First Edition

The aim of the present monograph is to give a systematic exposition of the theory of algebraic surfaces emphasizing the interrelations between the various aspects of the theory: algebro-geometric, topological and transcendental. To achieve this aim, and still remain inside the limits of the allotted space, it was necessary to confine the exposition to topics which are absolutely fundamental. The present work therefore makes no claim to completeness, but it does, however, cover most of the central points of the theory.

A presentation of the theory of surfaces, to be effective at all, must above all give the typical methods of proof used in the theory and their underlying ideas. It is especially true of algebraic geometry that in this domain the methods employed are at least as important as the results. The author has therefore avoided, as much as possible, purely formal accounts of results. The proofs given are of necessity condensed, for reasons of space, but no attempt has been made to condense them beyond the point of intelligibility. In many instances, due to exigencies of simplicity and rigor, the proofs given in the text differ, to a greater or less extent, from the proofs given in the original papers.

The author regrets that he has not been able, for the reasons outlined above, to include in his work two interesting and important developments of the theory: (I) the classification of surfaces by means of their invariants, due chiefly to ENRIQUES; (II) the theory of real algebraic surfaces, due to COMESSATTI. Fortunately, excellent and quite recent accounts of these two developments are available (I. GEPPERT, a; II. COMESSATTI, b; see "Bibliography").

Thanks are due to Dr. S. F. BARBER, National Research Fellow, and to Dr. R. J. WALKER of Princeton University, for careful reading of the manuscript and for many valuable suggestions.

Baltimore, June 12, 1934

O. ZARISKI

Preface to the Appendices

The many changes in mathematical taste and terminology and our limited knowledge of the literature have made all but impossible our task of satisfactorily updating ZARISKI's definitive account of the classical theory of algebraic surfaces. When, as the chief author of the appendices to the present edition, I sent the manuscripts to ZARISKI for his inspection, I felt acutely the deficiencies of our contributions. Is any potential reader skilled enough to be familiar with all the diverse foundations and abstract tools referred to in these appendices, patient enough to unwind the tangled relationships between old and new lines of argument, indulgent enough to forgive the gaps and gross oversimplifications caused by our parochial point of view and interested enough to want to read our hodge-podge that jumps back and forth between references and brief allusions?

The original edition of this book came at a very opportune moment. The Italian school, judged by its own standard, had completed a mature theory of algebraic surfaces. ZARISKI brought together the techniques of topology, analysis and algebraic geometry proper and put together a coherent and essentially complete account of this theory. It seems to us that the original text of the book is an excellent place for a student to learn the methods of classical geometry and to find the old results — some of them familiar results in the modern theory but still stated here clearly without the trappings of any abstract machinery. To help such a student one of the aims of our appendices is to clarify the connection between the modern and the Italian terminology and between essentially equivalent modern and Italian theorems. Chapters 2 and 3 particularly are hard for a modern reader to follow. The reason for this is that to the Italians, a surface was essentially a birational equivalence class and the models used were almost always (non-normal) surfaces in \mathbb{P}^3 ; whereas today 2 surfaces are thought to be "the same" only if they are biregularly equivalent (i.e. isomorphic as schemes), and since the models used must usually lie in at least \mathbb{P}^5 , the basic definitions are made by appealing to methods familiar in differential and analytic geometry (charts, tangent vectors and differentials, line bundles) rather than by projective methods. Thus for instance the text uses extensively the concept of a linear system *with assigned base points*. This is a complicated notion which is forced on you if you want to set up a birationally invariant theory of linear systems.

However it seems to me that it has relatively few applications — the main one in this book is ENRIQUES' proof in Chapter 4 of the birational invariance of the arithmetic genus — and the reader should be advised that for most of Chapter 4 and Chapter 5 through 8, the concept of an ordinary linear system will suffice. On the other hand, as the reader of LIPMAN's appendix to Chapter 2 will see, it can be used together with the "ZARISKI RIEMANN surface" of the function field to give a very clear idea of the birationally-invariant geometry of the surface.

The largest change in the scope of the modern theory (as opposed to changes in foundation or in technique or generalizations of results on surfaces to varieties of arbitrary dimension) is probably the extension to characteristic p and the consequent tie-up with arithmetic problems over finite fields. In almost every appendix, we have come back repeatedly to the question: which results are still true in char. p and where and how are they proven. For instance the extension to characteristic p of the theory of PICARD varieties with its application to the theory of correspondences lead WEIL to initiate the abstract theory of abelian varieties and to seek methods in char. p of constructing auxiliary spaces which could be defined in char. 0 by analytic methods. In the case of the topology of a surface, the extension to char. p has gone hand-in-hand with a whole new technique for defining cohomology groups — the theory of étale cohomology — and we have sketched a few of its parallels with the simplicial homology of a surface in the appendix to Chapter 6. Another major advance since this book first appeared is HODGE's theory of KÄHLER manifolds and the (p, q) -decomposition of their cohomology. This is briefly outlined in the appendix to Chapter 7. Finally the theory of the resolution and structure of singularities, in the hands of ZARISKI, ABHYANKAR, HIRONAKA and others, has grown tremendously. In fact, it has grown way beyond the scope of this book and it seemed impossible in a short appendix to do justice to the results as well as the many new geometric ideas that have been introduced here. Therefore, with regret, we abandoned the project of adding an Appendix to Chapter 1.

Like the original text, we have tried to point out clearly the main gaps in the classical theory — e.g. the problem of the completeness of the characteristic systems of various algebraic families (cf. Chapter 5 and its appendix), and the "strong LEFSCHETZ theorem" (cf. the appendix to Chapter 6). The first of these is now known to be sometimes true, sometimes false, but much work remains to be done before we have a clear idea which of these is the exception and which the rule. The second has been proven in char. 0 by transcendental means but is still unknown in char. p . In the case of char. p , I would like to mention what seem to me to be the 2 most important outstanding problems in the theory of algebraic surfaces: a) to find a theory of "integrals" in char. p , leading to

DERHAM cohomology groups over a suitable p -adic coefficient field;
b) to prove or disprove TATE's conjecture on the existence of divisors
on an algebraic surface in terms of the eigenvalues of the FROBENIUS
acting on H^2 .

Warwick, December 1970

DAVID MUMFORD

Springer-Verlag and the Environment

We at Springer-Verlag firmly believe that an international science publisher has a special obligation to the environment, and our corporate policies consistently reflect this conviction.

We also expect our business partners – paper mills, printers, packaging manufacturers, etc. – to commit themselves to using environmentally friendly materials and production processes.

The paper in this book is made from low- or no-chlorine pulp and is acid free, in conformance with international standards for paper permanency.

Table of Contents

Chapter I. Theory and Reduction of Singularities	1
1. Algebraic varieties and birational transformations	1
2. Singularities of plane algebraic curves	5
3. Singularities of space algebraic curves	10
4. Topological classification of singularities	12
5. Singularities of algebraic surfaces	13
6. The reduction of singularities of an algebraic surface	17
Chapter II. Linear Systems of Curves	24
1. Definitions and general properties	24
2. On the conditions imposed by infinitely near base points	27
3. Complete linear systems	29
4. Addition and subtraction of linear systems	31
5. The virtual characters of an arbitrary linear system	34
6. Exceptional curves	36
7. Invariance of the virtual characters	41
8. Virtual characteristic series. Virtual curves	43
Appendix to Chapter II by JOSEPH LIPMAN	45
Chapter III. Adjoint Systems and the Theory of Invariants	51
1. Complete linear systems of plane curves	51
2. Complete linear systems of surfaces in S_3	51
3. Subadjoint surfaces	53
4. Subadjoint systems of a given linear system	55
5. The distributive property of subadjunction	58
6. Adjoint systems	60
7. The residue theorem in its projective form	66
8. The canonical system	66
9. The pluricanonical systems	70
Appendix to Chapter III by DAVID MUMFORD	71
Chapter IV. The Arithmetic Genus and the Generalized Theorem of RIEMANN-ROCH	75
1. The arithmetic genus p_a	75
2. The theorem of RIEMANN-ROCH on algebraic surfaces	77

3. The deficiency of the characteristic series of a complete linear system	80
4. The elimination of exceptional curves and the characterization of ruled surfaces	83
Appendix to Chapter IV by DAVID MUMFORD	88
Chapter V. Continuous Non-linear Systems	92
1. Definitions and general properties	92
2. Complete continuous systems and algebraic equivalence	95
3. The completeness of the characteristic series of a complete continuous system	98
4. The variety of PICARD	104
5. Equivalence criteria	104
6. The theory of the base and the number ρ of PICARD	107
7. The division group and the invariant σ of SEVERI	111
8. On the moduli of algebraic surfaces	113
Appendix to Chapter V by DAVID MUMFORD	118
Chapter VI. Topological Properties of Algebraic Surfaces	129
1. Terminology and notations	129
2. An algebraic surface as a manifold M_4	129
3. Algebraic cycles on F and their intersections	130
4. The representation of F upon a multiple plane	131
5. The deformation of a variable plane section of F	132
6. The vanishing cycles δ_i and the invariant cycles	133
7. The fundamental homologies for the 1-cycles on F	135
8. The reduction of F to a cell	137
9. The three-dimensional cycles	138
10. The two-dimensional cycles	139
11. The group of torsion	140
12. Homologies between algebraic cycles and algebraic equivalence. The invariant ρ_0	142
13. The topological theory of algebraic correspondences	143
Appendix to Chapter VI by DAVID MUMFORD	147
Chapter VII. Simple and Double Integrals on an Algebraic Surface	156
1. Classification of integrals	156
2. Simple integrals of the second kind	157
3. On the number of independent simple integrals of the first and of the second kind attached to a surface of irregularity q . The fundamental theorem	159
4. The normal functions of POINCARÉ	165
5. The existence theorem of LEFSCHETZ-POINCARÉ	169

6. Reducible integrals. Theorem of POINCARÉ	173
7. Miscellaneous applications of the existence theorem	177
8. Double integrals of the first kind. Theorem of HODGE	182
9. Residues of double integrals and the reduction of the double integrals of the second kind	186
10. Normal double integrals and the determination of the number of independent double integrals of the second kind	191
Appendix to Chapter VII by DAVID MUMFORD	197
Chapter VIII. Branch Curves of Multiple Planes and Continuous Systems of Plane Algebraic Curves	
1. The problem of existence of algebraic functions of two variables	207
2. Properties of the fundamental group G	210
3. The irregularity of cyclic multiple planes	211
4. Complete continuous systems of plane curves with d nodes	214
5. Continuous systems of plane algebraic curves with nodes and cusps	219
Appendix 1 to Chapter VIII by SHREERAM SHANKAR ABHYANKAR	224
Appendix 2 to Chapter VIII by DAVID MUMFORD	229
Appendix A. Series of Equivalence	232
1. Equivalence between sets of points	232
2. Series of equivalence	233
3. Invariant series of equivalence	235
4. Topological and transcendental properties of series of equivalence	237
5. (Added in 2nd edition, by D. MUMFORD)	238
Appendix B. Correspondences between Algebraic Varieties	239
1. The fixed point formula of LEFSCHETZ	239
2. The transcendental equations and the rank of a correspondence	241
3. The case of two coincident varieties. Correspondences with valence	244
4. The principle of correspondence of ZEUTHEN-SEVERI	245
Bibliography	248
Supplementary Bibliography for Second Edition	256
Index	269

Chapter I

Theory and Reduction of Singularities

1. **Algebraic varieties and birational transformations** (KRONCKER, 1; MACAULAY, a; VAN DER WAERDEN, a₂, 4, 5; BERTINI, a; SEVERI, c, 26). Let x_1, x_2, \dots, x_{r+1} be homogeneous point coordinates in a complex projective r -dimensional space S_r . An *algebraic variety* V in S_r is the locus of points (x) satisfying a system of algebraic equations,

$$(1) \quad f_1(x_1, \dots, x_{r+1}) = 0, \dots, f_n(x_1, \dots, x_{r+1}) = 0,$$

where f_1, f_2, \dots, f_n are homogeneous polynomials. If φ is a homogeneous polynomial in the x 's which vanishes at all the common zeros of f_1, \dots, f_n , i. e. at every point of V , we say briefly that φ *vanishes* ($\varphi = 0$) *on* V . The variety V is *irreducible*, if from $\varphi\psi = 0$ on V it follows necessarily that one at least of the polynomials φ, ψ vanishes on V . In the language of the theory of ideals this definition can be formulated as follows: V is *irreducible* if the homogeneous polynomial ideal (f_1, \dots, f_n) (H -ideal) is a *primary ideal*¹ (MACAULAY, a, p. 33; VAN DER WAERDEN, a₂, p. 54). From the theorem of HILBERT-NETTO (MACAULAY, a, p. 48; VAN DER WAERDEN, a₂, p. 11) it follows then that either φ^q or ψ^q is a member of (f_1, \dots, f_n) , where q is a convenient integer.

The set of all homogeneous polynomials which vanish on V constitutes an ideal, and, if V is irreducible, this ideal is *prime*, i. e. if $\varphi\psi$ is a member of the ideal, then at least one of the polynomials φ and ψ is a member of the ideal. There is only one prime polynomial ideal associated with a given irreducible algebraic variety (VAN DER WAERDEN, a₂, p. 53).

The application of KRONECKER's method of elimination (KRONCKER, 1) toward the actual determination of the solutions of the system of equations (1) leads to important conclusions concerning the *parametric representation of an irreducible variety* V and allows us to give a rigorous definition of the *dimension* of V (MACAULAY, a, p. 27).

¹ If we were dealing with non-homogeneous coordinates x_1, x_2, \dots, x_r and with a system of non-homogeneous equations $\psi_1 = 0, \dots, \psi_n = 0$, then from the hypothesis that (ψ_1, \dots, ψ_n) is a primary ideal it also would follow that V is irreducible, provided that we include in V only such points at infinity which are limit points of points at finite distance on V .

It is then found that for a generic choice of the coördinate system in S_r an irreducible variety V admits a representation of the type:

$$(2) \quad \begin{cases} \varrho y_i = t_i \varphi', & i = 1, 2, \dots, k+2, k \leq r, \\ \varrho y_{k+2+j} = \psi_j(t_1, \dots, t_{k+2}), & j = 1, 2, \dots, r-k-1, \\ \varphi(t_1, t_2, \dots, t_{k+2}) = 0, \end{cases}$$

where φ and the ψ_j 's are homogeneous polynomials of like degree, ϱ is a factor of proportionality, φ is irreducible and $\varphi' = \frac{\partial \varphi}{\partial t_1}$. This representation gives all the points of V provided that we include all the limit points, which arise when φ' and the polynomials ψ_j vanish simultaneously at a zero of φ . This statement follows from the following lemma, due to RITT: *If a polynomial g does not vanish on an irreducible variety V , then any point of V , at which $g = 0$, is a limit point of such points of V , at which $g \neq 0$.* In the present case the polynomial g is φ' . A simple proof of the above lemma was given by VAN DER WAERDEN (4). The integer k which occurs in (2) is called the *dimension* of V . To indicate that V is of dimension k we denote the variety by V_k . The form of the equations (2) permits us to prove without difficulties the following theorem: *An arbitrary linear subspace S_{r-k} of S_r has always points in common with V_k (and for a generic S_{r-k} the number of common points is finite), while there exist subspaces S_{r-k-1} which have no points in common with V_k .* This property of an irreducible V_k can be used for the definition of k .

The dimension k can also be defined as follows: $k+1$ is the maximum number of variables x_i , say x_1, x_2, \dots, x_{k+1} , such that no homogeneous polynomial in these $k+1$ variables is a member of the prime H -ideal associated with V .

A neighborhood on V of a point $P_0(x_1^0, \dots, x_{r+1}^0)$ of V is the set of all points $P(x_1, \dots, x_{r+1})$ on V such that $|x_i - x_i^0| < \varepsilon$, where ε is a small real positive number. An irreducible V_k is also characterized by the property that if P_0 is any point of V_k , any algebraic variety which contains a neighborhood of P_0 contains the whole V_k (MACAULAY, a, p. 28).

In agreement with the above definitions it can be proved that: (1) the locus of points common to two or more algebraic varieties in S_r , or, more generally, satisfying a system of algebraic equations involving the point coördinates x_i and possibly certain parameters, consists of a finite number of irreducible algebraic varieties, possibly of different dimensions; (2) a V_{r-1} in S_r , i. e., an *hypersurface* in S_r , can be represented by one equation between the point coördinates x_i .

The parametric representation of an irreducible V_k is frequently used as the point of departure for the definition of a V_k (see, for instance,

SEVERI, b, Introduction; d, p. 14). An irreducible V_k in S_r is then defined as the locus of points admitting a parametric representation

$$(3) \quad \varrho x_i = \varphi_i(t_1, t_2, \dots, t_{k+2}), \quad i = 1, 2, \dots, r+1,$$

where the $k+2$ parameters t_j satisfy an *irreducible* homogeneous algebraic equation

$$(4) \quad f(t_1, t_2, \dots, t_{k+2}) = 0,$$

and where the φ_i 's are forms of like degree such that the Jacobian matrix of the $r+2$ polynomials φ_i and f is of rank $k+2$ on (4). This last condition implies that there exists a $(k+2)$ -rowed determinant, which is not divisible by the irreducible polynomial f , and is necessary in order to insure that the variety be exactly of dimension k and not less. The points (x) represented by (3) and (4), together with their limit points, which arise when the polynomials φ_i vanish simultaneously at a zero of f , constitute an irreducible V_k also in the sense of the first definition. The associated prime polynomial ideal is the set of all homogeneous polynomials $\psi(x_1, \dots, x_{r+1})$ such that $\psi(\varphi_1, \dots, \varphi_{r+1})$ is divisible by f .

Another definition of the dimension of an irreducible variety, independent of the elimination theory, has been given by VAN DER WAERDEN (a₂, pp. 61–64).

Let V_k and V'_i be two irreducible algebraic varieties in $S_r(x_1, x_2, \dots, x_{r+1})$ and $S'_e(y_1, y_2, \dots, y_{e+1})$ respectively. An *algebraic correspondence* T between the two varieties is a correspondence between their points such that the coördinates of corresponding points (x) and $(y) (= T(x))$ satisfy a system of algebraic equations

$$(5) \quad \begin{cases} \psi_1(x_1, \dots, x_{r+1}; y_1, \dots, y_{e+1}) = 0, \\ \psi_2(x_1, \dots, x_{r+1}; y_1, \dots, y_{e+1}) = 0, \dots, \end{cases}$$

where ψ_1, ψ_2, \dots are polynomials homogeneous in each set of variables: $x_1, \dots, x_{r+1}; y_1, \dots, y_{e+1}$. The possibility of the correspondence not being defined for all points of V_k is not excluded, in consequence of the fact that for generic points (x) on V_k the equations (5) may be inconsistent on V'_i . In this case the points (x) of V_k for which homologous points $(y) = T(x)$ on V'_i exist, lie on a finite number of algebraic varieties on V_k of dimension $< k$. Similar considerations apply to V'_i and to the inverse correspondence T^{-1} .

In various questions concerning algebraic correspondences between two varieties V_k and V'_i it is found useful to consider the *variety of pairs of points* of V_k and V'_i . We put

$$(5') \quad \sigma X_{ij} = x_i y_j, \quad i = 1, 2, \dots, r+1; j = 1, 2, \dots, e+1,$$