Mathematics Monograph Series 16

Association Schemes of Matrices

Yangxian Wang Yuanji Huo Changli Ma

Translated by Jianmin Ma

(矩阵结合方案)

SCIENCE PRESS
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Note from the Translator

This book is translated from the Chinese monograph *Association Schemes of Matrices* published by Science Press (Beijing, 2006).

This book is not intended as a comprehensive introduction to the theory of association schemes. We refer the reader to [2, 3, 29] for a good treatment of such theory. Rather, the main emphasis of this book is on the association schemes of various types of matrices.

I would like to say a few words about the tradition of association schemes of matrices in China. The first such scheme was constructed on $n \times n$ Hermitian matrices and the parameters were calculated for the lower dimensional ones by Z. Wan in 1965 [20]. Y. Wang derived a recurrence for the parameters in the general case in late 1970s. This result was published in his 1980 paper [23], which also included recurrences for the parameters of association schemes of alternate and rectangular matrices. The methods used here are based on the matrix method in the study of classical groups by L. K. Hua and Z. Wan started in the late 1940s [8, 22, 21]. Subsequently, the same methods have been applied to symmetric matrices in odd characteristic, even characteristic, and then to quadratic forms by the authors and their collaborators. Some of these results were published in Chinese, and hence are not widely available.

The matrix method here is elementary and requires little mathematical background. An undergraduate student who has taken a course in linear and abstract algebra will be able to take on a research project on matrix groups and their geometries with some of the matrix techniques in this book. The matrix method has proven to be effective when the underlying field has fewer elements or in lower dimensions.

There are other families of association schemes that are related to those of matrices. These include the association schemes of dual polar spaces and q-analog Johnson schemes, which are distance regular graphs. A recent development is the characterization of distance-subgraphs of these graphs [11, 12, 27, 28]. The spectrum of the distance-2 subgraph of the bilinear forms graph was determined in [4].

In this edition, I have added some remarks to help English readers. Certain notations are simplified. The authors have corrected all errors and mistakes found in the Chinese edition.

We would like to thank Misha Klin for providing us the article [19].

Atlanta, May, 2009 *Jianmin Ma*

Preface

The concept of the association scheme together with partially balanced incomplete block designs was defined in its own right by R. C. Bose and T. Shimamoto in 1952. It was introduced to describe the balance relations among the treatments of partially balanced incomplete block designs. Association schemes have close connections with coding theory, graph theory, and finite group theory, and in particular, provide a framework for studying codes and designs. By the 1980s, association scheme theory was an important branch of algebraic combinatorics, and the research work on association scheme theory had grown tremendously.

The study of association schemes in China was started by Professors L. C. Chang and Pao-Lu Hsu in the late 1950s. Later, my students and I began to construct association schemes and block designs using various subspaces of vector spaces under the action of classical groups. These results were collected in the monograph *Studies in Finite Geometries and the Construction of Partially Incomplete Block Designs* by Z. Wan, Z. Dai, X. Feng, and B. Yang published by Science Press (Beijing, 1966). In the mid-1960s, I constructed a family of association schemes on Hermitian matrices and computed the parameters of the lower dimensional ones [20] and started a new direction of construction of association schemes on matrices. The association scheme theory developed later indicates that the association schemes of maximal totally isotropic subspaces and of Hermitian matrices are known as primitive P- and Q-polynomial association schemes.

In the late 1970s, Professor Yangxian Wang continued the study of association schemes of matrices. He derived formulas for the parameters of association schemes of Hermitian matrices and constructed association schemes using rectangular matrices and alternate matrices. Later, Professors Yuanji Huo, Xueli Zhu and I studied the association schemes of symmetric matrices in odd characteristic. In the 1990s, Professor Yangxian Wang and his students Jianmin Ma and Changli Ma at that time studied the association schemes of symmetric matrices and quadratic forms in even characteristic. Besides the parameters of these association schemes, they discussed the subschemes, quotient schemes, and duality and automorphisms. So the study of association schemes of matrices has reached a more complete stage. In this monograph, Professors Yangxian Wang, Yuanji Huo, and Dr. Changli Ma collect the results on association schemes of matrices in a systematic way. The aim of this monograph is to study the association schemes of matrices, including construction, parameter calculation, primitivity, duality, automorphisms and polynomial properties, etc. I hope this monograph will provide readers with some methods and tools to study association schemes and bring new results.

Beijing, April, 2009 *Zhexian Wan*

Foreword to the Chinese Edition

Following Professor Zhexian Wan's suggestion, we have collected the results by us and our collaborators on association schemes of matrices over a finite field. We have also added some new material to make the presentation more complete.

This monograph consists of eight chapters. Chapter 1 introduces the basic theory of association schemes taken from Bannai and Ito's book [2]. The topics include Bose-Menser algebra, Krein parameters, duality, primitivity, subschemes and quotient schemes, the polynomial property, and automorphisms. Some topics have been worked out in greater detail. Chapters 2 to 7 cover association schemes based on various types of matrices: rectangular, alternate, Hermitian, symmetric, and quadratic forms. We cover their construction, parameters, duality, primitivity, polynomial property and automorphisms. Finally, Chapter 8 discusses the eigenvalues of the association scheme of quadratic forms and its fusions and their dual schemes.

These association schemes of matrices are from transitive permutation groups with a regular abelian subgroup. Their constructions are based on the normal forms of matrices under the general linear group. When treating the parameters, an effective approach is to use the enumeration formulas of subspaces of various types in the geometries of classical groups. We use Wan's book [22] extensively. In order to help the reader become familiar with the matrix method, we calculate the parameters of several distance regular graphs with the matrix method. The automorphism group of these association schemes can be obtained from the fundamental theorems of the geometry of matrices [21]. For quadratic forms in even characteristic, we use their matrix representation [22]. When determining the automorphisms of the association scheme of quadratic forms in even characteristic for $n \geqslant 3$, we use a result on the automorphism group of the quadratic forms graph [18]. The case $n = 2$ is obtained by an argument based on the matrix method. In terms of contents and historical background, the matrix method was developed by Lou-keng Hua and Zhe-xian Wan in the study of classical groups. In this monograph, we use the matrix method to study the association schemes based on matrices.

Chapters 2 to 7 are independent of each other. The reader should be able to read any other chapter once he reads Chapter 1. Of course, the reader may still need certain materials that can be found in [22] and [21]. In addition, some topics require certain basic character theory, which can be found in many books such as [2] or [10].

Some of the materials in this monograph have never been published. We are interested in any comments and suggestions.

We would like to acknowledge the support and guidance from Prof. Zhe-xian Wan and thank him for writing the preface. We also thank the following colleagues: Prof. Rongquan Feng and Dr. Jianmin Ma for help with the draft of Chapter 8, Prof. Suogang Gao for using some draft materials in his graduate course and Prof. Kaishun Wang for reading the manuscript and for checking certain calculations. They have provided us valuable comments. Finally, we acknowledge the financial support of Hebei Normal University and the scientific publishing foundation of the Chinese Academy of Science. We also appreciate the support of our editor Hong Lü of Science Press.

Beijing
Qionghai
Shijiazhuang
April, 2009

Yangxian Wang
Yuanji Huo
Changli Ma

List of Symbols

$\|X\|$	cardinality of set X
R_i	i-th relation or association class
$\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$	association scheme of class d
k_i	valency of R_i
p_{ij}^k	intersection number of $\mathfrak{X}$
A_i	adjacency matrix of R_i
$\mathfrak{A}$	adjacency algebra of $\mathfrak{X}$
$M_n(\mathbb{C})$	full matrix algebra of degree n over complex numbers $\mathbb{C}$
$\mathfrak{B}$	intersection algebra of $\mathfrak{X}$
Ω	finite set
G_x	stabilizer of x in G
Λ_i	orbit of G on $\Omega \times \Omega$
$\mathbb{A}(\sigma)$	permutation matrix of a group element σ
$\mathcal{X}_i$	formal sum of a finite set X_i in G
E_i	i-th primitive central idempotent of $\mathfrak{X}$
$p_i(j)$	eigenvalue of A_i on V_j
m_i	multiplicity of $\mathfrak{X}$, or rank of E_i
$q_i(j)$	dual eigenvalue
P	first eigenmatrix of an association scheme
Q	second eigenmatrix of an association scheme
$\circ$	Hadamard product of matrices
C_i	conjugacy class of a finite group
χ_i	character of a finite group
$\widehat{\mathfrak{A}}$	dual algebra of $\mathfrak{A}$
q_{ij}^k	Krein parameter
$\mathfrak{S}$	Schur ring
$\mathbb{X}_i$	formal sum of the group elements in X_i
Δ_i, Δ_i^*	linear mapping from $\mathbb{C}(X)$ to $\mathbb{C}$
$\mathfrak{S}^*$	dual Schur ring
$\mathfrak{X}(\Sigma)$	quotient scheme of $\mathfrak{X}$ on the system of imprimitivity Σ
$\partial(x,y)$	distance between vertices x and y
$\Gamma_i(x)$	set of vertices at distant i from x in Γ
$\mathrm{Aut}(\Gamma)$	automorphism group of the graph Γ
a_i, c_i, b_i	parameters of a distance regular graph
$\mathrm{Aut}\,\mathfrak{X}$	automorphism group of $\mathfrak{X}$
$\mathrm{Inn}\mathfrak{X}$	inner automorphism group of $\mathfrak{X}$
$\mathbb{F}_q$	field of q elements
$n_i(m \times n, q)$	number of $m \times n$ rectangular matrices of rank i over $\mathbb{F}_q$

$\mathrm{Mat}(m \times n, q)$	association scheme of $m \times n$ rectangular matrices over $\mathbb{F}_q$
$\mathrm{GL}_n(\mathbb{F}_q)$	general linear group of degree n over $\mathbb{F}_q$
ϕ_A	mapping defined by a matrix A
$\mathcal{K}(n,q)$	totality of alternate matrices of degree n over $\mathbb{F}_q$
$\mathrm{Alt}(n,q)$	association scheme of $n \times n$ alternate matrices over $\mathbb{F}_q$
$\mathrm{Sp}_{2\nu}(K_n, \mathbb{F}_q)$	symplectic group of degree 2ν over $\mathbb{F}_q$ with respect to K_n
$N(m,s;2\nu)$	number of subspaces of type (m,s)
$\mathcal{K}_i(n,q)$	number of $n \times n$ alternate matrices of rank $2i$ over $\mathbb{F}_q$
$\mathscr{H}(n,q^2)$	totality of Hermitian matrices of degree n over $\mathbb{F}_q^2$
$\mathrm{U}_n(H,\mathbb{F}_q^2)$	unitary group of degree n with respect to H over $\mathbb{F}_q^2$
$\mathrm{U}_n(\mathbb{F}_{q^2})$	unitary group of degree n over $\mathbb{F}_{q^2}$
$N(m,r;n)$	number of subspaces of type (m,r) in the unitary space $\mathbb{F}_q^{(n)}$
$\mathcal{H}_i(n,q)$	number of $n \times n$ Hermitian matrices of rank i over $\mathbb{F}_q$
$\mathbb{F}_q^*$	multiplicative group of $\mathbb{F}_q$
$\mathcal{S}(n,q)$	totality of $n \times n$ symmetric matrices over $\mathbb{F}_q$
$\mathrm{Sym}(n,q)$	association scheme of $n \times n$ symmetric matrices over $\mathbb{F}_q$
$C_{(i,\xi)}, C_{(i,\xi)}(n)$	cogredient class of type (i,ξ)
$R_{(i,\xi)}$	association class defined by $C_{(i,\xi)}$
$k_{(i,\xi)}$	valency of $R_{(i,\xi)}$
$\mathrm{O}_{2\nu+\delta}(\mathbb{F}_q)$	orthogonal group of degree $2\nu+\delta$ over $\mathbb{F}_q$
$p^{(k,\xi)}_{(i,\eta)(j,\zeta)}, p^{(k,\xi)}_{(i,\eta)(j,\zeta)}(n)$	intersection number
$\overline{R}_i$	a merged association class
$\mathrm{Quad}(n,q)$	Egawa scheme of quadratic forms in n variables over $\mathbb{F}_q$
$\mathrm{Ps}_{2\nu+\delta}(\mathbb{F}_q)$	pseudo-symplectic group of degree $2\nu+\delta$ over $\mathbb{F}_q$
$t(S)$	type of symmetric matrix S
$\overline{\mathrm{Sym}}(n,q)$	fusion scheme of $\mathrm{Sym}(n,q)$
B_f	symmetric bilinear form polarized by f
$\mathcal{Q}(n,q)$	totality of quadratic forms in n variables in even characteristic
$\mathrm{Qua}(n,q)$	association scheme of quadratic forms in n variables over $\mathbb{F}_q$
$\mathrm{O}_n(\mathbb{F}_q, G)$	orthogonal group of degree n with respect to G over $\mathbb{F}_q$
$N(m,2s+\tau,s,\varepsilon;2\nu+\delta)$	number of subspaces of type $(m,2s+\tau,s,\varepsilon)$
$N(m,2s+\tau,s,\Gamma;2\nu+\delta)$	number of subspaces of type $(m,2s+\tau,s,\Gamma)$
$\mathcal{Q}_i(n,q)$	totality of quadratic forms in n variables of type i over $\mathbb{F}_q$
$\widetilde{\mathrm{Qua}}(n,q)$	fusion scheme of $\mathrm{Qua}(n,q)$
$\widetilde{\mathrm{Sym}}(n,q)$	fusion scheme of $\mathrm{Sym}(n,q)$
$f_r^{(n)}$	character defined by a nonalternate matrix of rank r
$f_{2k^*}^{(n)}$	character defined by an alternate matrix of rank r

Contents

Chapter 1
Basic Theory of Association Schemes

1.1 Definition of Association Scheme

Let X be a nonempty set of cardinality n and $R_0, R_1, \ldots, R_d$ be subsets of $X \times X$ that satisfy the following conditions:

(i) $R_0 = \{(x,x) | x \in X\}$;
(ii) $X \times X = R_0 \cup R_1 \cup \cdots \cup R_d$, $R_i \cap R_j = \emptyset$ $(i \neq j)$;
(iii) for each $i \in \{0,1,\ldots,d\}$, there exists some $i' \in \{0,1,\ldots,d\}$ such that ${}^tR_i = R_{i'}$, where ${}^tR_i = \{(x,y) | (y,x) \in R_i\}$;
(iv) for any $i,j,k \in \{0,1,\ldots,d\}$, the number

$$p_{ij}^k = |\{z \in X | (x,z) \in R_i, \ (z,y) \in R_j\}|$$

is constant whenever $(x,y) \in R_k$.

Such a configuration $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ is called an association scheme of class d on X. R_0 is called the trivial or diagonal relation, while the others are called nontrivial relations. Note that d is the number of nontrivial relations. The numbers p_{ij}^k are called the intersection numbers of $\mathfrak{X}$. The association scheme $\mathfrak{X}$ is said to be commutative if

(v) $p_{ij}^k = p_{ji}^k$ for all $i,j,k \in \{0,1,\ldots,d\}$.

Further, $\mathfrak{X}$ is said to be symmetric (or the Bose-Mesner type) if

(vi) $i' = i$ for all $i \in \{0,1,\ldots,d\}$.

If $\mathfrak{X}$ is symmetric, then it is commutative. But the converse does not always hold. In the rest of this book, association schemes are assumed to be commutative unless specified otherwise.

Let $k_i = p_{ii'}^0$. The number k_i is the number of $y \in X$ such that $(x,y) \in R_i$ for any fixed $x \in X$. It is called the valency of R_i. Clearly,

$$k_0 = 1, \quad k_i = k_{i'}, \quad |X| = k_0 + k_1 + \cdots + k_d.$$

Let δ be the Kronecker delta: $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$.

Proposition 1.1. *The following hold*:

(i) $p_{0j}^k = \delta_{jk}$.
(ii) $p_{i0}^k = \delta_{ik}$.

(iii) $p_{ij}^0 = k_i \delta_{ij'}$.
(iv) $p_{ij}^k = p_{i'j'}^{k'}$.
(v) $\sum_{j=0}^d p_{ij}^k = k_i$.
(vi) $k_\gamma p_{\alpha\beta}^\gamma = k_\beta p_{\alpha'\gamma}^\beta = k_\alpha p_{\gamma\beta'}^\alpha$.
(vii) $\sum_{\alpha=0}^d p_{ij}^\alpha p_{k\alpha}^l = \sum_{\beta=0}^d p_{ki}^\beta p_{\beta j}^l$.

Proof. (i) For any $(x,y) \in R_k$,

$$p_{0j}^k = |\{z \in X \mid (x,z) \in R_0, (z,y) \in R_j\}|.$$

Since $(x,z) \in R_0$, $z = x$. If $j \neq k$, no such z exists; if $j = k$, $z = x$.

(ii) Similar to (i).

(iii) For $x \in X$,

$$p_{ij}^0 = |\{z \in X \mid (x,z) \in R_i, (z,x) \in R_j\}|.$$

If $j' \neq i$, no such z exists; if $j' = i$, there are k_i such z.

(iv) If $(x,y) \in R_k$, $(y,x) \in R_{k'}$. Thus,

$$\begin{aligned} p_{ij}^k &= |\{z \in X \mid (x,z) \in R_i, (z,y) \in R_j\}| \\ &= |\{z \in X \mid (z,x) \in R_{i'}, (y,z) \in R_{j'}\}| = p_{j'i'}^{k'}. \end{aligned}$$

Since $\mathfrak{X}$ is commutative, $p_{j'i'}^{k'} = p_{i'j'}^{k'}$.

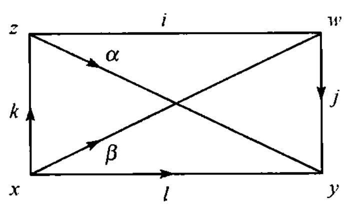
(v) For a fixed pair $(x,y) \in R_k$, count the number of $z \in X$ such that $(x,z) \in R_i$.

(vi) Count in different ways the number of triples (x,y,z) with

$$(x,y) \in R_\gamma, \quad (x,z) \in R_\alpha, \quad (z,y) \in R_\beta.$$

For example, for each fixed x, there are k_γ vertices y such that $(x,y) \in R_\gamma$. For each y, there are $p_{\alpha\beta}^\gamma$ vertices z such that $(x,z) \in R_\alpha$ and $(z,y) \in R_\beta$. We have $|X| k_\gamma p_{\alpha\beta}^\gamma$ such triples.

(vii) For a fixed pair $(x,y) \in R_l$, count the number of pairs (z,w) such that $(x,z) \in R_k$, $(z,w) \in R_i$, $(w,y) \in R_j$.



For $\alpha \in \{0,1,\ldots,d\}$, there are $p_{k\alpha}^l$ vertices z such that $(x,z) \in R_k$ and $(z,y) \in R_\alpha$; for each z, there are p_{ij}^α vertices w such that $(z,w) \in R_i$ and $(w,y) \in R_j$. Hence, there are $\sum_{\alpha=0}^d p_{ij}^\alpha p_{k\alpha}^l$ pairs (z,w). Similarly, there are $\sum_{\beta=0}^d p_{ki}^\beta p_{\beta j}^l$ pairs (z,w) if we count w first and then z. □

For a commutative association scheme $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$, we merge R_i and $R_{i'}$: $\widetilde{R}_i := R_i \cup R_{i'}$. Then $\widetilde{\mathfrak{X}} = (X, \{\widetilde{R}_i\}_{0 \leqslant i \leqslant e})$ is a symmetric association scheme. Let $\widetilde{p}_{ij}^k$ be the intersection number of $\widetilde{\mathfrak{X}}$. Then, by Proposition 1.1 (iv),

$$\widetilde{p}^k_{ij} = p^k_{ij} + (1-\delta_{ii'})p^k_{i'j} + (1-\delta_{jj'})p^k_{ij'} + (1-\delta_{ii'})(1-\delta_{jj'})p^k_{i'j'}.$$

$\widetilde{\mathfrak{X}}$ is called the symmetrization of $\mathfrak{X}$.

Let $\mathfrak{X} = (X, \{R_i\}_{0\leqslant i\leqslant d})$ be a commutative association scheme, $|X| = n$. The adjacency matrix A_i of R_i is the matrix of degree n whose rows and columns are indexed by the elements of X and whose (x,y) entry

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } (x,y) \in R_i, \\ 0, & \text{otherwise.} \end{cases}$$

Conditions (i) to (v) in the definition of an association scheme are equivalent to the following conditions (i′) to (v′):

(i′) $A_0 = I$, the identity matrix;
(ii′) $A_0 + A_1 + \cdots + A_d = J$, the all-one matrix;
(iii′) ${}^tA_i = A_{i'}$;
(iv′) $A_iA_j = \sum_{k=0}^d p^k_{ij} A_k$, $i, j \in \{0, 1, \ldots, d\}$;
(v′) $A_iA_j = A_jA_i$, $i, j \in \{0, 1, \ldots, d\}$.

If there are (0,1) matrices $A_0, A_1, \ldots, A_d$ satisfying conditions (i′) to (v′) with p^k_{ij} non-negative integers, then there exists a commutative association scheme $\mathfrak{X} = (X, \{R_i\}_{0\leqslant i\leqslant d})$ such that $|X| = n$, A_i are the adjacency matrices of R_i and p^k_{ij} are the intersection numbers of $\mathfrak{X}$.

Let $M_n(\mathbb{C})$ be the full matrix algebra of degree n over the complex numbers $\mathbb{C}$. Conditions (i′) to (v′) imply that the adjacency matrices $A_0, A_1, \ldots, A_d$ generated a commutative subalgebra of $M_n(\mathbb{C})$:

$$\mathfrak{A} = \mathbb{C}A_0 + \mathbb{C}A_1 + \cdots + \mathbb{C}A_d.$$

$\mathfrak{A}$ has dimension $\dim_{\mathbb{C}} \mathfrak{A} = d+1$. The algebra $\mathfrak{A}$ is called the adjacency algebra or Bose-Mesner algebra of $\mathfrak{X}$. Moreover, $A_0, A_1, \ldots, A_d$ is a basis of $\mathfrak{A}$.

From Proposition 1.1 (iv), the mapping $A_i \mapsto A_{i'}$ is an automorphism of $\mathfrak{A}$ of period 2. Consider the left regular representation of $\mathfrak{A}$: each A in $\mathfrak{A}$ corresponds to the linear transformation A_L of $\mathfrak{A}$, where $A_L : Y \mapsto AY$ for all $Y \in \mathfrak{A}$. Now, consider the matrix of A_L with respect to the basis $A_0, A_1, \ldots, A_d$. Let $AA_j = \sum_{k=0}^d a_{jk}A_k$. Then $\mathbb{A} = (a_{jk})$ is the matrix of A_L. Further, the correspondence A to ${}^t\mathbb{A}$ is a homomorphism from $\mathfrak{A}$ to $M_{d+1}(\mathbb{C})$. Since $\mathfrak{A}$ has the identity I, it is an isomorphism. For a basis element A_i,

$$A_iA_j = \sum_{k=0}^d p^k_{ij} A_k.$$

Let B_i be the matrix of degree $d+1$ whose (j,k) entry is p^k_{ij}. Then ${}^tB_i$ is the matrix of the linear transformation $(A_i)_L$. Since $\mathfrak{A}$ is commutative, the correspondence $A_i \mapsto B_i$ $(i = 0, 1, \ldots, d)$ is an isomorphism of $\mathfrak{A}$ to $M_{d+1}(\mathbb{C})$. So, we have proved:

Theorem 1.1. *Let $\mathfrak{X} = (X, \{R_i\}_{0\leqslant i\leqslant d})$ be a commutative association scheme. Let A_0, $A_1, \ldots, A_d$ be the adjacency matrices and p^k_{ij} be the intersection numbers. Let $\mathfrak{B}$ be the* *subalgebra of $M_{d+1}(\mathbb{C})$ spanned by $B_0, B_1, \ldots, B_d$. Then the adjacency algebra $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$ by the correspondence $A_i \mapsto B_i$. In particular, A_i and B_i have the same minimal polynomial.* □

The matrices $B_i = (p_{ij}^k)$ are called the intersection matrices of $\mathfrak{X}$, and the algebra $\mathfrak{B}$ is called the intersection algebra of $\mathfrak{X}$.

If $\mathfrak{X}$ is commutative, then

$$B_i B_j = \sum_{k=0}^{d} p_{ij}^k B_k, \quad i,j = 0,1,\ldots,d.$$

On the other hand, if $\mathfrak{X}$ is not necessarily commutative, then

$${}^tB_i {}^tB_j = \sum_{k=0}^{d} p_{ij}^k {}^tB_k, \quad i,j = 0,1,\ldots,d.$$

1.2 Examples

We give two examples of association schemes. Since groups and association schemes share certain natural connections, we start with permutation groups.

Example 1.2. *Let G be a finite group acting transitively on a finite set Ω. This induces an action on $\Omega \times \Omega$: for $(x,y) \in \Omega \times \Omega$ and $\sigma \in G$,*

$$(x,y)^\sigma = (x^\sigma, y^\sigma).$$

Then G no longer acts transitively on $\Omega \times \Omega$ if $|\Omega| = n > 1$. Let $\Lambda_0, \Lambda_1, \ldots, \Lambda_d$ be the orbits of G on $\Omega \times \Omega$, where $\Lambda_0 = \{(x,x) | x \in \Omega\}$. Then $\mathfrak{X} = (\Omega, \{\Lambda_i\}_{0 \leqslant i \leqslant d})$ is an association scheme (not necessarily commutative). □

In fact, the axioms (i) to (iii) for the association scheme hold by definition. Now, consider (iv). For any two pairs $(x,y), (x',y')$ in Λ_k, there is some $\sigma \in G$ such that $x^\sigma = x'$ and $y^\sigma = y'$. We have

$$\{z' | (x',z') \in \Lambda_i, (z',y') \in \Lambda_j\} = \{z | (x,z) \in \Lambda_i, (z,y) \in \Lambda_j\}^\sigma.$$

Therefore, the number p_{ij}^k does not depend on the choice (x,y) in Λ_k.

For any $x \in \Omega$, let

$$\Lambda_i(x) = \{y \in \Omega | (x,y) \in \Lambda_i\}.$$

Then $\Lambda_0(x) = \{x\}, \Lambda_1(x), \ldots, \Lambda_d(x)$ are the orbits of the stabilizer G_x of x in G on Ω. Moreover, $k_i = |\Lambda_i(x)|$.

We now consider the adjacency algebra of $\mathfrak{X}$. Let $\mathbb{A}$ be the permutation representation of G on Ω. For each $\sigma \in G$, define the matrix $\mathbb{A}(\sigma)$ as follows:

$$(\mathbb{A}(\sigma))_{xy} = \delta_{x^\sigma y}.$$

Let $\mathbb{A}(G) = \{\mathbb{A}(\sigma) | \sigma \in G\} \subset M_n(\mathbb{C})$. Then $\mathbb{A}$ is a homomorphism from G to $\mathbb{A}(G)$. For any matrix $X = (X_{xy})$ in $M_n(\mathbb{C})$,

$$(\mathbb{A}(\sigma)X\mathbb{A}(\sigma)^{-1})_{xy} = \sum_{s,t\in\Omega} \delta_{x^\sigma s} X_{st} \delta_{t^{\sigma^{-1}} y} = X_{x^\sigma y^\sigma}.$$

This means that $X = (X_{xy})$ commutes with all $\mathbb{A}(\sigma)$ $(\sigma \in G)$ if and only if the condition holds: $X_{xy} = X_{x^\sigma y^\sigma}$ for all $\sigma \in G$. Note that each adjacency matrix A_i satisfies this condition. Since $A_0 + A_1 + \cdots + A_d = J$, any matrix X that commutes with all $\mathbb{A}(\sigma)$ $(\sigma \in G)$ is a linear combination of $A_0, A_1, \ldots, A_d$. Thus, the adjacency algebra $\mathfrak{A}$ of $\mathfrak{X}$ is the centralizer algebra of $\mathbb{A}(G)$ in $M_n(\mathbb{C})$.

According to the (ordinary) representation theory, $\mathbb{A}$ is the direct sum of irreducible representations of G. By Schur's Lemma, $\mathfrak{A}$ is commutative if and only if each irreducible representation in $\mathbb{A}$ has multiplicity one.

Theorem 1.3. *Suppose that G is a finite group acting transitively on a finite set Ω of cardinality n. (G,Ω) determines an association scheme $\mathfrak{X} = (\Omega, \{\Lambda_i\}_{0\leqslant i\leqslant d})$ (not necessarily commutative), where $d+1$ is the number of orbits of the stabilizer G_x on Ω for any $x \in \Omega$. The adjacency algebra $\mathfrak{A}$ of $\mathfrak{X}$ is precisely the centralizer algebra of the permutation representation $\mathbb{A}$ of G on Ω in $M_n(\mathbb{C})$. Moreover, $\mathfrak{X}$ is commutative if and only if each irreducible representation appearing in $\mathbb{A}$ has multiplicity one.* □

Now let us discuss the second example.

Example 1.4. *Let X be a finite group. Each element $b \in X$ determines a permutation of X by right multiplication: $t_b : x \mapsto xb$. It is called the right translation of b. All right translations of X become a group under the function composition, denoted by $T(X)$. Clearly, $T(X)$ acts transitively on X. Let G_0 be an automorphism group of X. Let G be the group generated by G_0 and $T(X)$. In fact, $G = G_0 \cdot T(X)$, the semi-direct product of $T(X)$ and G_0. Each element in G is the unique product of an element in G_0 and an element in $T(X)$: σt_b with $\sigma \in G_0$ and $t_b \in T(X)$. Each σt_b defines a permutation on X by*

$$x^{\sigma t_b} = x^\sigma b \quad \textit{for all } x \in X.$$

G acts on X transitively. By the above example, it determines an association scheme $\mathfrak{X}$ on X. □

Let us first determine the relations of $\mathfrak{X}$. If (x,y) and (x',y') are in the same orbit of G, then there exists some element σt_b in G such that $(x,y)^{\sigma t_b} = (x',y')$:

$$x' = x^\sigma b, \quad y' = y^\sigma b.$$

Hence, $y'x'^{-1} = (yx^{-1})^\sigma$, and thus $y'x'^{-1}$ and yx^{-1} are in the same orbit of G_0. Conversely, if two pairs (x,y) and (x',y') are such that yx^{-1} and $y'x'^{-1}$ are in the same orbit of G_0, then there is some element σ in G_0 such that $y'x'^{-1} = (yx^{-1})^\sigma = y^\sigma(x^\sigma)^{-1}$. Set $b = (y^\sigma)^{-1}y'(= (x^\sigma)^{-1}x')$. Hence, $x' = x^\sigma b$ and $y' = y^\sigma b$, i.e., $(x',y') = (x,y)^{\sigma t_b}$. So (x,y) and (x',y') are in the same orbit of G. Now we see that the orbits of (G_0, X) determine the relations of $\mathfrak{X}$. Let $X_0 = \{1\}, X_1, \ldots, X_d$ be all the orbits of G_0. The relation R_i of X_i is

$$R_i = \{(x,y) \in X \times X | yx^{-1} \in X_i\}.$$

Hence, $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$.

Now we consider the adjacency algebra $\mathfrak{A}$ of $\mathfrak{X}$. Let $R = \mathbb{C}[X]$ be the group algebra of X over the complex numbers $\mathbb{C}$. It consists of all formal $\mathbb{C}$-linear combinations of elements of X, with the obvious rule for addition and with multiplication defined by extending the multiplication in X. For each orbit X_i of G_0, let

$$\mathbb{X}_i = \sum_{x \in X_i} x.$$

$\mathbb{X}_0, \mathbb{X}_1, \ldots, \mathbb{X}_d$ are elements of $\mathbb{C}[X]$ and

$$\mathbb{X}_i \mathbb{X}_j = \sum_{x \in X_i} \sum_{y \in X_j} xy.$$

If an element in X_k appears as a product term xy $(= c)$ on the right-hand side of the above expression, every other element c' in X_k also appears. In fact, all elements of X_k appear the same number of times in this product. Therefore,

$$\mathbb{X}_i \mathbb{X}_j = \sum_{k=0}^{d} c_{ij}^k \mathbb{X}_k, \tag{1.1}$$

where c_{ij}^k are nonnegative integers. $\mathbb{X}_0, \mathbb{X}_1, \ldots, \mathbb{X}_d$ generates a subalgebra of $\mathbb{C}[X]$, denoted by $\mathfrak{S}$. It is called a Schur ring on X.

Let p_{ij}^k be the intersection numbers of $\mathfrak{X}$. We claim: $c_{ij}^k = p_{ji}^k$. Fix a pair $(x,y) \in R_k$, i.e., $yx^{-1} \in X_k$. By definition,

$$p_{ji}^k = |\{z \in X | (x,z) \in R_j,\ (z,y) \in R_i\}|.$$

We count the number of elements z with $zx^{-1} \in X_j$ and $yz^{-1} \in X_i$. Since $yz^{-1} \cdot zx^{-1} = yx^{-1}$, by (1.1) there are exactly c_{ij}^k pairs $(a,b) \in X_i \times X_j$ with $ab = yx^{-1}$. Set $z = a^{-1}y$. Now, $yz^{-1} = a \in X_i$, $zx^{-1} = a^{-1}yx^{-1} = b \in X_j$. So, $(x,z) \in R_j$, $(z,y) \in R_i$. The pairs $(a,b) \in X_i \times X_j$ with $ab = yx^{-1}$ are in one-to-one correspondence with the elements z with $zx^{-1} \in X_j$ and $yz^{-1} \in X_i$. This proves the claim.

This claim means that the adjacency algebra $\mathfrak{A}$ of $\mathfrak{X}$ is anti-isomorphic to the Schur ring $\mathfrak{S}$. $\mathfrak{X}$ is commutative if and only if $\mathfrak{S}$ is commutative. In particular, $\mathfrak{X}$ is symmetric if each X_i is inverse-closed. A subset C of X is inverse-closed if $x^{-1} \in C$ whenever $x \in C$. We get the following result.

Theorem 1.5. *Let X be a finite group and $T(X)$ be the group of right translations of X. Let G_0 be an automorphism group of X. Suppose $G = \langle G_0, T(X) \rangle$, the group generated by G_0 and $T(X)$. Then G acts transitively on X. Hence, this action determines a (not necessarily commutative) association scheme $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$, where $d+1$ is the number of orbits of G_0 on X. Let $X_0 = \{1\}, X_1, \ldots, X_d$ be the orbits of G_0 on X. Then $\mathbb{X}_0, \mathbb{X}_1, \ldots, \mathbb{X}_d$ generate a Schur ring $\mathfrak{S}$ over X, where $\mathbb{X}_i = \sum_{x \in X_i} x$. The adjacency algebra of $\mathfrak{X}$ and $\mathfrak{S}$ are anti-isomorphic. $\mathfrak{X}$ is commutative if and only if $\mathfrak{S}$ is commutative. In particular, $\mathfrak{X}$ is symmetric if each X_i is inverse-closed, i.e., $x \in X_i$ if and only if $x^{-1} \in X_i$.* □