

V. F. Mukhanov, S. Winitzki

# Introduction to Quantum Effects in Gravity

Our history



引力的量子效应导论

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# Introduction to Quantum Effects in Gravity

DAVID H. KAYE

*Department of Physics, University of  
Sussex, Brighton, UK*

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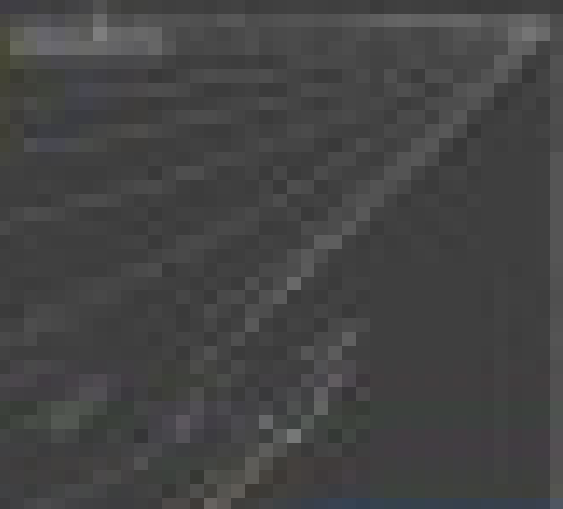
*Correspondence:* Dr David H. Kaye,  
Department of Physics, University of  
Sussex, Brighton BN1 9QJ, UK.  
Email: d.h.kaye@sussex.ac.uk

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*Abstract* Quantum effects in gravity are  
discussed in the context of the  
gravitational redshift, the Casimir  
effect and the Unruh effect.

**Keywords:** quantum effects, gravity,  
gravitational redshift, Casimir  
effect, Unruh effect.



**Fig. 1** Gravitational redshift. A photon emitted from a source at a high gravitational potential (top) is redshifted as it moves to a detector at a lower potential (bottom).

Quantum effects in gravity are discussed in the context of the gravitational redshift, the Casimir effect and the Unruh effect.

The gravitational redshift is a quantum effect that arises from the equivalence principle.

The Casimir effect is a quantum effect that arises from the quantization of the electromagnetic field.

The Unruh effect is a quantum effect that arises from the acceleration of an observer.

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# INTRODUCTION TO QUANTUM EFFECTS IN GRAVITY

VIATCHESLAV MUKHANOV AND SERGEI WINITZKI

*Ludwig-Maximilians University, Munich*



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## Preface

This book is an expanded and reorganized version of the lecture notes for a course taught at the Ludwig-Maximilians University, Munich, in the spring semester of 2003. The course is an elementary introduction to the basic concepts of quantum field theory in classical backgrounds. A certain level of familiarity with general relativity and quantum mechanics is required, although many of the necessary concepts are introduced in the text.

The audience consisted of advanced undergraduates and beginning graduate students. There were 11 three-hour lectures. Each lecture was accompanied by exercises that were an integral part of the exposition and encapsulated longer but straightforward calculations or illustrative numerical results. Detailed solutions were given for all the exercises. Exercises marked by an asterisk (\*) are more difficult or cumbersome.

The book covers limited but essential material: quantization of free scalar fields; driven and time-dependent harmonic oscillators; mode expansions and Bogolyubov transformations; particle creation by classical backgrounds; quantum scalar fields in de Sitter spacetime and the growth of fluctuations; the Unruh effect; Hawking radiation; the Casimir effect; quantization by path integrals; the energy-momentum tensor for fields; effective action and backreaction; regularization of functional determinants using zeta functions and heat kernels. Topics such as quantization of higher-spin fields or interacting fields in curved spacetime, direct renormalization of the energy-momentum tensor, and the theory of cosmological perturbations are left out.

The emphasis of this course is primarily on concepts rather than on computational results. Most of the required calculations have been simplified to the barest possible minimum that still contains all relevant physics. For instance, only free scalar fields are considered for quantization; background spacetimes are always chosen to be conformally flat; the Casimir effect, the Unruh effect, and the Hawking radiation are computed for massless scalar fields in suitable

$1 + 1$ -dimensional spacetimes. Thus a fairly modest computational effort suffices to explain important conceptual issues such as the nature of vacuum and particles in curved spacetimes, thermal effects of gravitation, and backreaction. This should prepare students for more advanced and technically demanding treatments suggested below.

The authors are grateful to Josef Gaßner and Matthew Parry for discussions and valuable comments on the manuscript. Special thanks are due to Alex Vikman who worked through the text and prompted important revisions, and to Andrei Barvinsky for his assistance in improving the presentation in the last chapter.

The entire book was typeset with the excellent  $\text{LyX}$  and  $\text{T}_{\text{E}}\text{X}$  document preparation system on computers running Debian GNU/Linux. We wish to express our gratitude to the creators and maintainers of this outstanding free software.

### Suggested literature

The following books offer a more extensive coverage of the subject and can be studied as a continuation of this introductory course.

- N. D. BIRRELL and P. C. W. DAVIES, *Quantum Fields in Curved Space* (Cambridge University Press, 1982).
- S. A. FULLING, *Aspects of Quantum Field Theory in Curved Space-Time* (Cambridge University Press, 1989).
- A. A. GRIB, S. G. MAMAEV, and V. M. MOSTEPANENKO, *Vacuum Quantum Effects in Strong Fields* (Friedmann Laboratory Publishing, St. Petersburg, 1994).

# Contents

<i>Preface</i>	<i>page ix</i>
<b>Part I Canonical quantization and particle production</b>	<b>1</b>
<b>1 Overview: a taste of quantum fields</b>	<b>3</b>
1.1 Classical field	3
1.2 Quantum field and its vacuum state	4
1.3 The vacuum energy	7
1.4 Quantum vacuum fluctuations	8
1.5 Particle interpretation of quantum fields	9
1.6 Quantum field theory in classical backgrounds	9
1.7 Examples of particle creation	10
<b>2 Reminder: classical and quantum theory</b>	<b>13</b>
2.1 Lagrangian formalism	13
2.1.1 Functional derivatives	15
2.2 Hamiltonian formalism	17
2.3 Quantization of Hamiltonian systems	19
2.4 Hilbert spaces and Dirac notation	20
2.5 Operators, eigenvalue problem and basis in a Hilbert space	22
2.6 Generalized eigenvectors and basic matrix elements	26
2.7 Evolution in quantum theory	29
<b>3 Driven harmonic oscillator</b>	<b>33</b>
3.1 Quantizing an oscillator	33
3.2 The “in” and “out” states	35
3.3 Matrix elements and Green’s functions	38



<b>4</b>	<b>From harmonic oscillators to fields</b>	<b>42</b>
4.1	Quantum harmonic oscillators	42
4.2	From oscillators to fields	43
4.3	Quantizing fields in a flat spacetime	45
4.4	The mode expansion	48
4.5	Vacuum energy and vacuum fluctuations	50
4.6	The Schrödinger equation for a quantum field	51
<b>5</b>	<b>Reminder: classical fields</b>	<b>54</b>
5.1	The action functional	54
5.2	Real scalar field and its coupling to the gravity	56
5.3	Gauge invariance and coupling to the electromagnetic field	58
5.4	Action for the gravitational and gauge fields	59
5.5	Energy-momentum tensor	61
<b>6</b>	<b>Quantum fields in expanding universe</b>	<b>64</b>
6.1	Classical scalar field in expanding background	64
6.1.1	Mode expansion	66
6.2	Quantization	67
6.3	Bogolyubov transformations	68
6.4	Hilbert space; “ <i>a</i> - and <i>b</i> -particles”	69
6.5	Choice of the physical vacuum	71
6.5.1	The instantaneous lowest-energy state	71
6.5.2	Ambiguity of the vacuum state	74
6.6	Amplitude of quantum fluctuations	78
6.6.1	Comparing fluctuations in the vacuum and excited states	80
6.7	An example of particle production	81
<b>7</b>	<b>Quantum fields in the de Sitter universe</b>	<b>85</b>
7.1	De Sitter universe	85
7.2	Quantization	88
7.2.1	Bunch–Davies vacuum	90
7.3	Fluctuations in inflationary universe	92
<b>8</b>	<b>Unruh effect</b>	<b>97</b>
8.1	Accelerated motion	97
8.2	Comoving frame of accelerated observer	100
8.3	Quantum fields in inertial and accelerated frames	103
8.4	Bogolyubov transformations	106
8.5	Occupation numbers and Unruh temperature	107

<b>9</b>	<b>Hawking effect. Thermodynamics of black holes</b>	<b>109</b>
9.1	Hawking radiation	109
9.1.1	Schwarzschild solution	110
9.1.2	Kruskal–Szekeres coordinates	111
9.1.3	Field quantization and Hawking radiation	115
9.1.4	Hawking effect in $3 + 1$ dimensions	118
9.2	Thermodynamics of black holes	120
9.2.1	Laws of black hole thermodynamics	121
<b>10</b>	<b>The Casimir effect</b>	<b>124</b>
10.1	Vacuum energy between plates	124
10.2	Regularization and renormalization	125
<b>Part II</b>	<b>Path integrals and vacuum polarization</b>	<b>129</b>
<b>11</b>	<b>Path integrals</b>	<b>131</b>
11.1	Evolution operator. Propagator	131
11.2	Propagator as a path integral	132
11.3	Lagrangian path integrals	137
11.4	Propagators for free particle and harmonic oscillator	138
11.4.1	Free particle	139
11.4.2	Quadratic potential	139
11.4.3	Euclidean path integral	142
11.4.4	Ground state as a path integral	144
<b>12</b>	<b>Effective action</b>	<b>146</b>
12.1	Driven harmonic oscillator (continuation)	146
12.1.1	Green's functions and matrix elements	146
12.1.2	Euclidean Green's function	148
12.1.3	Introducing effective action	150
12.1.4	Calculating effective action for a driven oscillator	152
12.1.5	Matrix elements	154
12.1.6	The effective action "recipe"	157
12.1.7	Backreaction	158
12.2	Effective action in external gravitational field	159
12.2.1	Euclidean action for scalar field	161
12.3	Effective action as a functional determinant	163
12.3.1	Reformulation of the eigenvalue problem	164
12.3.2	Zeta function	166
12.3.3	Heat kernel	167

<b>13</b>	<b>Calculation of heat kernel</b>	<b>170</b>
13.1	Perturbative expansion for the heat kernel	171
13.1.1	Matrix elements	172
13.2	Trace of the heat kernel	176
13.3	The Seeley–DeWitt expansion	178
<b>14</b>	<b>Results from effective action</b>	<b>180</b>
14.1	Renormalization of the effective action	180
14.2	Finite terms in the effective action	183
14.2.1	EMT from the Polyakov action	185
14.3	Conformal anomaly	187
<i>Appendix 1</i>	Mathematical supplement	193
A1.1	Functionals and distributions (generalized functions)	193
A1.2	Green’s functions, boundary conditions, and contours	202
A1.3	Euler’s gamma function and analytic continuations	206
<i>Appendix 2</i>	Backreaction derived from effective action	212
<i>Appendix 3</i>	Mode expansions cheat sheet	216
<i>Appendix 4</i>	Solutions to exercises	218
<i>Index</i>		272

# **Part I**

## **Canonical quantization and particle production**



# 1

## Overview: a taste of quantum fields

*Summary* Quantum fields as a set of harmonic oscillators. Vacuum state. Particle interpretation of field theory. Examples of particle production by external fields.

We begin with a few elementary observations concerning the vacuum in quantum field theory.

### 1.1 Classical field

A classical field is described by a function  $\phi(\mathbf{x}, t)$ , where  $\mathbf{x}$  is a three-dimensional coordinate in space and  $t$  is the time. At every point the function  $\phi(\mathbf{x}, t)$  takes values in some finite-dimensional “configuration space” and can be a scalar, vector, or tensor.

The simplest example is a real scalar field  $\phi(\mathbf{x}, t)$  whose strength is characterized by real numbers. A free massive scalar field satisfies the Klein–Gordon equation

$$\frac{\partial^2 \phi}{\partial t^2} - \sum_{j=1}^3 \frac{\partial^2 \phi}{\partial x_j^2} + m^2 \phi \equiv \ddot{\phi} - \Delta \phi + m^2 \phi = 0, \quad (1.1)$$

which has a unique solution  $\phi(\mathbf{x}, t)$  for  $t > t_0$  provided that the initial conditions  $\phi(\mathbf{x}, t_0)$  and  $\dot{\phi}(\mathbf{x}, t_0)$  are specified.

Formally one can describe a free scalar field as a set of decoupled “harmonic oscillators.” To explain why this is so it is convenient to begin by considering a field  $\phi(\mathbf{x}, t)$  not in infinite space but in a box of finite volume  $V$ , with some boundary conditions imposed on the field  $\phi$ . The volume  $V$  should be large enough to avoid artifacts induced by the finite size of the box or by physically irrelevant boundary conditions. For example, one might choose the box as a cube

with sides of length  $L$  and volume  $V = L^3$ , and impose the *periodic* boundary conditions,

$$\phi(x=0, y, z, t) = \phi(x=L, y, z, t)$$

and similarly for  $y$  and  $z$ . The Fourier decomposition is then

$$\phi(\mathbf{x}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \phi_{\mathbf{k}}(t) e^{i\mathbf{k} \cdot \mathbf{x}}, \quad (1.2)$$

where the sum goes over three-dimensional wavenumbers  $\mathbf{k}$  with components

$$k_x = \frac{2\pi n_x}{L}, \quad n_x = 0, \pm 1, \pm 2, \dots$$

and similarly for  $k_y$  and  $k_z$ . The normalization factor  $\sqrt{V}$  in equation (1.2) is chosen to simplify formulae (in principle, one could rescale the modes  $\phi_{\mathbf{k}}$  by any constant). Substituting (1.2) into equation (1.1), we find that this equation is replaced by an infinite set of decoupled ordinary differential equations:

$$\ddot{\phi}_{\mathbf{k}} + (k^2 + m^2) \phi_{\mathbf{k}} = 0,$$

with one equation for each  $\mathbf{k}$ . In other words, each complex function  $\phi_{\mathbf{k}}(t)$  satisfies the harmonic oscillator equation with the frequency

$$\omega_{\mathbf{k}} \equiv \sqrt{k^2 + m^2},$$

where  $k \equiv |\mathbf{k}|$ . The “oscillators” with complex coordinates  $\phi_{\mathbf{k}}$  “move” not in real three-dimensional space but in the *configuration space* and characterize the strength of the field  $\phi$ . The total energy of the field  $\phi$  in the box is simply equal to the sum of energies of all oscillators  $\phi_{\mathbf{k}}$ ,

$$E = \sum_{\mathbf{k}} \left[ \frac{1}{2} |\dot{\phi}_{\mathbf{k}}|^2 + \frac{1}{2} \omega_{\mathbf{k}}^2 |\phi_{\mathbf{k}}|^2 \right].$$

In the limit of infinite space when  $V \rightarrow \infty$  the sum in (1.2) is replaced by the integral over all wavenumbers  $\mathbf{k}$ ,

$$\phi(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k} \cdot \mathbf{x}} \phi_{\mathbf{k}}(t). \quad (1.3)$$

## 1.2 Quantum field and its vacuum state

The quantization of a free scalar field is mathematically equivalent to quantizing an infinite set of decoupled harmonic oscillators.

**Harmonic oscillator** A classical harmonic oscillator is described by a coordinate  $q(t)$  satisfying

$$\ddot{q} + \omega^2 q = 0. \quad (1.4)$$

The solution of this equation is unique if we specify initial conditions  $q(t_0)$  and  $\dot{q}(t_0)$ . We may identify the “ground state” of an oscillator as the state without motion, i.e.  $q(t) \equiv 0$ . This lowest-energy state is the solution of the classical equation (1.4) with the initial conditions  $q(0) = \dot{q}(0) = 0$ .

When the oscillator is quantized, the classical coordinate  $q$  and the momentum  $p = \dot{q}$  (for simplicity, we assume that the oscillator has a unit mass) are replaced by operators  $\hat{q}(t)$  and  $\hat{p}(t)$  satisfying the Heisenberg commutation relation

$$[\hat{q}(t), \hat{p}(t)] = [\hat{q}(t), \dot{\hat{q}}(t)] = i\hbar. \quad (1.5)$$

The solution  $\hat{q}(t) \equiv 0$  does not satisfy the commutation relation. In fact, the oscillator’s coordinate always fluctuates. The ground state with the lowest energy is described by the normalized wave function

$$\psi(q) = \left[ \frac{\omega}{\pi\hbar} \right]^{\frac{1}{4}} \exp\left(-\frac{\omega q^2}{2\hbar}\right).$$

The energy of this minimal excitation state, called the *zero-point energy*, is  $E_0 = \frac{1}{2}\hbar\omega$ . The typical amplitude of fluctuations in the ground state is  $\delta q \sim \sqrt{\hbar/\omega}$  and the measured trajectories  $q(t)$  resemble a random walk around  $q = 0$ .

**Field quantization** In the case of a field, each mode  $\phi_{\mathbf{k}}(t)$  is quantized as a separate harmonic oscillator. The classical “coordinates”  $\phi_{\mathbf{k}}$  and the corresponding conjugated momenta  $\pi_{\mathbf{k}} \equiv \dot{\phi}_{\mathbf{k}}^*$  are replaced by operators  $\hat{\phi}_{\mathbf{k}}$ ,  $\hat{\pi}_{\mathbf{k}}$ . In a finite box they satisfy the following equal-time commutation relations:

$$[\hat{\phi}_{\mathbf{k}}(t), \hat{\pi}_{\mathbf{k}'}(t)] = i\delta_{\mathbf{k}, -\mathbf{k}'},$$

where  $\delta_{\mathbf{k}, -\mathbf{k}'}$  is the Kronecker symbol equal to unity when  $\mathbf{k} = -\mathbf{k}'$  and zero otherwise. In the limit of infinite volume the commutation relations become

$$[\hat{\phi}_{\mathbf{k}}(t), \hat{\pi}_{\mathbf{k}'}(t)] = i\delta(\mathbf{k} + \mathbf{k}'), \quad (1.6)$$

where  $\delta(\mathbf{k} + \mathbf{k}')$  is the Dirac  $\delta$  function. To simplify the formulae, we shall almost always use the units in which  $\hbar = c = 1$ .

**Vacuum state** The *vacuum* is a state corresponding to the intuitive notions of “the absence of anything” or “an empty space.” Generally, the vacuum is defined as the state with the lowest possible energy. In the case of a classical field the vacuum is a state where the field is absent, that is,  $\phi(\mathbf{x}, t) = 0$ . This is a solution of the classical equations of motion. When the field is quantized it becomes impossible to satisfy simultaneously the equations of motion for the operator  $\hat{\phi}$  and the commutation relations by  $\hat{\phi}(\mathbf{x}, t) = 0$ . Therefore, the field always fluctuates and has a nonvanishing value even in a state with the minimal possible energy.



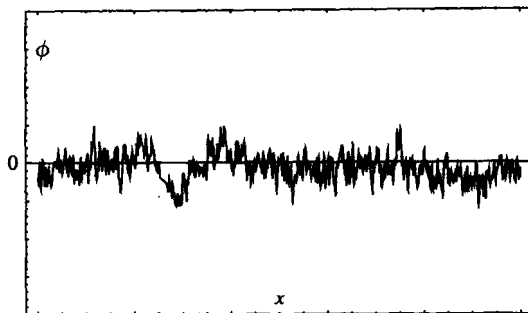


Fig. 1.1 A field configuration  $\phi(x)$  that could be measured in the vacuum state.

Since all modes  $\phi_{\mathbf{k}}$  are decoupled, the ground state of the field can be characterized by a *wave functional* which is the product of an infinite number of wave functions, each describing the ground state of a harmonic oscillator with the corresponding wavenumber  $\mathbf{k}$ :

$$\Psi[\phi] \propto \prod_{\mathbf{k}} \exp\left(-\frac{\omega_{\mathbf{k}} |\phi_{\mathbf{k}}|^2}{2}\right) = \exp\left[-\frac{1}{2} \sum_{\mathbf{k}} \omega_{\mathbf{k}} |\phi_{\mathbf{k}}|^2\right]. \quad (1.7)$$

The ground state of the field has the minimum energy and is called the vacuum state. Strictly speaking, equation (1.7) is valid only for a field quantized in a box. Note that if we had normalized the Fourier components  $\phi_{\mathbf{k}}$  in equation (1.2) differently, then there would be a volume factor in front of  $\omega_{\mathbf{k}}$ .

The square of the wave function (1.7) gives us the probability density for measuring a certain field configuration  $\phi(\mathbf{x})$ . This probability is independent of time  $t$ . The field fluctuates in the vacuum state and the field configurations can be visualized as small random deviations from zero (see Fig. 1.1).

When the volume of the box becomes very large, we have to replace sums by integrals,

$$\sum_{\mathbf{k}} \rightarrow \frac{V}{(2\pi)^3} \int d^3\mathbf{k}, \quad \phi_{\mathbf{k}} \rightarrow \sqrt{\frac{(2\pi)^3}{V}} \phi_{\mathbf{k}}, \quad (1.8)$$

and the wave functional (1.7) becomes

$$\Psi[\phi] \propto \exp\left[-\frac{1}{2} \int d^3\mathbf{k} |\phi_{\mathbf{k}}|^2 \omega_{\mathbf{k}}\right]. \quad (1.9)$$

### Exercise 1.1

The vacuum wave functional (1.9) contains the integral

$$I \equiv \int d^3\mathbf{k} |\phi_{\mathbf{k}}|^2 \sqrt{k^2 + m^2}, \quad (1.10)$$