ALM 10
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Trends in Partial Differential Equations

偏微分方程进展

Editors: Baojun Bian • Shenghong Li • Xu-Jia Wang

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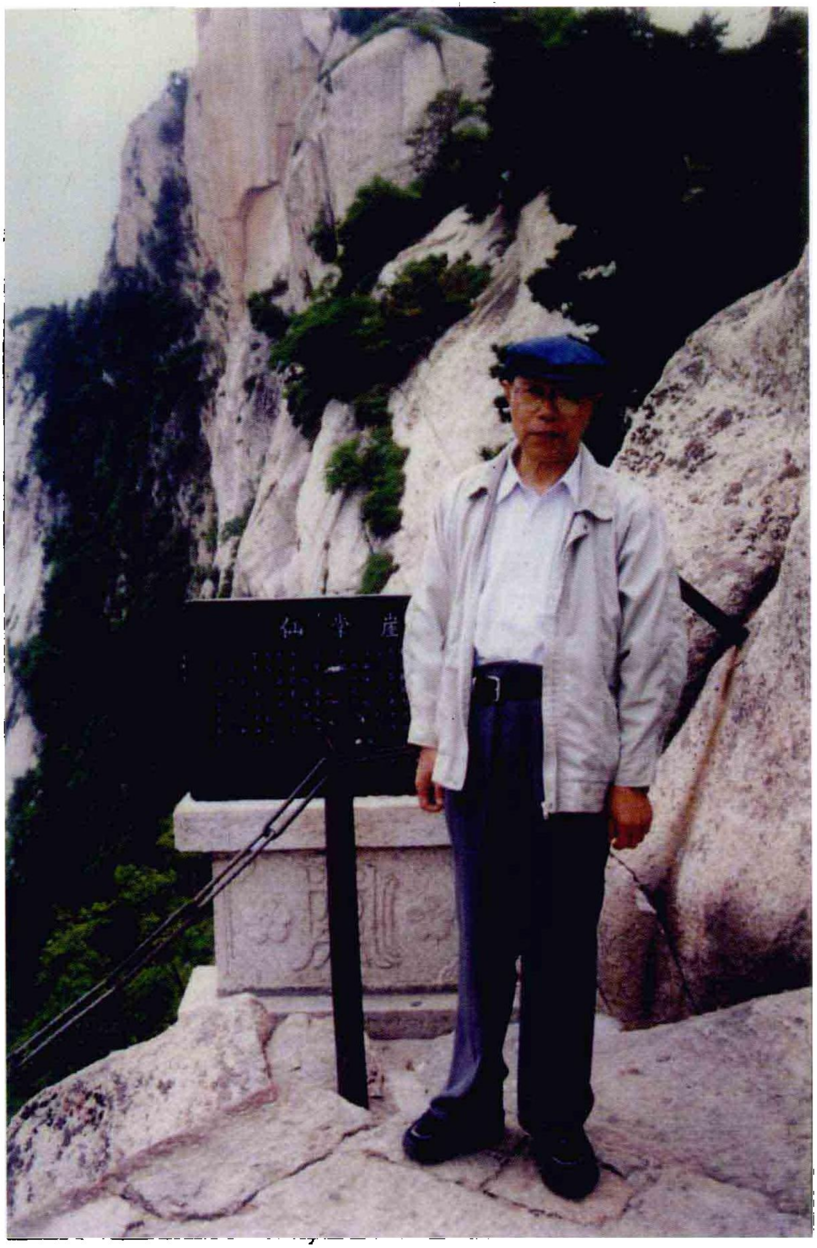
ADVANCED LECTURES IN MATHEMATICS

Dedicated to Professor Guangchang Dong
on the occasion of his 80th birthday



Professor Guangchang Dong

Professor Dong and his wife (1998)

Professor Dong and his wife (2008)

Professor Dong and his students and colleagues

International Conference on Elliptic and Parabolic Equations and Applications, August 2008, Hangzhou, Zhejiang University

Preface

This volume of 15 contributed papers is a tribute to Professor Guangchang Dong on the occasion of his 80th birthday. In the course of his almost 60 year career in mathematical research, his influence on the development of partial differential equations in China has been immersed, at both the teaching and the research level. To celebrate his 80th birthday, an international conference "Elliptic and Parabolic Equations and Applications" was held at the Center for Mathematical Sciences, Zhejiang University, Hangzhou, in August 2008. Some papers in this volume are contributed by speakers of the conference, and others by his friends and former students.

Professor Dong was born in January 1927 and received his Bachelor degree in 1950 at the Department of Mathematics, Zhejiang University, where he has been a faculty member since then. In 1952 he published his first two papers in analytic number theory in Acta Mathematica Sinica, a top mathematical journal in China. He kept working in number theory and by 1956 he had published a series of papers, see the list of his publications at the end of this book.

We have to mention an event which interrupted his research by a great deal. The Mathematics Department of Zhejiang University was a top one in China in early 1950s. But in 1952 the Chinese government decided to move all its faculty members, and also the library books, to Fudan University, Shanghai, except a small group of people remaining for engineering mathematics teaching. Prof. Dong was one of them. Considering the inconvenience in communication those days, one can imagine how difficult it was for a person at his twenties to work alone in mathematics. But his enthusiasm and talent in mathematics got him rewarded; his work in number theory was recognized and highly praised by Hua Loo-Keng, the leader of Chinese mathematics, which was undoubtedly a great impetus for him to continue his research.

However his research direction changed in the middle of 1950s. In 1954~1955 there was a program in partial differential equations (PDEs) at Beijing and Prof. Dong was sent to study PDEs. He was doing so well in the program that soon after he was able to write papers in the new area, and two were published again in Acta Mathematica Sinica in 1956. He then worked in the area for nearly 10 years, on different topics but mainly in degenerate elliptic and hyperbolic equations, and equations of mixed type.

Unfortunately his research was suspended in early 1960s. From early 1960s and during the Cultural Revolution, for more than 15 years, he was assigned to study practical mathematical problems arising in industry, and for many days he had to live in factories. One of the projects he was working on was to establish and implement smooth approximations of the shape of ships by spline functions; another one was computer aided design. He had no publications in this period but some of his work is described in book [1].

The Cultural Revolution ended in late 1976, and he was able to return to mathematics at his fifties. It was since then he was able to focus on his research. In the early 1980s, he got the precious opportunity to visit USA for two years, mainly at the Courant Institute. Since then his interest changed from degenerate elliptic and hyperbolic equations to elliptic and parabolic equations, and he has done work in variational problems, subsonic flows, degenerate parabolic equations, viscosity solutions of fully nonlinear elliptic equations. Since the 1990's he also found interests in applications of PDEs in image processing.

A list of his publications is included at the end of the book, among which there are 4 textbooks or research monographs, and more than 60 papers, most of them were published after the 1980s. Looking back to the political era of the 1950s and 1960s, it is hard to imagine the difficulty for a person to concentrate on research, in particular for a person like Dong who had only a Bachelor degree and worked in an isolated environment. All his achievements should be contributed to his talent and his passion in mathematics and his unremitting effort. After nearly 10 years of retirement, he can still be seen everyday in his office in the Institute of Mathematics, Zhejiang University. His persistent effort has also brought him numerous honors and awards. In 1981 he was among the first group of academic people who received the eligibility from the Chinese government to supervise PhD students. Also mainly due to him, Zhejiang University was one of the few top institutions that obtained the eligibility to host mathematical postdoctoral fellows.

He has not only made great achievement in research. He also made Zhejiang University a center of partial differential equations in China and attracted many young people to Hangzhou to study or work with him. He was the principal proponent of the creation of the Mathematical Institute of Zhejiang University. He was a steering committee member of the Chinese Mathematical Society, and was a founding member for the Chinese SIAM (Society for Industry and Applied Mathematics). He has also served as the first Editor-in-Chief of the journal: *Applied Mathematics*, a Journal of Chinese Universities, and in the editorial board for many other journals, including the *Chinese Annals of Mathematics* and *Journal of Partial Differential Equations.* Currently he is the honorary president of the Mathematical Society of Zhejiang Province.

There are 15 papers in the volume; most of them are survey or expository papers. We hope these papers can help readers to catch up recent developments in related topics in elliptic and parabolic partial differential equations and their applications. We would like to take this opportunity to thank the Center for Mathematical Sciences for sponsorship of the international conference for Professor Dong's 80th birthday, and thank all participants of the conference. In particular we wish to thank all the authors for their efforts to make this book possible.

Baojun Bian, Shenghong Li, Xu-Jia Wang (editors)
March 31, 2009

Contents

Trends in
Partial Differential Equations
ALM 10, pp. 1–40

Viscosity Solutions of Bellman Equations Arising in Pricing Passport Options*

Baojun Bian[†] Yang Wang[‡]

Abstract

We consider the viscosity solutions of Hamilton-Jacobi-Bellman equations and integro-differential Hamilton-Jacobi-Bellman equations. These equations are from the pricing problems of European passport option in diffusion model and jump-diffusion model. We characterize the value function for European passport option as the unique viscosity solutions of associated Hamilton-Jacobi-Bellman equations. We prove the comparison principle, uniqueness and convexity preserving for the viscosity solutions, and establish a mathematical foundation for pricing analysis of passport option.

2000 Mathematics Subject Classification: 91B02, 35B05, 49L25.

Keywords and Phrases: Passport option, HJB equation, viscosity solution, uniqueness, convexity preserving.

1 Introduction

In this paper, we consider the viscosity solutions of the following Hamilton-Jacobi-Bellman(HJB) equations

$$V_t + \sup_{|q|\le 1}\left\{\frac{1}{2}\sigma^2 S^2(V_{SS} + 2qV_{SX} + q^2 V_{XX}) - aqSV_X\right\} + (r-a)SV_S$$

$$+rXV_X - rV = 0, \qquad (S,X,t) \in J_T = \mathbf{R}^+ \times \mathbf{R} \times [0,T) \tag{1.1}$$

and related integro-differential HJB equations

$$V_t + \sup_{|q|\le 1}\left\{\frac{1}{2}\sigma^2 S^2(V_{SS} + 2qV_{SX} + q^2 V_{XX}) - (a+\lambda\kappa)qSV_X\right.$$

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$$+\lambda\int_{-1}^{\infty}V(S(1+\eta),X+qS\eta,t)p(\eta)d\eta-\lambda V\Big\}+(r-a-\lambda\kappa)SV_S$$
$$+rXV_X-rV=0,\qquad (S,X,t)\in J_T \tag{1.2}$$

Equations (1.1) and (1.2) are second order fully nonlinear equations of parabolic type. Equations (1.1) are derived from the pricing problem of European Passport options where underlying assets prices follow the geometric Brownian motion. When we consider a financial market where underlying assets prices are modeled by a Lévy process $S(\tau)$ (see Merton [33], see also [16], [43] and [6]), the pricing equations become equations (1.2).

The HJB equations we will study in this paper are fully nonlinear degenerate parabolic equations. In general, the value function is not smooth enough to satisfy the HJB equation in the classical sense. Therefore it is natural to ask for a weak solution such that the value function is unique even though it is not smooth. One such weak solution, called a viscosity solution, was introduced by Crandall, Ishii and Lions ([17], [18]). We refer to [43], [6] for the viscosity solution of integro-differential equations. In this paper, we will discuss the viscosity solutions of Equations (1.1) and (1.2) for Cauchy problem and related state constraint boundary value problem, prove the comparison principle, uniqueness and convexity preserving, and therefore give a mathematical analysis for the valuation of European passport option by a viscosity approach. These results(especially comparison principle) also give a mathematical foundation for further study in numerical analysis. We refer Soner([44], [45]) for more applications of viscosity solutions in mathematical finance.

The passport option, introduced and marketed by Banker Trust, is a call option on the balance of a trading account. In this option, the option holder is allowed to trade a particular asset S_t during the life of the option $[0,T]$. The number of assets he holds at time t is denoted q_t and there is a constraint on q_t: $|q_t|\le C$. So q_t can be any value in $[-C,C]$ and it is the holder who decides its value, he can change it whenever he wants and as many times as he wants. The holder doesn't actually buy or sell anything, he just tells his strategy q_t to the issuer, and the issuer has to deal with it. At expiry T, the issuer has to pay $X_T^+\max(X_T,0)$ to the option holder, where X_t represents the value of the trading account. This implies that the holder can not lose more than the premium paid. This option allows a customer to pay an upfront premium in exchange for immunity from any losses he may incur in trading, while retaining any gains. It has the effect of a guaranteed stop-loss, limiting the option holder's downside to the premium he spent at $t=0$. In this sense, passport option is an insurance against trading loss.

The passport option typifies a new class of derivatives which allows the holder of the option to switch during the life of the option among various positions in an underlying asset. This feature enables investors to avoid extreme financial loss. It allows one to apply derivative-based tools for principle protection, which previously could be used only by passive index-tracking portfolios, to actively managed funds. A straightforward application of passport option is on active portfolio management. Investor should buy option to make sure his "mistakes on trading strategy" are insured, and issuer of the option will gain when option holder's strategy differs from the optimal trading strategy.

There are two main approaches to study the pricing problem of the passport option: the partial differential equation (PDE in short), and the probabilistic approaches. There exist many literature on PDE approach. Hyer, Lipton-Lifschitz and Pugachevsky [23] derive formally the pricing equation for the European passport option. They gain that the price of the European passport option satisfies a Hamilton-Jacobi-Bellman (HJB in short) equation. They obtain an analytic solution for symmetric cases, and use numerical methods for asymmetric cases. Andersen, Andreasen and Brotherton-Ratcliffe [2] use the change of numéraire to give the one-factor PDE for the European type, and extend it to the American type. They also provide numerical algorithms for solving the PDEs. Ahn, Penaud and Wilmott [5], and Penaud, Wilmott and Ahn [39] further extend the plain vanilla European passport options to incorporate various exotic features. Nagayama [36] uses the change of numéraire and the Skorohod theorem to give a closed form solution for symmetric cases of the European type, and use numerical methods for asymmetric cases. Henderson and Hobson [24] use a local time construction to obtain a direct relationship between the prices of the passport and the lookback options for the symmetric case. Shreve and Vecer [47] derive a variant of the European passport option and obtain the analytic solutions using probability arguments for symmetric cases. Chan [14] studies the valuation problem of the American passport option by using the notion of viscosity solutions and the linear complementarity method. Henderson and Hobson [25] analyze passport option pricing under stochastic volatility models. The theory of passport options has also been applied to different areas. For instance, commodity hedging strategies based on passport options are described in [26], while a connection to Asian options is made in [49]. Passport option theory was also used to examine trader bonus issues in [4]. In this paper, we study this pricing problem by PDE approach.

This paper is concerned with the viscosity solutions of Cauchy problem for the pricing equations (1.1) and (1.2). Our goal is to derive and study the pricing equations by the viscosity solutions approach. Assume that the stock price S_t follows the diffusion or jump-diffusion process. Let X_t be the value of the trading account for option holder. Then the value function is given by

$$V(S,X,t) = \sup_{|q|\le C} \left\{ e^{-r(T-t)} E\Big((X_T)^+ | S_t = S, X_t = X \Big) \right\} \tag{1.3}$$

By the Dynamic Programming Principle, we can associate Hamilton-Jacobi-Bellman equations to the value function of our pricing problem. Our main result is the following theorem, which follows immediately from the results stated and proven in section 2 and section 3.

Theorem 1.1 *The value function $V(S,X,t)$ is the unique viscosity solution of (1.1)(or (1.2)) with terminal condition $V(S,X,T) = X^+$. Furthermore, $V(S,X,t)$ is convex in X for fixed S and t.*

The rest of the present paper is organized as follows. In section 2, we derive the pricing equation for the European passport option by using the dynamic