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Albert C.J. Luo
Valentin Afraimovich
Editors

Long-range Interactions, Stochasticity and Fractional Dynamics

Dedicated to George M. Zaslavsky (1935–2008)

长距离相互作用、随机及分数维动力学

高等教育出版社
HIGHER EDUCATION PRESS

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Preface

George M. Zaslavsky was born in Odessa, Ukraine in 1935 in a family of an artillery officer. He received education at the University of Odessa and moved in 1957 to Novosibirsk, Russia. In 1965, George joined the Institute of Nuclear Physics where he became interested in nonlinear problems of accelerator and plasma physics. Roald Sagdeev and Boris Chirikov were those persons who formed his interest in the theory of dynamical chaos. In 1968 George introduced a separatrix map that became one of the major tools in theoretical study of Hamiltonian chaos. The work "Stochastical instability of nonlinear oscillations" by G. Zaslavsky and B. Chirikov, published in Physics Uspekhi in 1971, was the first review paper "opened the eyes" of many physicists to power of the theory of dynamical systems and modern ergodic theory. It was realized that very complicated behavior is possible in dynamical systems with only a few degrees of freedom. This complexity cannot be adequately described in terms of individual trajectories and requires statistical methods. Typical Hamiltonian systems are not integrable but chaotic, and this chaos is not homogeneous. At the same values of the control parameters, there coexist regions in the phase space with regular and chaotic motion. The results obtained in the 1960s were summarized in the book "Statistical Irreversibility in Nonlinear Systems" (Nauka, Moscow, 1970).

The end of the 1960s was a hard time for George. He was forced to leave the Institute of Nuclear Physics in Novosibirsk for signing letters in defense of some Soviet dissidents. George got a position at the Institute of Physics in Krasnoyarsk, not far away from Novosibirsk. There he founded a laboratory of the theory of nonlinear processes which exists up to now. In Krasnoyarsk George became interested in the theory of quantum chaos. The first rigorous theory of quantum resonance was developed in 1977 in collaboration with his co-workers. They introduced the important notion of quantum break time (the Ehrenfest time) after which quantum evolution begins to deviate from a semiclassical one. The results obtained in Krasnoyarsk were summarized in the book "Chaos in Dynamical Systems" (Nauka, Moscow and Harwood, Amsterdam, 1985).

In 1984, R. Sagdeev invited George to the Institute of Space Research in Moscow. There he has worked on the theory of degenerate and almost degenerate Hamiltonian systems, anomalous chaotic transport, plasma physics, and theory of chaos in waveguides. The book "Nonlinear Physics: from the Pendulum to Turbulence and Chaos" (Nauka, Moscow and Harwood, New York, 1988), written with R. Sagdeev, is now a classical textbook for everybody who studies chaos theory. When studying interaction of a charged particle with a wave packet, George with colleagues from the Institute discovered that stochastic layers of different separatrices in degenerated Hamiltonian systems may merge producing a stochastic web. Unlike the famous Arnold diffusion in non-degenerated Hamiltonian systems, that appears only if the number of degrees of freedom exceeds 2, diffusion in the Zaslavsky webs is possible at one and half degrees of freedom. This diffusion is rather universal phenomenon and its speed is much greater than that of Arnold diffusion. Beautiful symmetries of the Zaslavsky webs and their properties in different branches of physics have been described in the book "Weak chaos and Quasi-Regular Structures" (Nauka, Moscow, 1991 and Cambridge University Press, Cambridge, 1991) coauthored with R. Sagdeev, D. Usikov, and A. Chernikov.

In 1991, George emigrated to the USA and became a Professor of Physics and Mathematics at Physical Department of the New York University and at the Courant Institute of Mathematical Sciences. The last 17 years of his life he devoted to principal problems of Hamiltonian chaos connected with anomalous kinetics and fractional dynamics, foundations of statistical mechanics, chaotic advection, quantum chaos, and long-range propagation of acoustic waves in the ocean. In his New York period George published two important books on the Hamiltonian chaos: "Physics of Chaos in Hamiltonian Systems" (Imperial College Press, London, 1998) and "Hamiltonian chaos and Fractional Dynamics" (Oxford University Press, NY, 2005). His last book "Ray and wave chaos in ocean acoustics: chaos in waveguides" (World Scientific Press, Singapore, 2010), written with D. Makarov, S. Prants, and A. Virovlynsky, reviews original results on chaos with acoustic waves in the underwater sound channel.

George was a very creative scientist and a very good teacher whose former students and collaborators are working now in America, Europe and Asia. He authored and coauthored 9 books and more than 300 papers in journals. Many of his works are widely cited. George worked hard all his life. He loved music, theater, literature and was an expert in good vines and food. Only a few people knew that he loved to paint. In the last years he has spent every summer in Provence, France, working, writing books and papers and painting in water colors. The album with his water colors was issued in 2009 in Moscow.

George Zaslavsky was one of the key persons in the theory of dynamical chaos and made many important contributions to a variety of other subjects. His books and papers influenced very much in advancing modern nonlinear science.

Sergey Prants
Albert C.J. Luo
Valentin Afraimovich

March, 2010

Contents

Chapter 1
Fractional Zaslavsky and Hénon Discrete Maps

Vasily E. Tarasov

Abstract This paper is devoted to the memory of Professor George M. Zaslavsky passed away on November 25, 2008. In the field of discrete maps, George M. Zaslavsky introduced a dissipative standard map which is called now the Zaslavsky map. G. Zaslavsky initialized many fundamental concepts and ideas in the fractional dynamics and kinetics. In this chapter, starting from kicked damped equations with derivatives of non-integer orders we derive a fractional generalization of discrete maps. These fractional maps are generalizations of the Zaslavsky map and the Hénon map. The main property of the fractional differential equations and the correspondent fractional maps is a long-term memory and dissipation. The memory is realized by the fact that their present state evolution depends on all past states with special forms of weights.

1.1 Introduction

There are a number of distinct areas of mechanics and physics where the basic problems can be reduced to the study of simple discrete maps. Discrete maps have been used for the study of dynamical problems, possibly as a substitute of differential equations (Sagdeev et al., 1988; Zaslavsky, 2005; Chirikov, 1979; Schuster, 1988; Collet and Eckman, 1980). They lead to a much simpler formalism, which is particularly useful in computer simulations. In this chapter, we consider discrete maps that can be used to study the evolution described by fractional differential equations (Samko et al., 1993; Podlubny, 1999; Kilbas et al., 2006).

The treatment of nonlinear dynamics in terms of discrete maps is a very important step in understanding the qualitative behavior of continuous systems described by differential equations. The derivatives of non-integer orders (Samko et al., 1993) are

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a natural generalization of the ordinary differentiation of integer order. Note that the continuous limit of discrete systems with power-law long-range interactions gives differential equations with derivatives of non-integer orders with respect to coordinates (Tarasov and Zaslavsky, 2006; Tarasov, 2006). Fractional differentiation with respect to time is characterized by long-term memory effects that correspond to intrinsic dissipative processes in the physical systems. The memory effects to discrete maps mean that their present state evolution depends on all past states. The discrete maps with memory are considered in the papers (Fulinski and Kleczkowski, 1987; Fick et al., 1991; Giona, 1991; Hartwich and Fick, 1993; Gallas, 1993; Stanislavsky, 2006; Tarasov and Zaslavsky, 2008; Tarasov, 2009; Edelman and Tarasov, 2009). The interesting question is a connection of fractional equations of motion and the discrete maps with memory. This derivation is realized for universal and standard maps in (Tarasov and Zaslavsky, 2008; Tarasov, 2009).

It is important to derive discrete maps with memory from equations of motion with fractional derivatives. It was shown (Zaslavsky et al., 2006) that perturbed by a periodic force, the nonlinear system with fractional derivative exhibits a new type of chaotic motion called the fractional chaotic attractor. The fractional discrete maps (Tarasov and Zaslavsky, 2008; Tarasov, 2009) can be used to study a new type of attractors that are called pseudochaotic (Zaslavsky et al., 2006).

In this chapter, fractional equations of motion for kicked systems with dissipation are considered. Correspondent discrete maps are derived. The fractional generalizations of the Zaslavsky map and the Hénon map are suggested.

In Sect. 1.2, we give a brief review of fractional derivatives to fix notation and provide a convenient reference. In Sect. 1.3, the fractional generalizations of the Zaslavsky map are suggested. A brief review of well-known discrete maps is considered to fix notations and provide convenient references. In Sect. 1.4, the fractional generalizations of the Hénon map are considered. The differential equations with derivatives of non-integer orders with respect to time are used to derive generalizations of the discrete maps. In Sect. 1.5, a fractional generalization of differential equation in which we use a fractional derivative of the order $0 \leq \beta < 1$ in the kicked term, i.e. the term of a periodic sequence of delta-function type pulses (kicks). The other generalization is suggested in (Tarasov and Zaslavsky, 2008). The discrete map that corresponds to the suggested fractional equation of order $0 \leq \beta < 1$ is derived. This map can be considered as a generalization of universal map for the case $0 < \beta < 1$. In Sect. 1.6, a fractional generalization of differential equation for a kicked damped rotator is suggested. In this generalization, we use a fractional derivative in the kicked damped term, i.e. the term of a periodic sequence of delta-function type pulses (kicks). The other generalization is also suggested in (Tarasov and Zaslavsky, 2008). The discrete map that corresponds to the suggested fractional differential equation is derived. Finally, a short conclusion is given in Sect. 1.7.

1.2 Fractional derivatives

In this section a brief introduction to fractional derivatives are suggested. Fractional calculus is a theory of integrals and derivatives of any arbitrary order. It has a long history from 1695, when the derivative of order $\alpha = 1/2$ has been described by Gottfried Leibniz. The fractional differentiation and fractional integration goes back to many mathematicians such as Leibniz, Liouville, Grunwald, Letnikov, Riemann, Abel, Riesz, Weyl. The integrals and derivatives of non-integer order, and the fractional integro-differential equations have found many applications in recent studies in theoretical physics, mechanics and applied mathematics. There exists the remarkably comprehensive encyclopedic-type monograph by Samko, Kilbus and Marichev, which was published in Russian in 1987 and in English in 1993. The works devoted substantially to fractional differential equations are the book by Miller and Ross (1993), and the book by Podlubny (1999). In 2006 Kilbas, Srivastava and Trujillo published a very important and remarkable book, where one can find a modern encyclopedic, detailed and rigorous theory of fractional differential equations. The first book devoted exclusively to the fractional dynamics and application of fractional calculus to chaos is the book by Zaslavsky published in 2005.

Let us give a brief review of fractional derivatives to fix notation and provide a convenient reference.

1.2.1 Fractional Riemann-Liouville derivatives

Let $[a,b]$ be a finite interval on the real axis $\mathbb{R}$. The fractional Riemann-Liouville derivatives D^{α}_{a+} and D^{α}_{b-} of order $\alpha > 0$ are defined (Kilbas et al., 2006) by

$$\begin{aligned}(D^{\alpha}_{a+}f)(x) &= D^n_x(I^{n-\alpha}_{a+})(x)\\ &= \frac{1}{\Gamma(n-\alpha)} D^n_x \int_a^x \frac{f(z)dz}{(x-z)^{\alpha-n+1}} \quad (x>a),\end{aligned}$$

$$\begin{aligned}(D^{\alpha}_{b-}f)(x) &= (-1)^n D^n_x(I^{n-\alpha}_{b-})(x)\\ &= \frac{(-1)^n}{\Gamma(n-\alpha)} D^n_x \int_x^b \frac{f(z)dz}{(z-x)^{\alpha-n+1}} \quad (x<b),\end{aligned}$$

where $n = [\alpha]+1$ and $[\alpha]$ means the integral part of α. Here D^n_x is the usual derivative of order n. In particular, when $\alpha = n \in N$, then

$$(D^0_{a+}f)(x) = (D^0_{b-}f)(x) = f(x),$$

$$(D^n_{a+}f)(x) = D^n_x f(x), \quad (D^n_{b-}f)(x) = (-1)^n D^n_x f(x).$$

The fractional Riemann-Lioville differentiation of the power functions $(x-a)^{\beta}$ and $(b-x)^{\beta}$ yields power functions of the same form:

$$D^{\alpha}_{a+}(x-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(x-a)^{\beta-\alpha},$$
$$D^{\alpha}_{b-}(b-x)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(b-x)^{\beta-\alpha},$$

where $\beta > -1$ and $\alpha > 0$. In particular, if $\beta = 0$ and $\alpha > 0$, then the fractional Riemann-Liouville derivatives of a constant C are not equal to zero:

$$D^{\alpha}_{a+}C = \frac{1}{\Gamma(\alpha+1)}(x-a)^{-\alpha},$$
$$D^{\alpha}_{b-}C = \frac{1}{\Gamma(\alpha+1)}(b-x)^{-\alpha}.$$

On the other hand, for $k = 1, 2, ..., [\alpha]+1$, we have

$$D^{\alpha}_{a+}(x-a)^{\alpha-k} = 0, \quad D^{\alpha}_{b-}(b-x)^{\alpha-k} = 0.$$

The equality

$$(D^{\alpha}_{a+}f)(x) = 0$$

is valid if and only if,

$$f(x) = \sum_{k=1}^{n} C_k (x-a)^{\alpha-k},$$

where $n = [\alpha]+1$ and C_k are real arbitrary constants. The equation

$$(D^{\alpha}_{b-}f)(x) = 0$$

is satisfied if and only if,

$$f(x) = \sum_{k=1}^{n} C_k (b-x)^{\alpha-k},$$

where $n = [\alpha]+1$ and C_k are real arbitrary constants.

1.2.2 Fractional Caputo derivatives

The fractional Caputo derivatives ${}^{C}D^{\alpha}_{a+}$ and ${}^{C}D^{\alpha}_{b-}$ are defined for functions for which the Riemann-Liouville derivatives exists. Let $\alpha > 0$ and let n be given by $n = [\alpha]+1$ for $\alpha \notin \mathbb{N}$, and $n = \alpha$ for $\alpha \in \mathbb{N}$. If $\alpha \notin \mathbb{N}$, then the fractional Caputo derivatives is defined by the equations

$$({}^{C}D^{\alpha}_{a+}f)(x) = (I^{n-\alpha}_{a+}D^{n}f)(x) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{x} dz \frac{D^{n}_{z}f(z)}{(x-z)^{\alpha-n+1}},$$

$$({}^C D^{\alpha}_{b-} f)(x) = (-1)^n (I^{n-\alpha}_{b-} D^n f)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b dz \frac{D^n_z f(z)}{(z-x)^{\alpha-n+1}},$$

where $n = [\alpha] + 1$. If $\alpha = n \in \mathbb{N}$, then

$$({}^C D^{\alpha}_{a+} f)(x) = D^n_x f(x), \quad ({}^C D^{\alpha}_{b-} f)(x) = (-1)^n D^n_x f(x).$$

If $\alpha \notin \mathbb{N}$ and $n = [\alpha] + 1$, then fractional Caputo derivatives coincide with the fractional Riemann-Liouville derivatives in the following cases:

$$({}^C D^{\alpha}_{a+} f)(x) = (D^{\alpha}_{a+} f)(x),$$

if

$$f(a) = (D^1_x f)(a) = \cdots = (D^{n-1} f)(a) = 0,$$

and

$$({}^C D^{\alpha}_{b-} f)(x) = (D^{\alpha}_{b-} f)(x),$$

if

$$f(b) = (D^1_x f)(b) = \cdots = (D^{n-1} f)(b) = 0.$$

It can be directly verified that the fractional Caputo differentiation of the power functions $(x-a)^{\beta}$ and $(b-x)^{\beta}$ yields power functions of the form

$$\begin{aligned} {}^C D^{\alpha}_{a+} (x-a)^{\beta} &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} (x-a)^{\beta-\alpha}, \\ {}^C D^{\alpha}_{b-} (b-x)^{\beta} &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} (b-x)^{\beta-\alpha}, \end{aligned}$$

where $\beta > -1$ and $\alpha > 0$. In particular, if $\beta = 0$ and $\alpha > 0$, then the fractional Caputo derivatives of a constant C are equal to zero:

$${}^C D^{\alpha}_{a+} C = 0, \quad {}^C D^{\alpha}_{b-} C = 0.$$

For $k = 0, 1, 2, \ldots, n-1$,

$${}^C D^{\alpha}_{a+} (x-a)^k = 0, \quad D^{\alpha}_{b-} (b-x)^k = 0.$$

The Mittag-Leffler function $E_{\alpha}[\lambda (x-a)^{\alpha}]$ is invariant with respect to the Caputo derivatives ${}^C D^{\alpha}_{a+}$,

$${}^C D^{\alpha}_{a+} E_{\alpha}[\lambda (x-a)^{\alpha}] = \lambda E_{\alpha}[\lambda (x-a)^{\alpha}],$$

but it is not the case for the Caputo derivative ${}^C D^{\alpha}_{b-}$.

1.2.3 Fractional Liouville derivatives

Let us define the fractional Liouville derivatives on the whole real axis $\mathbb{R}$. The fractional Liouville derivatives D^{α}_{+} and D^{α}_{-} of order $\alpha > 0$ are defined (Kilbas et al., 2006) by

$$(D^{\alpha}_{+}f)(x) = D^{n}_{x}(I^{n-\alpha}_{+})(x) = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_{-\infty}^{x}\frac{f(z)\mathrm{d}z}{(x-z)^{\alpha-n+1}},$$

$$(D^{\alpha}_{-}f)(x) = (-1)^n D^{n}_{x}(I^{n-\alpha}_{-})(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_{x}^{+\infty}\frac{f(z)\mathrm{d}z}{(z-x)^{\alpha-n+1}},$$

where $n = [\alpha]+1$ and $[\alpha]$ means the integral part of α. Here D^{n}_{x} is the usual derivative of order n. In particular, when $\alpha = n \in N$, then

$$(D^{0}_{+}f)(x) = (D^{0}_{-}f)(x) = f(x),$$

$$(D^{n}_{+}f)(x) = D^{n}_{x}f(x), \quad (D^{n}_{-}f)(x) = (-1)^n D^{n}_{x}f(x).$$

If $f(x)$ is an integrable function and $\beta > \alpha > 1$, then

$$(D^{\alpha}_{\pm}I^{\alpha}_{\pm}f)(x) = f(x),$$
$$(D^{\alpha}_{\pm}I^{\beta}_{\pm}f)(x) = (I^{\beta-\alpha}_{\pm}f)(x),$$
$$(D^{k}D^{\alpha}_{\pm}f)(x) = (\pm 1)^k(D^{\alpha+k}_{\pm}f)(x).$$

If $\alpha > 0$, then the following relations hold:

$$(\mathscr{F}D^{\alpha}_{\pm}f)(k) = (\mp ik)^{\alpha}(\mathscr{F}f)(k),$$

where

$$(\mp ik)^{\alpha} = |k|^{\alpha}\exp\{\pm\alpha\pi i \operatorname{sgn}(x)/2\}.$$

Here $\mathscr{F}$ is the Fourier transform.

1.2.4 Interpretation of equations with fractional derivatives

To describe the physical interpretation of equations with fractional derivatives and integrals with respect to time, we consider the memory effects and limiting cases widely used in physics: (1) the absence of the memory; (2) the complete memory; (3) the power-like memory.

Let us consider the evolution of a dynamical system in which some quantity $F(t)$ is related to another quantity $f(t)$ through a memory function $M(t)$:

$$F(t) = \int_0^t M(t-\tau) f(\tau) \mathrm{d}\tau. \tag{1.1}$$

Equation (1.1) means that the value $F(t)$ is related with $f(t)$ by the convolution operation

$$F(t) = M(t) * f(t).$$

Equation (1.1) is a typical non-Markovian equation obtained in studying the systems coupled to an environment, with environmental degrees of freedom being averaged. Let us consider special cases of Eq. (1.1).

(1) For a system without memory, we have the Markov processes, and the time dependence of the memory function is

$$M(t-\tau) = \delta(t-\tau), \tag{1.2}$$

where $\delta(t-\tau)$ is the Dirac delta-function. The absence of the memory means that the function $F(t)$ is defined by $f(t)$ at the only instant t. For this limiting case, the system loses all its states except for one with infinitely high density. Using (1.1) and (1.2), we have

$$F(t) = \int_0^t \delta(t-\tau) f(\tau) \mathrm{d}\tau = f(t). \tag{1.3}$$

The expression (1.3) corresponds to the process with complete absence of memory. This process relates all subsequent states to previous states through the single current state at each time t.

(2) If memory effects are introduced into the system the delta-function turns into some function, with the time interval during which $f(t)$ affects on the function $F(t)$. Let $M(t)$ be the step function

$$M(t-\tau) = t^{-1}, \quad (0 < \tau < t);$$
$$M(t-\tau) = 0, \quad (\tau > t).$$

The factor t^{-1} is chosen to get normalization of the memory function to unity:

$$\int_0^t M(\tau) \mathrm{d}\tau = 1.$$

Then in the evolution process the system passes through all states continuously without any loss. In this case,

$$F(t) = \frac{1}{t} \int_0^t f(\tau) \mathrm{d}\tau,$$

and this corresponds to complete memory.

(3) The power-like memory function