冯康文集

Collected Works of Feng Kang

国防工业出版社

（京）新登字 106 号

图书在版编目（CIP）数据

冯康文集/冯康著.-北京：国防工业出版社，1994
ISBN 7-118-01301-3

Ⅰ.冯… Ⅱ.冯… Ⅲ.计算数学-文集 Ⅳ.024-53

中国版本图书馆 CIP 数据核字（94）第 04336 号

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国防工业出版社出版发行
（北京市海淀区紫竹院南路 23 号）
（邮政编码 100044）
新华书店经售
国防工业出版社印刷厂印装

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开本 787×1092 1/16 印张 24 550 千字
1994 年 8 月第 1 版 1994 年 8 月北京第 1 次印刷 印数 1—400册

ISBN 7-118-01301-3/TP・171

冯　康（1920—1993）

冯 康 教 授 生 平

冯康，中国科学院学部委员，中国科学院计算中心名誉主任，数学和物理学家，计算数学家，中国计算数学的奠基人和开拓者。1920 年 9 月 9 日出生于江苏省南京市；因患脑蛛网膜腔下出血，经多方抢救无效，于 1993 年 8 月 17 日 13 时 45 分逝世，享年 73 岁。

冯康于 1939 年春考入福建协和学院数理系；同年秋又考入重庆中央大学电机工程系，两年后转物理系学习直到 1944 年毕业。1945 年到 1951 年，他先后在复旦大学物理系、清华大学物理系和数学系任助教；1951 年调到刚组建的中国科学院数学研究所任助理研究员；1951 年到 1953 年在苏联斯捷克洛夫数学研究所工作。从 1945 年到 1953 年他曾先后在当代著名数学大师陈省身、华罗庚和庞特利亚金等人指导下工作。1957 年根据国家十二年科学发展计划，他受命调到中国科学院计算技术研究所，参加了我国计算技术和计算数学的创建工作，成为我国计算数学和科学工程计算学科的奠基者和学术带头人。1978 年调到中国科学院计算中心任中心主任，1987 年改任计算中心名誉主任直到逝世。冯康教授事业心极强，刻苦工作，成就卓著，受到了党和人民的尊敬，以及国内外学者的赞誉。1959 年被评为全国先进工作者，1965 年被选为第三届全国人大代表，1979 年被评为全国劳动模范，1980 年当选为中国科学院学部委员。曾任全国计算机学会副主任委员；全国计算数学会理事长、名誉理事长，国际计算力学会创始理事，英国伦敦凯莱计算与信息力学研究所科技顾问、国际力学与数学交互协会名誉成员、英国爱丁堡国际数学研究中心科学顾问等多个学会、协会职务。他是全国四种计算数学杂志的主编，先后担任美国“计算物理”，日本“应用数学”，荷兰“应用力学与工程的计算方法”，美国“科学与工程计算”，“中国科学”等杂志的编委，并任中国大百科全书数学卷副主编。

冯康的科学成就是多方面的和非常杰出的，1957 年前他主要从事基础数学研究，在拓扑群和广义函数理论方面取得了卓越的成就。1957 年以后他转向应用数学和计算数学研究，由于其具有广博而扎实的数学、物理基础，使得他在计算数学这门新兴学科上做出了一系列开创性和历史性的贡献。

50 年代末与 60 年代初，冯康在解决大型水坝计算问题的集体研究实践的基础上，独立于西方创造了一套求解偏微分方程问题的系统化、现代化的计算方法，当时命名为基于变分原理的差分方法，即现时国际通称的有限元方法。有限元方法的创立是计算数学的一项划时代成就，它已得到国际上的公认。

70 年代，冯康建立了间断有限元函数空间的嵌入理论，并将椭圆方程的经典理论推广到具有不同维数的组合流形，为弹性组合结构提供了严密的数学基础，在国际上为首创。

与此同时，冯康对传统的椭圆方程归化为边界积分方程的理论作了重要的发展，提出自然边界元方法，这是当今国际上边界元方法的三大流派之一。1978 年以来，冯康先后应邀赴法国、意大利、日本、美国等十多所著名的科研机构及大学主讲有限元和自然边界元方法，受到高度评价。

1984年起冯康将其研究重点从以椭圆方程为主的稳态问题转向以哈密顿方程和波动方程为主的动态问题。他于1984年首次提出基于辛几何计算哈密顿体系的方法，即哈密顿体系的保结构算法，从而开创了哈密顿体系计算方法这一富有活力及发展前景的新领域；冯康指导和带领了中国科学院计算中心一个研究组投入了此领域的研究，取得了一系列优秀成果。新的算法解决了久悬未决的动力学长期预测计算方法问题，正在促成天体轨道、高能加速器、分子动力学等领域计算的革新，具有更为广阔的发展前景。冯康多次应邀在国内及西欧、苏联、北美等多国讲学或参加国际会议作主题报告，受到普遍欢迎及高度评价，国际和国内已兴起了许多后继研究。1995年国际工业与应用数学大会已决定邀请冯康就此主题作一小时大会报告。

由于他在科学上的突出贡献，曾先后获得1978年全国科学大会重大成果奖、全国自然科学二等奖、科技进步二等奖及科学院自然科学一等奖等。

冯康除了本人的研究工作外，还承担了众多的行政工作。他花了大量的心血做了大量学术指导工作。早在60年代，他亲自为当时中国科学院计算技术研究所三室200多人讲授现代计算方法和具体指导科学研究，他们中的许多人已成为我国计算数学的业务骨干。冯康费尽心血，大力培养年轻优秀人材，他亲自培养的研究生目前已遍布国内外，有的已成为国际知名学者。

冯康非常关心全国计算数学学科的发展及队伍的建设，多次提出重要的指导性意见，向中央领导同志提出紧急建议，呼吁社会各方重视科学与工程计算，倡议将科学与工程计算列入国家基础研究重点项目等等。冯康用极大的热情，从科学技术发展的战略高度上阐明了科学与工程计算的地位和作用，有力地促进了计算数学在我国四化建设中发挥其应有的作用。“科学与工程计算的方法和理论”已列为八五期间国家基础性研究重大关键项目，冯康任首席科学家。

冯康的一生，是为科学事业奋斗不息的一生，是为祖国繁荣昌盛无私奉献的一生。他在研究工作中，积极倡导理论联系实际，并身体力行，自觉运用辩证法，把握住事物的本质，成功地开创了科学的新方向、新道路、新领域，带领一批又一批人在新方向上做出卓越的贡献。他从不满足，具有强烈的进取心和为国争光的使命感，这使他一直走在世界计算数学队伍前列。在他年已古稀之时，仍经常废寝忘食、通宵达旦地工作。就在冯康患病住院的前一个小时，他还在为一项新的工作奔波、伏案疾书，在他从昏迷中清醒的片刻，首先询问的是九三华人青年学者“科学与工程计算”学术讨论会的准备工作，关心着下一代的成长。他心里只有科学事业。他是“将军”，总是运筹帏幄；他又是士兵，一直在冲锋陷阵。他是导师，总是开辟方向，指导我们前进；他又是益友，总和我们研究人员在一起。他是老一代知识分子的优秀代表，是我们学习的榜样。

中国科学院计算中心

1993年8月18日

纪念冯康先生

P. Lax[①]

(原载 SIAM NEWS 26 卷(93 年)11 期)

FengKang, China's leading applied mathematician, died suddenly on August17, inhis 73rd year, after a long and distinguished career that had shown on sign of slowing.

Feng's early education was in electrical engineering, physics, andmathematics, abackground that subtly shaped his later interests. He spent the early 1950s at the Steklov Institute in Moscow. Under the influence of Pontryagin, he began by working on problems of topological groups and Lie groups. On his return to China, he was among the first to popularize the theory of distributions.

In the late 1950s, Feng turned his attention to applied mathematics, where his most important contributions lie. Independently of parallel developments in the West, he created a theory of the finite element method. He was instrumental in both the implementation of the method and the creation of its theoretical foundation using estimates in Sobolev spaces. He showed how to combine boundary and domain finite elements effectively, taking advantage of integral relations satisfied by solutions of partial differential equations. In particular, he showed how radiation conditions can be satisfied in this way. He oversaw the application of the method to problems in elasticity as they occur in structural problems of engineering.

In the late 1980s, Feng proposed and developed so-called symplectic algorithms for solving evolution equations in Hamiltonian form. Combining theoretical analysis and computer experimentation, he showed that such methods, over long times, are much superior to standard methods. At the time of his death, he was at work on extensions of this idea to other structures.

Feng's significance for the scientific development of China cannot be exaggerated. He not only put China on the map of applied and computational mathematics, through his own research and that of his students, but he also saw to it that the needed resources were made available. After the collapse of the Cultural Revolution, he was ready and able to help the country build again from the ashes of this selfinflicted conflagration. Visitors to China were deeply impressed by his familiarity with new developments everywhere.

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① 美国科学院院士,柯朗研究所教授,原美国总统科学顾问,原美国数学会会长,原柯朗研究所所长.

Throughout his life, Feng was fiercely independent, utterly courageous, and unwilling to knuckle under to authority. That such a person did survive and thrive shows that even in the darkest days, the authorities were aware of how valuable and irreplaceable he was.

In Feng's maturity the well-deserved honors were bestowed upon him — membership in the Academia Sinica, the directorship of the Computing Center, the editorship of important journals, and other honors galore.

By that time his reputation had become international. Many remember his small figure at international conferences, his eyes and mobile face radiating energy and intelligence. He will be greatly missed by the mathematical sciences and by his numerous friends. — *Peter Lax, Courant Institute of Mathematical Sciences, New York University.*

前　言

冯康教授于1993年8月17日与世长辞了。他作为著名的数学和物理学家，作为中国计算数学的奠基人和开拓者，为发展我国计算数学和科学与工程计算事业，倾注了毕生心血，建立了不朽的功勋；他培养了一批计算数学人才，创建了我国现代计算数学队伍；他开辟一个个新兴的学科方向，给我们留下了极为丰富和宝贵的科学遗产，已经发表了许多论文、专著，并留下了大量珍贵的手稿。

为了使更多的学者，特别是年轻的计算数学家，能够分享到冯康教授的科学遗产，促进计算数学的发展，我们除了把冯康教授的手稿整理出来，陆续在有关杂志上发表外，决定收集出版《冯康文集》。

本书是第一集，主要收集了冯康教授关于广义函数、有限元方法、边界元方法和弹性力学等方面的论文，已经出版的专著没有收入。

在此，我们敬告读者，如果您处珍藏有冯康教授尚未发表的论文或报告手稿，请复印一份给我们，并欢迎您协助整理，尽早发表。

让我们以冯康教授的科学成就为起点，开创我国计算数学繁荣昌盛的未来。

在此，向为《冯康文集》出版做出了贡献的所有人员表示感谢，感谢他们辛勤和卓有成效的工作。

《冯康文集》整理编辑组

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目　　录

ON THE MINIMALLY ALMOST PERIODIC TOPOLOGICAL GROUPS[①]

最小几乎周期拓扑群

摘　要

在拓扑群上如果对任意二不同元素必定有一个几乎周期函数在这二元素上取不等值，这个群就叫做最大几乎周期群．如果群上所有的几乎周期函数都是常数，它就叫做最小几乎周期群．Freudenthal 及 Weil 解决了最大几乎周期群的问题，它就是一个封闭群和一个向量加群的直接乘积．本文系致力于最小几乎周期群的问题，阐明一些最小几乎周期性的特征，知道它们相当于根本上不封闭和不可换的群．主要的结果是：线性（或单连通）连通李群是最小几乎周期群的充要条件是（一）它与它的换位群相重合，（二）它的最大半单李代数不包含相当于封闭群的直接因子．由此可见，对线性李群而言，最小几乎周期性可由局部完全决定．此外还列举若干最小几乎周期群的实例，并应用最小几乎周期性证明一个关于复数李群的定理．

The theory of almost periodic (a. p.) functions in arbitrary groups was first established by von Neumann[1]. In the following we shall confine ourselves to the case of topological groups, thus the a. p. functions and the representations are required to be continuous. The a. p. functions are intimately related with the representations by unitary matrices, in fact, a representation is equivalent to a unitary one if and only if all its matrix coefficients are a. p. functions, and every a. p. function generates a unitary representation[1,2]. As to the admissibility of the a. p. functions, *i. e.*, of the unitary representations, we have, after von Neumann, the following two extreme classes of groups: 1. A topological group is called maximally almost periodic if to each pair of distinct elements there is an a. p. function which takes different values at these two elements, or equivalently, to each non-identity element there is a unitary representation which carries it into a matrix different from the unit matrix. 2. A topological group is called minimally almost periodic if every a. p. function is a constant, or equivalently, every unitary representation is trivial. The maximally a. p. case was characterized by

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① *Science Record*, Vol. 3, No. 2-4, pp161-166, 1950.

Freudenthal and Weil: *a connected locally compact group is maximally a. p. if and only if it is a direct product of a compact group and an Euclidean vector group*[2],[3]. The present note is devoted to the characterization of the minimally a. p. groups. We obtain conditions, some necessary, some sufficient, for a connected Lie group to be minimally a. p., and in particular, a necessary and sufficient condition for a connected linear Lie group to be minimally a. p. We see that the minimal case, in contradistinction to the maximal one, corresponds to the "essentially" non-compact and non-abelian groups.

Let G be a topological group, and K be the subset of G which consists of all the elements a such that $f(a)$ is the unit matrix for every unitary representation f of G. K is called the unitary kernel of G and is a closed normal subgroup of G (this was first introduced by Weil[4], cf. also [5]). With this in view, the maximal and minimal cases correspond to $K=(e)$ and $K=G$ respectively. The factor group (here and henceforth the factor groups are understood in the topologico-group-theoretic sense) G/K is obviously maximally a. p., and every closed normal subgroup H of G such that G/H is maximally a. p. contains K. Thus it follows immediately that a topological group is minimally a. p. if and only if it has no proper closed normal subgroup whose corresponding factor group is maximally a. p. It is also evident that the direct product of a finite number of minimally a. p. topological groups is minimally a. p. and every factor group of a minimally a. p. topological group is minimally a. p.

Lemma 1 *Every connected semi-simple Lie group whose Lie algebra contains no simple ideal corresponding to a compact group is minimally a. p.* Thus, in particular, *all semi-simple complex Lie groups are minimally a. p.*

Proof. Let G be a non-compact, non-abelian, simple Lie group. All possible proper closed normal subgroups of G are discrete. Thus all possible non-trivial factor groups are locally isomorphic to G; they are also non-compact, non-abelian, simple Lie groups for which the Freudenthal-Weil decompositions are impossible. Therefore G is minimally a. p. Every connected semi-simple Lie group whose Lie algebra contains no simple ideal corresponding to a compact group is a factor group of a direct product of groups of the above type modulo a discrete normal subgroup. Therefore it is also minimally a. p.

Lemma 2 *Let G be a connected Lie group which coincides with its commutator subgroup, and G_1 be the maximal semi-simple subgroup of G which corresponds to the maximal semi-simple subalgebra of a Levi decomposition of the Lie algebra of G. Then every closed connected normal subgroup of G containing G_1 coincides with G.*

Proof. Let A be the Lie algebra of G, $A=A_1\dot{+}A_2$ be a Levi decomposition of A, where A_1 is a semi-simple subalgebra of A and A_2 is the maximal solvable ideal of A, and G_1 be the subgroup of G which corresponds to the subalgebra A_1. Let G' be a closed connected normal subgroup of G containing G_1, A' be an ideal of A corresponding to G'. Since G' is itself a connected Lie group, so the factor group G/G' has a Lie algebra isomorphic with A/A'. Let ϕ be the natural homomorphism of G onto G/G', ϕ induces a homomorphism ψ of A onto A/A'. It is easily seen that the contraction of ψ on the ideal A_2 is a homomorphism of A_2 onto A/A'. Thus A/A' is solvable, and so is G/G'. Suppose $G\neq G'$, then G/G' has a nontrivial abelian abstract homomorph, and the group G has also a non-trivial abelian abstract homomorph. This leads to a contradiction.

Theorem 3 *If G is a connected Lie group satisfying the following conditions:*

(I) *G coincides with its commutator subgroup.*

(II) *the maximal semi-simple subalgebra of the Lie algebra of G contains no simple direct factor which corresponds to a compact group, then G is minimally a. p.*

Proof. We keep the notations in the proof of lemma 2. In view of (II) and lemma 1, we see that the subgroup G_1 is minimally a. p. with respect to its intrinsic topology. The contraction of any continuous mapping of G on the subset G_1 is also a continuous mapping of G_1 with respect to its intrinsic topology. Thus every unitary representation of G is trivial on the subset G_1. Therefore G_1 is contained in the unitary kernel K of G. Since G_1 is connected, so it is contained in the closed identity-component of K. Then it follows from lemma 2 that G is minimally a. p.

As further examples of minimally a. p. groups we now enumerate all the complex Lie groups which coincide with their own commutator subgroups, since then the condition (II) is automatically satisfied. Furthermore, let G be one of the following linear Lie groups: the special linear groups $SL(n,R)$, $SL(n,C)$, $n\geqslant 2$; the special complex-orthogonal groups $SO(n,C)$, $n\geqslant 3$; the symplectic groups $Sp(2n,R)$, $Sp(2n,C)$, $n\geqslant l$; R and C denote the fields of real and of complex numbers respectively. Let E_pG be the group of all the matrices of the form

$$\begin{Vmatrix} A & P \\ O & I_p \end{Vmatrix},$$

where A is an arbitrary matrix of the group G (of degree, say, m), P is an arbitrary matrix of m rows and p columns over the appropriate field, and I_p is the unit matrix of degree p. It can be verified that $E_p G$ is a connected Lie group satisfying the conditions (I) and (II), so it is minimally a. p. Also the Lie group locally isomorphic to a direct product of groups of the type $E_p G$ and the type G is minimally a. p.

Lemma 4 *The Lie algebra of a minimally a. p. connected semi-simple Lie group contains no simple ideal which corresponds to a compact group.*

Proof. We have $G=G'/N$, where G' is the universal covering group of the group G in question, and N is a discrete normal subgroup of G' which is contained in the center Z of G'. We may write

$$G'=G_1\times G_2\times\cdots\times G_p,$$
$$Z=Z_1\times Z_2\times\cdots\times Z_p,$$

where $G_i(i=1,2,\cdots,p; p\geqslant 1)$ are non-abelian, simple, connected Lie groups, and Z_i is the center of G_i. Suppose, say $G_j(j=1,2,\cdots,q; q\geqslant 1)$ are compact and the remaining G_k are not compact. Let $f_j(j=1,\cdots,q)$ be the adjoint representation of G_j; without loss of generality, they may be assumed to be unitary. Let $f_k(k=q+1,\cdots,p)$ be the trivial representation of

G_k. Then the "sum" representation f, defined by

$$f \equiv f_1 + f_2 + \cdots + f_p,$$

is a unitary representation of G' such that $f(a)$ is the unit matrix if and only if a belongs to the subset

$$Z_1 \times \cdots \times Z_q \times G_{q+1} \times \cdots \times G_p.$$

Since N is contained in the above subset, so f induces a non-trivial unitary representation of G, this leads to a contradiction.

Theorem 5 *If G is a minimally a. p. connected non-solvable Lie group, then every maximal semi-simple subalgebra of the Lie algebra of G contains no simple direct factor which corresponds to a compact group.*

Proof. Let S be the maximal solvable normal subgroup of G. According to Malcev, S is a closed subgroup of G[6]. Then the factor group G/S is minimally a. p. and has a Lie algebra isomorphic with every maximal semi-simple subalgebra of the Lie algebra of G. Then our assertion follows from lemma 4.

Theorem 6 *Let G be a minimally a. p. connected Lie group which is a covering group of some Lie group whose commutator subgroup is closed. Then G coincides with its own commutator subgroup.*

Proof. Let G be a covering group of G' whose commutator subgroup C' is closed. Since G' is also minimally a. p., it coincides with its unitary kernel which is contained in C'. Then $C' = \overline{C'} = G'$. Thus the common Lie algebra of G and G' coincides with its derived algebra. Therefore G coincides with its own commutator subgroup.

The commutator subgroup of a simply-connected Lie group is closed. The linear Lie groups also enjoy the same property, as was shown by Malcev[6]. Thus the above theorem holds for the Lie groups which are covering groups of linear Lie groups (this includes the simply-connected case, as is easily seen from the well-known theorem of Ado on the representability of Lie algebra by matrices). In view of this we may deduce the following

Corollary 7 *Every connected solvable Lie group is not minimally a. p.*

Proof. We may assume that the group G is not abelian. The center Z of G is properly contained in G, thus the adjoint group G/Z is a non-trivial connected linear Lie group. Since G/Z is also solvable, it does not coincide with its own commutator subgroup. Therefore G/Z is not minimally a. p. and so is G.

In view of theorems 3, 5, 6 and corollary 7, we obtain:

Theorem 8 *A linear connected Lie group,* or more generally, *a connected Lie group which is a covering group of some linear Lie group, is minimally a. p. if and only if its Lie algebra satisfies the following conditions:*

(I) *it coincides with its derived algebra,*

(II) *all its maximal semi-simple subalgebra contains no simple direct factor which corresponds to a compact group.*

It is highly probable that conditions (I) and (II) suffice to characterize the minimal almost periodicity of arbitray connected Lie groups. This amounts to say that minimal almost periodicity is an invariant under local isomorphism; but we are unable to prove this at present. However, in a weaker form, the minimal almost periodicity is an invariant of the equivalent classes introduced by Malcev[7] within a family of local isomorphic groups (two connected Lie groups are said to be equivalent if they are finite-multiple covering groups of a third group). Our assertion is justified by the following

Theorem 9 *The finite-multiple covering groups of a minimally a. p. connected Lie group are minimally a. p.*

Proof. Let $G'=G/N$, where N is a finite central subgroup of G. Suppose G is not minimally a. p. Let K be the unitary kernel of G, then G/K is a non-trivial maximally a. p. connected group and assumes the form $G/K=H_1\times H_2$, where H_1 is a compact group, and H_2 is an Euclidean vector group. Let ϕ be the natural homomorphism of G onto $G/K=\phi(G)$. Then $\phi(N)$ is a finite central subgroup of $\phi(G)$, and it is easily seen that $\phi(N)$ is contained in H_1. Thus we have $\phi(G)/\phi(N)=(H_1/\phi(N))\times H_2$, and $\phi(G)/\phi(N)$ is maximally a. p. and admits a non-trival unitary representation f. Let ψ be the natural homomorphism of $\phi(G)$ onto $\phi(G)/\phi(N)$. Then the unitary representation f' of G defined by $f'\equiv f\cdot\psi\cdot\phi$ is non-trivial and having a kernel containing N. Thus f' induces a non-trivial and having a kernel containing N. Thus f' induces a non-trivial unitary representation of G', so G' is not minimally a. p.

It may be of some interest to remark that the minimal almost periodicity may serve to prove a well-known theorem to the effect that every compact complex Lie group is necessarily abelian (cf. for example[8]). It is a consequence of the following

Theorem 10 *A connected complex Lie group G is solvable if and only if it contains a proper closed normal solvable subgroup S (not necessarily a complex subgroup) such that G/S is compact.*

Proof. Let G be solvable. In view of corollary 7, G is not minimally a. p. Let K be the unitary kernel of G, then G/K is of the form $G/K=H_1\times H_2$, where H_1 is a compact group, H_2 is an Euclidean vector group, and H_1 and H_2 do not reduce to the trival groups simultaneously. Evidently G/K has a non-trivial compact factor group, thus G has a proper closed normal solvable subgroup S such that G/S is compact. Conversely, let S be the subgroup in question. As G/S is compact, it admits a faithful unitary representation. Then the natural homomorphism of G onto G/K carries the unitary kernel K into the identity element of G/S, *i. e.*, the solvable subgroups contains K, which, in turn, contains all the maximal semi-simple subgroups of G. Thus the semi-simple part of G necessarily reduces to the trivial group. Therefore G is solvable.

If, furthermore, G is itself compact, then, in view of theorem 10, it is solvable. According to a theorem of Chevalley[9] and Malcev[7], G can be written in the form $A\cdot E$, where A is a compact abelian subgroup of G, and E is a subset of G, homeomorphic to an Euclidean space, and every element of G has a unique decomposition. Now, since G is compact, the Euclidean part vanishes, therefore G is abelian.

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