周毓麟论文集

Selected Papers of Zhou Yulin

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序

中国科学院学部委员，全国计算数学学会理事长周毓麟教授在数学这块科学园地上辛勤耕耘已经将近半个世纪了，所取得的成果是极为丰富的。由周毓麟教授自选的这本文集反映了作者在数学科学的若干不同领域里都有很高的造诣。

周毓麟教授走进数学研究的殿堂是从组合拓扑学的研究开始的，在同调论与同伦论方面作了很好的工作。解放后他去苏联学习、毅然地选择了被认为具有更高应用价值的偏微分方程理论作为自己攻读的专业，期望能将自己的知识更好地为新生的社会主义祖国的经济建设服务。这一时期他对非线性抛物型和椭圆型方程的各类问题做了很多有意义的工作。特别是他与苏联数学家 O. A. Олейник 等人合作的关于渗流方程的工作被认为具有开创性的经典型的研究，至今仍被国内外研究这一问题的学者不断地引用。回国后，周毓麟教授在开展研究工作的同时还从事教学工作，为我国偏微分方程理论研究培养出了一批高水平的教学和科研人员。

六十年代初，他奉调参加我国核武器的理论研究工作。周毓麟教授为了祖国的强盛，又一次在一夜之间改变了自己的研究方向，在一个对他说来完全崭新的科学领域内开始了自己新的征程。大规模科学计算是当代科学研究的三种手段之一，更是核武器理论研究的必不可少的重要手段。周毓麟教授主持了我国核武器的数值模拟及流体力学方面的研究工作，为我国核武器的研制作出了重大的贡献。他在长期从事大规模科学计算的基础上，从科学计算的角度对电子计算机提出了一系列要求并作了理论上的分析，对我国电子计算机的研制产生了深远的影响。

改革开放的十多年来，周毓麟致力于非线性偏微分方程及其差分方法的研究，获得了一系列完整的深刻的结果。他研究的很多方程都有很强的物理的和应用的背景，受到国内外数学家们的关注。特别是他发展了离散泛函分析的方法，用来研究各种非线性偏微分方程的差分方法，使差分方法的理论研究形成一个新的系统，打破了偏微分方程与差分格式的理论研究之间的界限，使之浑然一体，有其独到之处。

周毓麟教授治学十分严谨，一丝不苟。他在学术上深思熟虑，精益求精；对待科学实事求是，严肃认真。他诲人不倦，热情指导后辈。他平时工作勤奋，以致废寝忘食，在待人处世上，他是非分明，坦率真诚，耿直不阿，嫉恶如仇。我与周毓麟教授相识已有三十余年，常有机会向他讨教，时时在一起交流思想，切磋学问。多年来他的言行使我受益匪浅。

值此，《周毓麟论文集》出版之际，嘱我作序，回溯周毓麟教授四十多年的科学活动，一个数十年如一日地将自己对祖国富强和对科学发展的执著追求完满地结合起来的知识分子的形象跃然眼前。我们衷心希望为了祖国的繁荣昌盛，周毓麟教授在科学事业上有更大的成就。

李德元

1992 年 6 月 20 日

FOREWORD

Professor Zhou Yulin, a member of Chinese Academy of Sciences and chairman of Chinese Society of Computational Mathematics, has ploughed and weeded industriously in mathematics field for nearly half a century and gained plentiful achievements. This volume of collected papers , selected by professor Zhou himself , evidently shows his great academic attainments in various areas of mathematical study.

Professor Zhou Yulin walked into the palace of mathematics research beginning with the combinatorial topology, and did well in homology theory and homotopy theory. After liberation, he went to the Soviet Union to study. The theory of partial differential equations was chosen determinedly as his speciality because it is deemed to be more valuable in application. He was expecting to use the knowledge to economic construction for the new socialist China. During that period, he did a lot of significant works on the theory of nonlinear parabolic equations and elliptic equations. In particular, the work on the filtration equation cooperated with O.A.Oleinek et al., the mathematicians of the Soviet Union, is universally acknowledged as initiative and classical study which is often quoted by domestic and oversea scholars. After returning to his country, professor Zhou Yulin was engaged not only in research work but also in teaching and has educated a batch of high-level teachers and researchers for theoretical research of partial differential equations.

In the early nineteen sixties, professor Zhou Yulin was under order to be transferred to take part in the theoretical researches of Chinese nuclear weapons. For the prosperity of our homeland, he had suddenly, which happened only in one night, to change his research direction again and stepped his foot on a new journey in the scientific field which was completely new to him. The large scale scientific computing is one of the three sorts of methods for contemporary scientific research, and also it is absolutely necessary and more important for theoretical research of nuclear weapons. Professor Zhou Yulin took charge of the research work on numerical simulation and fluid dynamics of nuclear weapons and made great contributions to the development of Chinese nuclear weapons. Being engaged in large scale scientific calculation over a long period, he raised a series of requirements to computer's device from the view of scientific calculations. Moreover he carried out good theoretical analysis. It has had great influence on the developments of Chinese computers.

In the last over ten years, the period of reform and opening, professor Zhou Yulin has been devoting himself to scientific researches on the nonlinear partial differential equations and their difference methods and has obtained a series of systematic and profound results. Most of the equations he studied have very distinctive physical and applied background, and his research works have been paid close attention to by domestic and oversea mathematicians. Especially, in order to study the difference methods of nonlinear partial differential equations, he developed the methods of discrete functional analysis which have formed a new theoretical system of difference methods. His theory has broken down the line of demarcation between the theoretical studies of partial differential equations and the difference schemes, and made them a unified entity. Therefore, this theory possesses its originality.

Professor Zhou Yulin pursues his studies very rigorously and scrupulously. He gives careful and deep consideration to every scholastic problem, and constantly improves it. He is serious in science and seeks truth from facts. As a senior he is always tireless in teaching and zealously advises younger scientists. And it is usual for him to work so hard that he forgets food and sleep. On the way of getting along with people, he is a sincere and frank man, and he distinguishes clearly between right and wrong; he is an upright man who never flatters, and he hates evil as much as he hates an enemy. It is more than thirty years that I know professor Zhou Yulin so that I often have chances to ask him for advice, to discuss studies with him and to exchange ideas each other. As a result from his words and deeds I benefit a lot.

On the occasion that " Selected Papers of Zhou Yulin" is published, I am told to write the preface. As I look back to the scientific activities of professor Zhou Yulin in the past more than forty years, the figure of an intellectual who has insisted on pursuing the combination of the development of sciences and the prosperity of the homeland for several ten years as one day, appears vividly in front of my eyes. We heartily hope that professor Zhou Yulin will make more remarkable scientific achievements for the prosperity of our country

Li De-yuan

June 20,1992

目 录

Contents

On the Orientability of Differentiable Manifolds

SHIINGSHEN CHERN (陈省身) AND YUHLIN JOU (周毓麟)

Abstract

The aim of this note is to give a simple way by which is decided the orientability or non-orientability of some differentiable manifolds, namely, the Grassmann manifolds and the hyperquadrics in a real projective space. The method is based on a simple lemma.

1. A Lemma

We recall some definitions and notions of a differentiable manifold. A *differentiable manifold M* is a connected Hausdorff space, whose neighborhoods are n-dimensional open cells, such that the following further conditions are satisfied

1) There exists a finite or enumerable set of neighborhoods $\{U_i\}$, which cover M. Each U_i is the topological map θ_i of an open n-dimensional cell E^n defined by

$$|x^k| < 1, \qquad k = 1, \cdots, n.$$

If $p \in U_i$, the coordinates $x^k_{(i)}$ of $\theta_i^{-1}(p)$ are called the local coordinates of p, relative to U_i.

2) If $p \in U_i U_j^1$, the local coordinates $x^k_{(i)}, x^k_{(j)}$ of p relative to U_i, U_j are related by a differentiable transformation

$$x^k_{(j)} = f^k_{(j)}(x^1_{(i)}, \cdots, x^n_{(i)}), \qquad k = 1, \cdots, n$$

of class $\geq m > 0$, with non-vanishing Jacobian :

$$J_{ij} = \frac{\partial(x^1_{(j)}, \cdots, x^n_{(j)})}{\partial(x^1_{(i)}, \cdots, x^n_{(i)})} \neq 0.$$

The set of neighborhoods $\{U_i\}$, with the above conditions, is said to form a *differentiable structure* of M, and M is said to be of dimension n and class $\geq m$.

Let τ be a topological map of E^n onto itself of class $\geq m$. The map θ_i can be replaced by $\theta_i \tau$ and the local coordinates $x^k_{(i)}$ of $p \in U_i$ by the coordinates of $(\theta_i \tau)^{-1}(p)$. Such a change of local coordinates is called an allowable change of local coordinates.

The allowable changes of local coordinates are divided into two classes, according to the sign of the Jacobian. The neighborhood U_i, the map θ_i, and the class of allowable changes of local coordinates with positive (or negative) Jacobian are said to form an oriented neighborhood. Every U_i has exactly two orientations.

M is called orientable, if the neighborhoods U_i can be so oriented that at every point $p \in U_iU_j$ the Jacobian $J_{ij} > 0$. If this is not possible, M is called *non-orientable.*

With these preliminaries, the lemma we intend to prove can be stated as follows :

Lemma. *If there are two (oriented) neighborhoods U_1 and U_2 such that the Jacobian does not keep a constant sign in the intersection U_1U_2, then M is non-orientable. If the Jacobian in U_1U_2, keep the same sign and if $M-(U_1+U_2)$ is a subpolyhedron of dimension $\leq n-2$, then M is orientable.*

Proof. The first statement follows immediately from the definition. To prove the second statement we notice that $U_1 \times U_2$ is a regularly connected polyhedron, for which orientability or non-orientability has a sense. The hypotheses then imply that U_1+U_2 is orientable, and the same is thus true of M.

2. The Grassmann Manifold

The Grassmann manifold, to be denoted by $H(n, N)$, is the manifold of all n-dimensional planes X through the origin 0 of an $(n+N)$-dimensional real Euclidean space E^{n+N}. By adjoining the hyperplane at infinity to E^{n+N} and taking the section of X by a projective hyperplane not through 0, we can regard the manifold also as that of all the $(n-1)$-dimensional linear spaces in a projective space of $n+N-1$ dimensions.

Theorem 1. *$H(n, N)$ is orientable if $n+N$ is even, and non-orientable if $n+N$ is odd*[2].

Proof. We denote by $e_1, \cdots, e_{n+N}$ the coordinate vectors in E^{n+N}, and $x_1, \cdots, x_{n+N}$ the coordinates. Take N coordinate vectors $e_{i_1}, \cdots, e_{i_N}$ and consider the X which meet in 0 only, the linear space spanned by these N coordinate vectors. Such an X has the equations

$$x_{i_s} = \sum \xi_{i_s k} x_k, \qquad s = 1, \cdots, N,$$

where the summation is extended over the indices $k \neq i_1, \cdots, i_N$ All these X form a cell of dimension nN, with $\xi_{i_s k}$ as local coordinates. It is clear that such cells form a covering of $H(n, N)$ and define a differentiable structure on $H(n, N)$.

We take in particular the cells σ, σ^1 corresponding respectively to the indices $i_1 = n+1, i_2 = n+2, \cdots, i_N = n+N$ and $i_1 = n, i_2 = n+2, \cdots, i_N = n+N$. Then $H(n, N) - (\sigma + \sigma^1)$ is a subpolyhedron of $H(n, N)$ of dimension $\leq nN-2$. It follows from our lemma that $H(n, N)$ is orientable or non-orientable, according as the Jacobian of the transformation of local coordinates keeps the same sign or not in the intersection $\sigma \cdot \sigma^1$.

The equations of an $X \in \sigma$ are

$$x_{n+1} = \lambda_{11}x_1 + \cdots + \lambda_{1n}x_n,$$
$$\cdots$$
$$x_{n+N} = \lambda_{N1}x_1 + \cdots + \lambda_{Nn}x_n,$$

while those of an X in σ^1 are

$$\begin{aligned} x_n &= \mu_{11}x_1 + \cdots + \mu_{1n-1}x_{n-1} + \mu_{1n}x_{n+1}, \\ x_{n+2} &= \mu_{21}x_1 + \cdots + \mu_{2n-1}x_{n-1} + \mu_{2n}x_{n+1}, \\ x_{n+N} &= \mu_{N1}x_1 + \cdots + \mu_{Nn-1}x_{n-1} + \mu_{Nn}x_{n+1}. \end{aligned}$$

In the intersection $\sigma \cdot \sigma^1$ we have $\lambda_{1n} \neq 0$, and the relations between the two systems of local coordinates λ and μ are easily found to be

$$\begin{aligned} \mu_{1\alpha} &= -\frac{\lambda_{1\alpha}}{\lambda_{1n}}, \qquad & \mu_{1n} &= \frac{1}{\lambda_{1n}}, \qquad & \alpha &= 1, \cdots, n-1, \\ \mu_{r\alpha} &= \lambda_{r\alpha} - \frac{\lambda_{rn}\lambda_{1\alpha}}{\lambda_{1n}}, \qquad & \mu_{rn} &= \frac{\lambda_{rn}}{\lambda_{1n}}, \qquad & r &= 2, \cdots, n. \end{aligned}$$

It remains to evaluate the Jacobian

$$\triangle = \frac{\partial(\lambda_{11}, \cdots, \lambda_{1n}, \cdots, \lambda_{N1}, \cdots, \lambda_{Nn})}{\partial(\mu_{11}, \cdots, \mu_{1n}, \cdots, \mu_{N1}, \cdots, \mu_{Nn})}$$

and an easy computation gives

$$\triangle = (-1)^n \frac{1}{\lambda_{1n}^{n+N}}.$$

This proves the theorem.

3. The Real Hyperquadrics in Projective Space

We denote by $x_1, \cdots, x_{n+1}$ the homogeneous coordinates of a real projective space of dimension n, A real hyperquadric Σ is defined by an equation of the form

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{n-1}^2 - x_n x_{n+1} = 0.$$

For brevity we introduce the notation

$$Q(x) \equiv \varepsilon_1 x_1^2 + \cdots + \varepsilon_{n-1}x_{n-1}^2, \quad \varepsilon_1 = \cdots = \varepsilon_p = 1, \quad \varepsilon_{p+1} = \cdots = \varepsilon_{n-1} = -1,$$

so that the equation of the hyperquadric can be written

$$Q(x) - x_n x_{n+1} = 0.$$

Theorem 2. *Every even-dimensional real hyperquadric is orientable. An odd-dimensional real hyperquadric is orientable only when it is a hypersphere.*

Proof. We take the points $p(0, \cdots, 0, 1, 0)$ and $p'(0, \cdots, 0, 1)$ on the hyperquadric, and denote by π and π' the tangent hyperplanes at these points. Then each of the sets $\Sigma - (\pi\Sigma)$ and $\Sigma - (\pi'\Sigma)$ is an open cell of dimension $n - 1$. For take a hyperplane τ not through p; by projection from $p, \Sigma - (\pi\Sigma)$ is mapped topologically onto $\tau - (\pi\tau)$, which is an open cell of dimension $n - 1$.

In the set $\Sigma - (\pi\Sigma) - (\pi'\Sigma)$ we have $x_n \neq 0, x_{n+1} \neq 0$. We can therefore put $x_{n+1} = 1$, so that $x_n = Q(x)$. As the local coordinates in the cells $\Sigma - (\pi\Sigma)$ and $\Sigma - (\pi'\Sigma)$ it is possible to take $x_1, \cdots, x_{n-1}$ and $x'_1, \cdots, x'_{n-1}$, where

$$x'_i = \frac{x_i}{Q(x)}, \qquad i = 1, \cdots, n-1.$$

It remains to evaluate the Jacobian

$$\triangle = \frac{\partial(x'_1, \cdots, x'_{n-1})}{\partial(x_1, \cdots, x_{n-1})}$$

which is equal to the determinant

$$\frac{(-2)^{n-1}\varepsilon_1 \cdots \varepsilon_{n-1}}{[Q(x)]^{2(n-1)}} \begin{vmatrix} x_1^2 - \frac{1}{2\varepsilon_1}Q(x) & x_1x_2 & \cdots & x_1x_{n-1} \\ x_2x_1 & x_2^2 - \frac{1}{2\varepsilon_2}Q(x) & \cdots & x_2x_{n-1} \\ \cdots & & & \\ x_{n-1}x_1 & x_{n-1}x_2 & \cdots & x_{n-1}^2 - \frac{1}{2\varepsilon_{n-1}}Q(x) \end{vmatrix}.$$

A simple expansion gives

$$\triangle = \frac{-1}{[Q(x)]^{n-1}}.$$

Therefore $\triangle$ keeps a constant sign when and only when $n - 1$ is even or $n - 1$ is odd all terms of $Q(x)$ have the same sign. This proves the theorem.

References

[1] We shall denote by MN the intersection of the sets M and N.

[2] So far as the writers are aware, this theorem was first established by C. Ehresmann, by means of the theory of Lie groups; ef. C. Ehresmann, *Sur la topologie de certaines variétés algébriques réelles*, Jour. de Math. **16**, 73–75 (1937).

Pseudomanifold and Manifold Homotopy Groups[1)]

YUHLIN JOU

Introduction

The homotopy groups of an arcwise connected topological space were first defined by W. Hurewicz[2)].

Later in a paper written with N. E. Steenrod[11] the definition and a few theorems of the relative homotopy groups of a space modulo a subset were given. Independently both B. Eckmann[3] and J. H. Whitehead[14] introduced the same groups in their investigations on fibre spaces. In a recent work of A. L. Blakers and W. S. Massey[2] the triad homotopy groups of a space relative to two of its subsets had been discussed.

In the study of spherical mappings in a metric space, M. Abe[1] and S.T. Hu[8] characterized algebraically the structures of the Abe groups and the abhomotopy groups respectively. In order to study the Whitehead product[13] of the ordinary homotopy groups, R. H. Fox[6] constructed for each $r \geq 1$ a group τ_γ called the r-dimensional trous homotopy group which has the following interesting properties: (A) Every ordinary homotopy group of dimension less than $r+1$ can be mapped isomorphically into the group τ_γ. (B) If $\gamma = \alpha \cdot \beta \varepsilon \pi^{m+n-1}$ is the Whitehead product of every pair of elements $\alpha\varepsilon\pi^m, \beta\varepsilon\pi^n$ and $m+n-1 \geq r$, then the isomorphisms $\pi^m \longrightarrow \tau_\gamma, \pi^n \longrightarrow \tau_\gamma, \pi^{m+n-1} \longrightarrow \tau_\gamma$ can be so chosen that $\alpha \longrightarrow \bar{\alpha}, \beta \longrightarrow \bar{\beta}, \gamma \longrightarrow \bar{\gamma}$ and $\bar{\gamma} = \bar{\alpha}\,\bar{\beta}\,\bar{\alpha}^{-1}\,\bar{\beta}^{-1}$.

The object of this paper is to generalize and unify various homotopy groups, such as the torus homotopy groups, abhomotopy groups, Abe groups and ordinary homotopy groups of an arcwise connected topological space. For every closed orientable pseudomanifold M and every positive integer n, we shall define a group $\pi^n_M(Y)$ and call it the n-th pseudomanifold M-homotopy group of the space Y. The principal results are the following two generalizations of Fox's work.

(1) If M is an m-dimensional closed orientable manifold M having the property that the topological product $M \times E^k$ of the manifold M and a k-cell E^k can be imbedded into the Euclidean $(m+k)$-space R^{m+k}, then the $(m+n)$-dimensional ordinary homotopy group $\pi^{n+m}(Y)$ is isomorphic to a subgroup of the n-th manifold homotopy group $\pi^n_M(Y)$ of Y for every $n \geq k$ (§5).

1) The author is deeply grateful to Prof. T.H. Kiang, Prof. H.F. Tuan, Mr. Y.F. Sun, Mr. L. Ma and Mr. C, Chen for their encouragement and many helpful suggestions.

2) Hurewicz, [10]. The number in bracket refers to the bibliography at the end of the paper.

(2) A product, $\alpha\cdot\beta\varepsilon\pi^{m+n+1}_{K\times H\times L}(Y)$ for any $\alpha\varepsilon\pi^{m+1}_{K\times H}(Y)$ and $\beta\varepsilon\pi^{m+1}_{K\times L}(Y)$ is defined which generalizes the Whitehead product for three arbitrary closed orientable pseudomanifolds K, H, L (§6). The isomorphisms $\varphi:\pi^{m+1}_{K\times H}(Y)\longrightarrow\pi^{1}_{S^m\times S^n\times K\times H\times L}(Y), \varphi':\pi^{n+1}_{K\times L}(Y)\longrightarrow\pi^{1}_{S^m\times S^n\times K\times H\times L}(Y)$ and $\overline{\varphi}:\pi^{m+n+1}_{K\times H\times L}(Y)\longrightarrow\pi^{1}_{S^m\times S^n\times K\times H\times L}(Y)$ can be so chosen (§7) that

$$\varphi(\alpha\cdot\beta)=\varphi(\alpha)\varphi'(\beta)\varphi(\alpha^{-1})\varphi'(\beta^{-1}).$$

Parallel results are obtained for relative and triad pseudomanifold homotopy groups.

The pseudomanifold homotopy groups are defined as usual by an group operation introduced between the homotopy classes of mappings (§1). Perhaps it would be much better to introduce a lot of antecedents in defining the pseudomanifold homotopy groups. In fact, the proof of the theorems would be simplified by the choice of suitable antecedents. But it is difficult to get an antecedent which would be convenient for the whole homotopy theory.

The absolute and relative pseudomanifold homotopy groups have many properties which are the analogues of the familiar properties of the ordinary homotopy groups of spaces. The groups of different base points form a system of local groups (§2). An exact M-homotopy sequence is constituted by the totality of the absolute and relative M-homotopy groups for a given M of a space Y and its subset Y_0 together with the homeomorphisms induced by the inclusion mappings and the so-called boundary operation (§3).

As a matter of fact, a pseudomanifold homotopy groups of a space Y can be regarded as the fundamental group of a certain mapping space of Y. Generally, a pseudomanifold homotopy group may also be considered as certain pseudomanifold homotopy group of a mapping space (§4).

In addition to the imbeddings of the ordinary·homotopy groups into the manifold homotopy groups, we prove that the n-th m-sphere-homotopy group is also imbeddable into the manifold homotopy group $\pi^n_M(Y)$ for $n\geq k$, where k is the integer mentioned before. The theorems for the relative case follow immediately from the properties of mapping spaces and the theorems for the absolute case (§5).

As an application of group multiplication and its commutator representation, the $(S^{m_1}\times\cdots\times S^{m_r}\times M)$-homotopy groups are algebraically determined by the various M-homotopy groups and the product defined on them corresponding to $K=M$ and H, L being single points (§8).

It is clear that the M-homotopy groups $\pi^n_M(Y)$ reduce to the ordinary homotopy groups $\pi^n(Y)$ when M is a single point. This becomes the Abe group $k_r(Y)$ if we take $n=1$ and $M=S^{n-1}$. The (n,r)-th abhomotopy group $k^n_r(Y)$ is isomorphic to the $(r+1)$-th $(n-r-1)$-sphere-homotopy group for any $n>r>1$. Further the r-dimensional torus homotopy group $\tau_r(Y)$ is a pseudomanifold homotopy group for $n=1$ and M is an $(r-1)$-dimensional torus (§9).

In this paper the theories of the absolute and relative pseudomanifold homotopy groups are carried forward parallelly. But the triad pseudomaniflod homotopy groups are studied separately (§10). The analogous properties to the absolutes and relative cases are easily found.

1. The absolute and relative pseudomaniflod homotopy groups

In this section, the definitions of the absolute and relative pseudomanifold homotopy groups are given.

Let us consider in the Euclidean n-space, the n-cell E^n defined by

$$x_1^2 + \cdots + x_n^2 \leq 1.$$

Let S^{n-1} be the boundary of E^n; E_1^n, E_2^n the subsets of E^n defined by $x_n \geq 0$, $x_n \leq 0$ and Q_1^{n-1}, Q_2^{n-1} the subsets of S^{n-1} defined by $x_n \geq 0, x_n \leq 0$ respectively. Denote by ξ_0 the north pole $(1, 0, \cdots, 0)$ of S^{n-1}. Construct the topological product of the n-cell E^n and any closed orientable pseudomanifold M with a given orientation.

Let Y be an arcwise connected topological space, Y_0 an arcwise connected subset of Y and γ_0 a given point of Y_0. By $Y^X\{X_0, Y_0; X'_0, \gamma_0\}$[9] we denote the totality of mappings $f : X \longrightarrow Y$ such that $f(x_0) \subset Y_0, f(X'_0) = \gamma_0$ and $X'_0 \subset X_0 \subset X$. Consider the totality $Y^{E^m \times M}\{S^{n-1} \times M, Y_0;\ \xi_0 \times M, \gamma_0\}$ of mappings $f : E^n \times M \longrightarrow Y$ such that $f(S^{n-1} \times M) \subset Y_0$ and $f(\xi_0 \times M) = \gamma_0$. These mappings can be divided into homotopy classes as follows: Two mappings f, g are said to be homotopic relative to $\{S^{n-1} \times M, Y_0;\ \xi_0 \times M, \gamma_0\}$ or equivalent, if there exists a homotopy or a continuous family of mappings $h_t \varepsilon Y^{E^n \times M}\{S^{n-1} \times M, Y_0;\ \xi_0 \times M, \gamma_0\}, 0 \leq t \leq 1$, such that $h_0 = f, h_1 = g$. In notation:

$$f \cong g \quad \text{rel.} \quad \{S^{n-1} \times M, Y_0;\ \xi_0 \times M, \gamma_0\}.$$

We shall denote by $[f]$ the homotopy class in which the mapping f is contained and by 0 the constant mapping $0\ (E^m \times M) = \gamma_0$.

Let $f, g \varepsilon Y^{E^n \times M}\{S^{n-1} \times M, Y_0;\ \xi_0 \times M, \gamma_0\}$. Consider the partial mapping $f|S^{n-1} \times M$. There is a mapping $f_0 : S^{n-1} \times M \longrightarrow Y_0$ such that $f_0(Q_2^{n-1} \times M) = \gamma$, and

$$f_0 \cong f|S^{n-1} \times M \quad \text{rel.} \quad \{S^{n-1} \times M, Y_0;\ \xi_0 \times M, \gamma_0\}.$$

In fact, it is easy to construct a deformation $\delta_t : S^{n-1} \longrightarrow S^{n-1}, 0 \leq t \leq 1$; such that δ_0 =identity, $\delta_1(Q_2^{n-1}) = \xi_0$, $\delta_2(\xi_0) = \xi_0$ and $\delta_t(Q_1^{n-1}) \subset Q_1^{n-1}$. Define a deformation $\delta_t^* : S^{n-1} \times M \longrightarrow S^{n-1} \times M$ by taking

$$\delta_t^*(\xi, p) = (\delta_t(\xi), p), \quad \xi \varepsilon S^{n-1}, \quad p \varepsilon M.$$

clearly the homotopy $f\delta_t^* : S^{n-1} \times M \longrightarrow Y_0$ gives a mapping $f_0 = f\delta_1^*$ homotopic to $f|S^{n-1} \times M = f\delta_0^*$.

Define a mapping $f' : (E_2^n \times M) \cup (S^{n-1} \times M) \longrightarrow Y$ by taking

$$f' = \begin{cases} f_0 & \text{on } S^{n-1} \times M \\ \gamma_0 & \text{on } E_2^n \times M. \end{cases}$$

Similarly, it can be verified that

$$f' \cong f|(E_2^n \times M) \cup (S^{n-1} \times M) \quad \text{rel.} \quad \{S^{n-1} \times M, Y_0;\ \xi_0 \times M, \gamma_0\}.$$

Since $(E_2^n \times M) \cup (S^{n-1} \times M)$ is a closed subpolyhedron of $E^n \times M$, it has the homotopy extension property in $E^n \times M$ relative to any Y. So f' has an extension $f^* : E^n \times M \longrightarrow Y$ such that $f^*(E_2^n \times M) = \gamma_0$ and

$$f^* \cong f \quad \text{rel.} \quad \{S^{n-1} \times M, Y_0;\ \xi_0 \times M, \gamma_0\}.$$

Similarly, there is a mapping $g^* : E^n \times M \longrightarrow Y$ such that $g^*(E_1^n \times M) = \gamma_0$

$$g^* \cong g \quad \text{rel.} \quad \{S^{n-1} \times M, Y_0;\ \xi_0 \times M, \gamma_0\}.$$

Now we define a mapping $\varphi \varepsilon Y^{E^n \times M}\{S^{n-1} \times M, Y_0;\ \xi_0 \times M, \gamma_0\}$ by taking

$$\varphi = \begin{cases} f^* & \text{on } E_1^n \times M, \\ g^* & \text{on } E_2^n \times M. \end{cases}$$

It can be proved that the class $[\varphi]$ depends only on the classes $[f]$ and $[g]$. We define

$$[\varphi] = [f] + [g].$$

It can be seen that the operation + has all the properties of a group operation but is not necessarily commutative. For the non-commutative case, we shall write multiplicatively. the neutral element is the class [0]; and the inverse of $[f]$ is the class $[f\theta]$ where $\theta : E^n \times M \longrightarrow E^n \times M$ is defined by $\theta(\eta, p) = (\overline{\eta}, p)$ for $\eta \varepsilon E^n, p \varepsilon M$ with $\overline{\eta}$ the mirror image of η in the hyperplane $x_n = 0$.

The group $\pi_M^n(Y, Y_0, \gamma_0), n \geq 2$, obtained in this way is called the *nth relative pseudo-manitold M-homotopy group of Y modulo Y_0 with γ_0 as the base point* or simply the *n-th relative M-homotopy group.*

If $n \geq 1, Y_0 = \gamma_0$, the group is called the *n-th (absolute) pscudomanifold M-homotopy group of Y with γ_0 as the base point* or simple the *n-th M-homotopy group.* the symbol $\pi_M^n(Y, \gamma_0, \gamma_0)$ may be abridged as $\pi_M^n(Y, \gamma_0)$. the n-th absolute M-homotopy group $\pi_M^n(Y, \gamma_0)$ may be regarded as the group of the homotopy classes rel. $\{\xi_0 \times M, \gamma_0\}$ of mappings in $Y^{S^n \times M}\{\xi_0 \times M, \gamma_0\}$.

As a consequence of the definition, we have the following.

Theorem. *The n-th relative M-homotopy group* $\pi_M^n(Y, Y_0, \gamma_0)$ *is abelian if* $n \geq 3$ *and the n-th (absolute) M-homotopy group* $\pi_M^n(Y, \gamma_0)$ *is abelian if* $n \geq 2$. *The groups* $\pi_M^2(Y, Y_0, \gamma_0)$ *and* $\pi_M^1(Y, \gamma_0)$ *are in general non-abelian.*

The theorem is true since we may rotate E^n continuously keeping S^{n-1} on itself and ξ_0 fixed, until E_1^n and E_2^n are interchanged if $n \geq 3$ for the relative M-homotopy group; and we may rotate S^n continuously keeping ξ_0 fixed until Q_1^n and Q_2^n are interchanged if $n \geq 2$, for the absolute one.

As a general rule, any regularly connected finite polyhedron X together with its connected subsets X'_0, X_0, for which there is an onto mapping $\omega : X \longrightarrow E^n$ such that ω is a homeomorphism between $X - X'_0$ and $E^n - \xi_0$, may be taken to be the antecedent for defining the relative M-homotopy group. Denote $X_0 = \omega^{-1}(S^{n-1})$ and then $X'_0 \subset X_0 \subset X$. Consider the totality of mappings in $Y^{X \times M}\{X_0 \times M, Y_0; X'_0 \times M, \gamma_0\}$. The group operation is defined by taking $X_1 = \omega^{-1}(E_1^n), X_2 = \omega^{-1}(E_2^n)$ instead of $E_1^n, E_2^n,$.

A group thus obtained is clearly isomorphic to $\pi_M^n(Y, Y_0, \gamma_0)$.

For the absolute M-homotopy group, the regularly connected finite polyhedron X together with its connected subset X_0 may be taken as the antecedent, if there exists an onto mapping $\omega : X \longrightarrow S^n$ which maps $X - X_0$ homeomorphically onto $S^n - \xi_0$ with $X_0 = \omega^{-1}(\xi_0)$. A group operation can be defined in order to get the group $\pi_M^n(Y, \gamma_0)$ on the classes of mappings in $Y^{X \times M}\{X_0 \times M, \gamma_0\}$ by taking $X_1 = \omega^{-1}(Q_1^n), X_2 = \omega^{-1}(Q_2^n)$ instead of Q_{12}^n, Q_2^n.

Consider the n-cube E^n defined by $0 \leq x_i \leq 1, i = 1, \cdots, n$, in an Euclidean n-space. Denote by $\dot{E}^n$ the boundary of E^n and by $J^{n-1} = \dot{E}^n - E^{n-1}$ the set $(1 - x_1)\prod_{i=2}^n x_i(1 - x_i) = 0$ where E^{n-1} is defined by $x_1 = 0$. Let E_1^n, E_2^n denote the subsets of E^n defined by $x_n \geq \frac{1}{2}, x_n \leq \frac{1}{2}$. The relative M-homotopy group $\pi_M^n(Y, Y_0, \gamma_0)$ can be obtained by taking $J^{n-1} \subset \dot{E}^n \subset E^n$ as an antecedent and by considering the mappings in $Y^{E^n \times M}\{\dot{E}^n \times M, Y_0; J^{n-1} \times M, \gamma_0\}$. When $n \geq 1, Y_0 = \gamma_0$, the group $\pi_M^n(Y, \gamma_0)$ may be obtained by taking $\dot{E}^n \subset E^n$ as antecedent.

Let C^n for $n \geq 2$ denote the subset of E^{n+1} defined by the condition $\prod_{i=2}^{n+1} x_i(1 - x_i) = 0$. Let $\lambda : E^n \longrightarrow C^n$ be a mapping described as follows. Denote by P_t, Q_t the subsets of E^n, C^n given by $x_1 = t. p_t$ is an $(n-1)$-cube and Q_t is an $(n-1)$-sphere. Let λ map P_t onto Q_t with degree one in such a way that the boundary of P_t is mapped on the point $(t, 0, \cdots, 0)\varepsilon Q_t$. Let L be the subset of C^n defined by $x_i = 0$, $i = 2, \cdots, n+1$. Taking $(Q_1 \cup L) \subset (Q_0 \cup Q_1 \cup L) \subset C^n$ as antecedent and considering the totality of mappings in $Y^{C^n \times M}\{Q_0 \times M, Y_0;\ (Q_1 \cup L) \times M, \gamma_0\}$, we have also the group $\pi_M^n(Y, Y_0, \gamma_0)$.

Identify the subset $\xi_0 \times M$ of the topological product $S^n \times M$ to a single point x_0 to form the pinched topological product X_M^n. Similarly, we can divide the mappings in $Y^{X_M^n}\{x_0, \gamma_0\}$ into homotopy classes relative to $\{x_0, \gamma_0\}$ and define a group operation so that the absolute M-homotopy group $\pi_M^n(Y, \gamma_0)$ thus obtained is the same as before.