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李彦博 著

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Preface

Frobenius algebras are finite rank algebras over a commutative ring with identity, which have a certain self-dual property. These algebras appear in not only some branches of algebra, such as representation theory, Hopf algebra, algebraic geometry and so on, but also topology, geometry and coding theory, even in the work on the solutions of Yang-Baxter equation. Symmetric algebras including group algebras of finite groups are a large source of examples. Apart from its own importance, Frobenius algebras are useful because the symmetricity of an algebra is usually not easy to verify. Thus it is meaningful to develop methods that can be used to deal with algebras only known to be Frobenius.

Cellular algebras were introduced by Graham and Lehrer in [35] in 1996, motivated by previous work of Kazhdan and Lusztig in [45]. They were defined by a so-called cellular basis with some nice properties. The theory of cellular algebras provides a systematic framework for studying the representation theory of non-semisimple algebras which are deformations of semisimple ones. One can parameterize simple modules for a finite dimensional cellular algebra by methods in linear algebra.

Many classes of algebras from mathematics and physics are found to be cellular, including Hecke algebras of finite type, Ariki-Koike algebras, q-Schur algebras, q-rook monoid algebras, Temperley-Lieb algebras, cyclotomic Temperley-Lieb algebras, Jones algebras, Brauer algebras, partition algebras, Birman-Wenzl algebras and so on, see [27], [33], [35], [78], [82], [83], [87], [88] for details.

An equivalent basis-free definition was given by Koenig and Xi in [48], which is useful in dealing with structural problems. By using this definition, they showed that there are only two different kinds of cell ideals, one being the hereditary ideal and the other having square zero. Some homological properties were also investigated. Also by this definition, in [52], Koenig and Xi made explicit an inductive construction of cellular algebras called inflation, which could produce all cellular algebras. In [54], Brauer algebras were shown to be iterated inflations of group algebras of symmetric groups and then more information about these algebras was found.

There are some generalizations of cellular algebras. For example, Koenig and Xi in [55] introduced affine cellular algebras which contain cellular algebras as special cases. Affine Hecke algebras of type A and infinite dimensional diagram algebras like the affine Temperley-Lieb algebras were proved to be affine cellular. Study of affine cellularity of different classes of algebras is an active research area. Recently, Kleshchev and Loubert proved in [46] that all Khovanov-Lauda-Rouquier algebras of finite Lie types are affine cellular algebras. Moreover, Cui proved the affine cellularities of affine q-Schur algebras, affine Brauer algebras and affine Birman-Wenzl algebras in [9], [10] and [11]. Other generalizations include based algebras, procellular algebras, tabular algebras and cellular categories. We refer the reader to [20], [36], [37], [86] for details.

It is an open problem to find explicit formulas for the dimensions of simple modules of a cellular algebra. By the theory of cellular algebras, this is equivalent to determine the dimensions of the radicals of bilinear forms associated with cell modules. By a result of [35], the radicals of bilinear forms are related to the radical of the cellular algebra. This leads us to studying the radical of a cellular algebra. Moreover, it is well known that the radical of an algebra is equal to the direct sum of the radicals of all blocks of the algebra, each block corresponds to a central idempotent. Thus, to study the radical of a cellular algebra, we also need to investigate the central idempotents, namely, we are suggested to consider the center of a cellular algebra. However, we have no idea for dealing with general cellular algebras as so far. This book is devoted to investigate centers and radicals of symmetric cellular algebras. Note that Hecke algebras of finite types, Ariki-Koike algebras over any ring containing inverses of the parameters, Khovanov's diagram algebras are all symmetric cellular algebras. The trivial extension of a cellular algebra is also a symmetric cellular algebra. We refer the reader to [6], [68] and [90].

There are many papers on centers of Hecke algebras. Among these papers, two problems are mainly concerned. One is to find bases for centers. The other is to investigate the relations between the centers and the so-called Jucys-Murphy elements.

In [44], Jones found bases for centers of Hecke algebras of type A over $\mathbb{Q}[q, q^{-1}]$, where q is an indeterminant. This basis is an analog of conjugacy class sum in a group algebra. In [31], Geck and Rouquier found bases for the centers of generic Hecke algebras over $\mathbb{Z}[q, q^{-1}]$ with q an indeterminant. However, it is not easy to write the basis explicitly. Then one should ask, is there any basis which can be written explicitly? In [23], Francis gave an integral "minimal" basis for the center of a Hecke algebra. Then in [24], he used the minimal basis approach to provide a way of describing and calculating elements of the minimal basis for the center of an Iwahori-Hecke algebra which is entirely combinatorial. In [26], Francis and Jones found an explicit non-recursive expression for the coefficients appearing in these linear combinations for the Hecke algebras of type A.

The fact that Hecke algebras of finite type are all cellular leads us to considering how to describe the centers of Hecke algebras by cellular bases. Furthermore, how to describe the center of a cellular algebra in general? Clearly, most of the approaches for studying Hecke algebras can not be used directly for cellular algebras, since we have no Weyl group structure to use. Then we must look for some new method. In fact, the symmetry of Hecke algebras provides us a way.

Jucys-Murphy elements were constructed for the group algebras of symmetric groups first. The combinatorics of these elements allow one to compute simple representations explicitly and often easily in the semisimple case. Then Dipper, James and Murphy (see [15], [16], [17], [18], [19]) did a lot of work on representations of Iwahori-Hecke algebras and produced analogues of the Jucys-Murphy elements for Iwahori-Hecke algebras of types A and B. The constructions for other algebras can be found in [39], [80] and so on. In [16], Dipper and James conjectured that the center of a Hecke algebra of type A consists of symmetric polynomials in the Jucys-Murphy elements. In [71], Mathas introduced a family of polynomials indexed by pairs of partitions and proved that the conjecture hold if these polynomials are self-orthogonal. The conjecture was proved by Francis and Graham in [25] in 2006. In [5], Brundan proved that the center of each degenerate cyclotomic Hecke algebra consists of symmetric polynomials in the Jucys-Murphy elements. An analogous conjecture for Ariki-Koike Hecke algebra is open in non-semisimple case. Moreover, some authors expressed the Jucys-Murphy elements and their symmetric polynomials by diagrams, this makes them intuitive, see [38], [72], [73] for details.

The fact that most of the algebras which have Jucys-Murphy elements are cellular leads one to defining Jucys-Murphy elements for general cellular algebras. In [69], Mathas did some work in this direction. By the definition of Mathas, we will investigate the relations between the centers and the Jucys-Murphy elements of cellular algebras.

The Jacbson radical of an algebra is important and interesting. For example, it reflects the complexity of the algebra in some sense. For a cellular algebra, little has been done on it. In [57], for a quasi-hereditary cellular algebra, Lehrer and Zhang found a set that contains the radical. However, it is not easy to write the elements explicitly. In this book, for a symmetric cellular algebra A, we will constructed a nilpotent ideal, which is certainly contained in the radicals of A. It is helpful to note that sometimes the nilpotent ideal we constructed is just equal to the radical.

Now let us give an outline of the book.

In Chapter 1, we recall some basic notions and facts on Frobenius algebras. We first recall some equivalent definitions of Frobenius algebras and then give some examples. In Section 1.3, we construct some ideals of centers of Frobenius algebras by the so-called Nakayama twisted centers. In Section 1.4, we recall Schur elements and give a semisimple criterion for a symmetric algebra. Another class of Frobenius algebras called canonical mesh algebras is introduced in Section 1.5.

In Chapter 2, we develop the theory of cellular algebras. In Section 2.1, we recall two equivalent definitions of cellular algebras given by Graham and Lehrer and by Koenig and Xi. In Section 2.2, we give a quick review on representation theory of cellular algebras. In Section 2.3, we study the quasi-hereditity of cellular algebras. In Section 2.4, we construct a new class of diagram algebras which are cellular and quasi-hereditary. In Section 2.5, we study based algebras which are introduced by Du and Rui in [20]. We will construct simple modules for 0-Hecke algebras by using the theory of standard based algebras. In Section 2.6, we introduce two generalizations of cellular algebras including affine cellular algebras and procellular algebras.

In Chapter 3, we study the theory of Frobenius cellular algebras. In Section 3.1, we investigate the property of the dual basis of a cellular basis of a Frobenius cellular algebra. In particular, we give a sufficient and necessary condition for the dual basis of a cellular basis of a symmetric cellular algebra being cellular. In Section 3.2, we give some examples of non-symmetric Frobenius cellular algebras. In Section 3.3, we develop the theory of symmetric cellular algebras. It is proved that the dual basis of a cellular basis of a symmetric cellular is "almost" cellular. Furthermore, we introduce a constant for each cell module, which could be viewed as a generalization of the Schur element. It also connects the Gram matrices of a cell module and the dual cell module which is defined by the dual basis.

In Chapter 4, we learn the centers and radicals of symmetric cellular algebras. In Section 4.1, we construct an ideal of the center of a symmetric cellula algebra, which contains the so-called Higman ideal. In Section 4.2, we detect Nakayama twisted centers of Frobenius cellular algebras. In Section 4.3, we study relations between the Jucys-Murphy elements and centers of cellular algebras. In Section 4.4, we apply the result obtained in Section 4.3 to Ariki-Koike algebras, which give a new proof of a theorem on centers of semisimple Ariki-Koike algebras. In Section 4.5, we consider radicals of symmetric cellular algebras. A nilpotent ideal is constructed for a symmetric cellular algebra. The ideal connects the radicals of cell modules with the radical of the algebra. It also reveals some information on the dimensions of simple modules. As a by-product, we obtain some equivalent conditions for a finite dimensional symmetric cellular algebra to be semisimple.

In Chapter 5, we study Hecke algebras of type A. In Section 5.1, we recall some basic facts on Murphy basis. In Section 5.2, we give another classification of simple modules by using the dual Murphy basis and then give a series of sufficient and necessary conditions of the projectivity of Specht modules. In Section 5.3, we list some examples. In Section 5.4, we give an introduction on Kazhdan-Lusztig theory.

The book is based on the author's PhD. thesis [66] and papers [58]-[65]. I am indebted to my supervisor Professor Xi C.C. for his guidance. I wish to thank support from Fundamental Research Funds for the Central Universities (N130423011). I am also grateful to Mrs. Shi Yuling and other editors of Northeastern University Press for their diligent work.

Li Yanbo

January 2015

Contents

Chapter 1

Frobenius algebras

1.1 Definition of Frobenius algebras

In this section, we will give some equivalent definitions of Frobenius algebras. In particular, the so-called Nakayama automorphisms will also be described.

Let R be a commutative ring with identity and A an associative R-algebra. As an R-module, we assume that A is finitely generated and free. Let $f : A \times A \to R$ be an R-bilinear map . We say that f is *non-degenerate* if the determinant of the matrix $(f(a_i, a_j))_{a_i, a_j \in B}$ is a unit in R for some R-basis B of A. We say f is *associative* if

$$f(ab, c) = f(a, bc)$$

for all $a, b, c \in A$, and *symmetric* if

$$f(a, b) = f(b, a)$$

for all $a, b \in A$.

Definition 1.1.1. An R-algebra A is called a *Frobenius algebra* if there is a non-degenerate associative bilinear form f on A. We call A a *symmetric algebra* if f is symmetric, non-degenerate and associative.

It is helpful to point out that Frobenius algebras could be defined equivalently as follows.

Theorem 1.1.2. *Let A be a finite dimensional R-algebra. Then the following are equivalent.*

(1) *A is Frobenius.*

(2) *$A \simeq \hat{A}$ as left A-modules, where $\hat{A} := \mathrm{Hom}_R(A, R)$.*

(3) *For each right ideal $X \subseteq A$ and left ideal $Y \subseteq A$,*

$$\dim_R X + \dim_R ann_l(X) = \dim_K A = \dim_R Y + \dim_R ann_r(Y),$$

where $ann_l(X)$ is the left annihilator of X and $ann_r(Y)$ is the right annihilator of Y.

(4) *There exists a hyperplane (an R-space of A of codimension 1) $H \subset A$ which contains no nonzero left ideal.*

The following result is well known and we omit the proof here.

Theorem 1.1.3. *Let A be a Frobenius algebra with Jacbson radical $\mathrm{rad}\, A$ and M a left A-module. Then*

(1) *M is projective if and only if it is injective.*

(2) *$ann_l(\mathrm{rad}\, A) = ann_r(\mathrm{rad}\, A)$.*

For a Frobenius algebra A, there is an automorphism α of A such that

$$Hom_R(A, R) \simeq {}_{\alpha}A_1$$

as A-A-bimodules, where ${}_{\alpha}A_1$ is A as vector space, on which $a \in A$ acts by multiplication on the right and by $\alpha(a)$ on the left. This automorphism is unique up to inner automorphisms and is called *Nakayama automorphism*. In [40], Holm and Zimmermann proved the following lemma.

Lemma 1.1.4. ([40] Lemma 2.7) *Let A be a finite dimensional Frobenius algebra. Then an automorphism α of A is a Nakayama automorphism if and only if*

$$f(a,b) = f(\alpha(b), a)$$

for all $a, b \in A$.

Remarks 1.1.5. (1) The Nakayama automorphism of a Frobenius algebra is independent of the ground ring. See [75] for details.

(2) If there is some $1 \leq n \in \mathbb{N}$ such that $\alpha^n = \mathrm{id}_A$ and $\alpha^m \neq \mathrm{id}_A$ for all $1 \leq m < n$, then α is called a Nakayama automorphism of rank n.

Let A be a Frobenius algebra with a basis

$$B = \{a_i \mid i = 1, \cdots, n\}$$

and a non-degenerate associative bilinear form f. Define an R-linear map $\tau : A \to K$ by

$$\tau(a) = f(a, 1).$$

We call τ a *symmetrizing trace* if A is symmetric. Denote by

$$d = \{d_i \mid i = 1, \cdots, n\}$$

the basis which is uniquely determined by the requirement that

$$\tau(a_i d_j) = \delta_{ij}$$

and

$$D = \{D_i \mid i = 1, \cdots, n\}$$

the basis determined by the requirement that

$$\tau(D_j a_i) = \delta_{ij}$$

for all $i, j = 1, \cdots, n$. We will call d the *right dual basis* of B and D the *left dual basis* of B, respectively.

The following lemma is clear.

Lemma 1.1.6. *Suppose that A is a Frobenius R-algebra with a basis $\{a_i \mid i = 1, \cdots, n\}$. Let τ, τ' be two maps determined by two non-degenerate bilinear forms f and f' respectively. Denote by $\{d_i \mid i = 1, \cdots, n\}$ the right dual basis of B determined by f and $\{d_i' \mid i = 1, \cdots, n\}$ the right dual basis determined by f'. Then for any $1 \leq i \leq n$, we have*

$$d_i' = \sum_{j=1}^{n} \tau(a_j d_i') d_j.$$

Now let us give a Nakayama automorphism which connects dual bases. Fixing a τ, Define an R-linear map $\alpha : A \to A$ by

$$\alpha(d_i) = D_i.$$

We claim that α is a Nakayama automorphism of A. In fact, it is clear that

$$f(x, y) = f(\alpha(y), x)$$

for all $x, y \in A$. Then by Lemma 1.1.4, we only need to prove that α is an automorphism of A, that is, $\alpha(ab) = \alpha(a)\alpha(b)$.

Given $x \in A$, on one hand,

$$f(x, ab) = f(\alpha(ab), x).$$

On the other hand, the associativity of f implies that

$$\begin{aligned} f(x, ab) = f(xa, b) &= f(\alpha(b), xa) = f(\alpha(b)x, a) \\ &= f(\alpha(a), \alpha(b)x) = f(\alpha(a)\alpha(b), x). \end{aligned}$$

Hence

$$f(\alpha(ab), x) = f(\alpha(a)\alpha(b), x).$$

Note that f is non-degenerated. Then $\alpha(ab) = \alpha(a)\alpha(b)$ holds.

If A is a symmetric algebra and f is a symmetric, non-degenerate associative bilinear form, then α is the identity map and then the left and the right dual basis are the same.

1.2 Examples of Frobenius algebras

Let K be a field throughout. In this section, we give some examples of Frobenius algebras, including semisimple algebras, group algebras, Hopf algebras and so on.

Example 1.2.1. Let A be a semisimple K-algebra. Then A is Frobenius.

Example 1.2.2. Let G be a finite group and let $A = KG$ be the group algebra. Then A is Frobenius.

Example 1.2.3. Let H be a finite dimensional Hopf algebra. Then A is a Frobenius algebra and the orders of all Nakayama automorphisms of H are finite.

Example 1.2.4. Let $\mathcal{M} = (M_1, M_2, \cdots, M_n)$ be an n-tuple of $n \times n$ matrices $M_k = (a_{ij}^{(k)}) \in M_{n\times n}(K)$ satisfying the following conditions:

(1) $a_{ij}^{(k)} a_{il}^{(j)} = a_{il}^{(k)} a_{kl}^{(j)}$ for all $i, j, k, l \in \{1, \cdots, n\}$.

(2) $a_{kj}^{(k)} = a_{ik}^{(k)} = 1$ for all $i, j, k, \in \{1, \cdots, n\}$.

(3) $a_{ii}^{(k)} = 0$ for all $i, k \in \{1, \cdots, n\}$ such that $i \neq k$.

Let

$$A = \oplus_{1\leq i,j\leq n} K u_{ij}$$

be a K-space with basis

$$\{u_{ij} \mid 1 \leq i, j \leq n\}.$$

The multiplication of V is given by

$$u_{ik}u_{lj} := \begin{cases} a_{ij}^{(k)} u_{ij}, & \text{if} \quad k = l; \\ 0, & \text{otherwise.} \end{cases}$$

Now assume that $a_{ij}^{(k)} = 0$ or 1 for all $1 \leq i, j, k \leq n$. If there exists a permutation σ of the set $\{1, \cdots, n\}$ such that $\sigma(i) \neq i$ for all $i \in \{1, \cdots, n\}$ and that $a_{i\sigma(i)}^{(k)} = 1$ for all $i, k \in \{1, \cdots, n\}$, then A is Frobenius.

Example 1.2.5. Let A and B be Frobenius K-algebras. Then $A \otimes_K B$ is Frobenius.

Example 1.2.6. Let A be a finite dimensional K-algebra and M a A-A-bimodule. Define the *trivial extension algebra* $\mathcal{T}$ of A as follows: the elements are pairs (a, m), addition is componentwise and multiplication is given by

$$(a, m)(a', m') = (aa', am' + ma').$$

Then $\mathcal{T}$ is Frobenius.

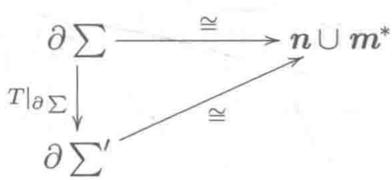
Example 1.2.7. 2-D topological quantum field theories.

Let Pre2-Cobord denote the 2-category defined as follows.

Objects: disjoint unions of labeled, oriented, compact one manifolds.

Morphisms: $\sum : \boldsymbol{n} \to \boldsymbol{m}$, oriented topological surfaces equipped with an orientation preserving homeomorphism from the boundary $\partial \sum$ to the disjoint union $\boldsymbol{n}^* \cup \boldsymbol{m}$. Here $\boldsymbol{n}^*$ indicates reversal of orientation.

2-Morphisms: orientation-preserving homeomorphisms $T : \sum \to \sum'$ of morphisms such that the following diagram commutes.



A 2-Cobord is a category whose objects are those of Pre2-Cobord and whose morphisms are the equivalence classes of morphisms induced by the 2-category structure of Pre2-Cobord.

Let Vec/K be the category consisting of finite dimensional K-vector spaces and linear maps, with the monoid structure given by tensor products. A 2-dimensional topological quantum theory is a monoidal functor

$$Z : 2\text{−Cobord} \to Vec/K$$

which is defined by taking $0 \mapsto K$ and $\boldsymbol{n} \mapsto V^{\otimes n}$. Then Z induces a Frobenius structure on $Z(\mathbf{1})$.