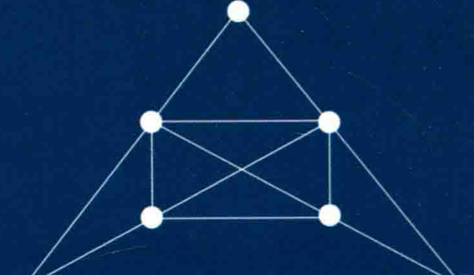
Factor Critical Graphs

◉ 周思中 (Sizhong Zhou) 著



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Abstract

This book mainly discusses the factor critical graph and fractional factor critical graph problems. This book is divided into five chapters. In Chapter 1, we show basic terminologies, definitions and graphic parameters. In Chapter 2, we present some sufficient conditions for graphs to be (a,b,k)-critical graphs. In Chapter 3, we study fractional critical graphs which are the generalizations of fractional factors in graphs. In Chapter 4, we investigate all fractional (a,b,k)-critical graphs. In Chapter 5, we discuss a generalization of fractional factors, i. e., fractional ID-factor-critical graphs from different perspectives.

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Preface

Graph theory is one of the branches of modern mathematics which has shown impressive advances in recent years. Graph theory is widely applied in physics, chemistry, biology, network theory, information science and other fields, and so it has attracted a great deal of attention.

Factor theory of graph is an important branch in graph theory. It has extensive applications in various areas, e. g. , combinatorial design, network design, circuit layout, scheduling problems, the file-transfering problems and so on.

The fractional factor problem in graphs can be considered as a relaxation of the well-known cardinality matching problem. The fractional factor problem has wide-range applications in areas such as network design, scheduling and combinatorial polyhedra. For instance, in a communication network if we allow several large data packets to be sent to various destinations through several channels, the efficiency of the network will be improved. The feasible assignment of data packets can be seen as a fractional flow problem and it becomes a fractional matching problem when the destinations and sources of a network are disjoint.

The factor critical graph and fractional factor critical graph is an extension of factor and fractional factor in a graph, respectively. The factor critical graph and fractional factor critical graph problems in network are used to measure whether there exists a feasible data packet transmission scheme from node to node when certain nodes are damaged.

In this book, we mainly discuss the factor critical graph and fractional factor critical graph problems. This book is divided into five chapters.

In Chapter 1, we show basic terminologies, definitions and graphic parameters.

In Chapter 2, we present some sufficient conditions for graphs to be (a,b,k)-critical graphs. This chapter is divided into three parts. Firstly, we give some sufficient conditions for the existence of (a, b, k)-critical graphs in terms of neighborhood, neighborhood and minimum degree, neighborhood union, etc. It is shown that this results in this part are sharp. Secondly, we obtain some binding number conditions for graphs to be (a, b, k)-critical graphs, and verify that the results are sharp. Finally, we investigate (a, b, k)-critical graphs, and use connectivity and independence number, independence number and minimum degree to obtain some sufficient conditions for graphs to be (a, b, k)-critical graphs, and these results are best possible in some sense.

In Chapter 3, we study fractional critical graphs which are the generalizations of fractional factors in graphs. We first pose a necessary and sufficient condition for a graph to be a fractional (f, n)-critical graph. Then we show some sufficient conditions for the existence of fractional (f, n)-critical graphs. Finally, we give a result on fractional (g, f, n)-critical graphs. Furthermore, it is shown that these results are sharp.

In Chapter 4, we investigate all fractional (a, b, k)-critical graphs. We first show a criterion for graphs to be all fractional (a, b, k)-critical graphs. Then we obtain some graphic parameter (such as neighborhood and binding number) conditions for graphs to be all fractional (a, b, k)-critical graphs by using the criterion. Our results on all fractional (a, b, k)-critical graphs are an extension of the previous results.

In Chapter 5, we discuss a generalization of fractional factors, i. e., fractional ID-factor-critical graphs from different perspectives, such as degree condition, neighborhood condition, binding number, independence number and connectivity. We present some sufficient conditions related to these parameters for the existence of fractional ID-factor-critical graphs. This chapter is divided into three parts. First, We present some sufficient conditions for graphs to be fractional ID-k-factor-critical graphs. Second, we show some results on fractional ID-$[a, b]$-factor-critical graphs. Finally, we gave a sufficient condition for a graph to be a fractional ID-(g, f)-factor-critical graph.

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Contents

Chapter 1 Basic Terminologies and Graphic Parameters

In this chapter, some basic terminologies, definitions and graphic parameters are given, which will be used throughout this book.

1.1 Basic Terminologies

All graphs considered in this book are finite undirected graphs which have neither loops nor multiple edges. Let G be a graph. We use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. For $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G. The neighborhood of $x \in V(G)$ is denoted by $N_G(x)$. For $S \subseteq V(G)$, we define the neighborhood of S as follows:

$$N_G(S) = \bigcup_{x \in S} N_G(x).$$

Note that $N_G(x)$ does not contain x, but it may happen that $N_G(S) \supseteq S$. For a set $S \subseteq V(G)$, let $G[S]$ be the subgraph of G induced by S, and $G - S = G[V(G) \setminus S]$. If $S, T \subseteq V(G)$, then $e_G(S, T)$ denotes the number of edges having one end-vertex in S and the other in T. If S is a singleton $\{x\}$, we write $S = x$ instead of $S = \{x\}$. For example, we write $e_G(x, T)$ instead of $e_G(\{x\}, T)$. Let r be a real number. Recall that $\lfloor r \rfloor$ is the greatest integer such that $\lfloor r \rfloor \leqslant r$ and $\lceil r \rceil$ is the smallest integer with $\lceil r \rceil \geqslant r$. Other definitions and terminology can be found in [4].

A graph H is called a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of G is called a spanning subgraph of G if $V(H) = V(G)$. Let g and f be two integer-valued functions such that $0 \leqslant g(x) \leqslant f(x)$ for any $x \in V(G)$. A spanning subgraph F of G with $g(x) \leqslant d_F(x) \leqslant f(x)$ for any $x \in V(G)$ is called a (g, f)-factor. A (g, f)-factor is said to be an $[a, b]$-factor if $g(x) = a$ and $f(x) = b$ for any $x \in V(G)$. When $a = b = k$, an $[a, b]$-factor is simply called a k-factor.

Let $h: E(G) \to [0, 1]$ be a function. If $g(x) \leqslant \sum_{e \ni x} h(e) \leqslant f(x)$ holds for every $x \in V(G)$, then we call $G[F_h]$ a fractional (g, f)-factor of G with indicator function h, where $F_h = \{e : e \in E(G), h(e) > 0\}$. If $g(x) \equiv a$ and $f(x) \equiv b$, then a fractional (g, f)-factor is said to be a fractional $[a, b]$-factor. A fractional (f, f)-factor is called simply a fractional f-factor. A fractional $[k, k]$-factor is simply called a fractional k-factor. A fractional 1-factor is also called a fractional perfect matching.

A graph G is called an (a, b, k)-critical graph if after deleting any k vertices of G the remaining graph of G has an $[a, b]$-factor. If G is an (a, b, k)-critical graph, then we also say that G is (a, b, k)-critical. If $a=b=r$, then an (a, b, k)-critical graph is simply called an (r, k)-critical graph. In particular, a $(1, k)$-critical graph is simply called a k-critical graph.

A graph G is said to be fractional (g, f, n)-critical if for any $N \subseteq V(G)$ with $|N|=n$, $G-N$ contains a fractional (g, f)-factor. A fractional (g, f, n)-critical graph is called a fractional (f, n)-critical graph if $g(x)=f(x)$ for every $x \in V(G)$. When f is a constant function taking a value k, a fractional (f, n)-critical graph is a fractional (k, n)-critical graph.

We say that G has all fractional (g, f)-factors if G has a fractional r-factor for every $r: V(G) \rightarrow Z^{+}$ such that $g(x) \leqslant r(x) \leqslant f(x)$ for each $x \in V(G)$. All fractional (g, f)-factors are said to be all fractional $[a, b]$-factors if $g(x)=a$ and $f(x)=b$ for each $x \in V(G)$. A graph G is all fractional (a, b, k)-critical if after deleting any k vertices of G the remaining graph of G has all fractional $[a, b]$-factors.

A graph G is said to be fractional ID-(g, f)-factor-critical if $G-I$ contains a fractional (g, f)-factor for every independent set I of G. If $g(x)=a$ and $f(x)=b$ for each $x \in V(G)$, then a fractional ID-(g, f)-factor-critical graph is simply called a fractional ID-$[a, b]$-factor-critical graph. A fractional ID-(f, f)-factor-critical graph is a fractional ID-f-factor-critical graph. If $f(x) \equiv k$, then we say a fractional ID-k-factor-critical graph instead of a fractional ID-f-factor-critical graph. If $k=1$, then a fractional ID-k-factor-critical graph is called a fractional ID-factor-critical graph.

Many authors have investigated graph factors, and the basic results on graph factors can be found in [1,28].

1.2 Graphic Parameters

The graphic parameters play important roles in the research of (a, b, k)-

critical graphs, fractional (f,n)-critical graphs, fractional (g,f,n)-critical graphs, all fractional (a,b,k)-critical graphs, fractional ID-k-factor-critical graphs, fractional ID-$[a,b]$-factor-critical graphs and fractional ID-(g,f)-factor-critical graphs, they are used frequently as sufficient conditions for the existence of (a,b,k)-critical graphs, fractional (f,n)-critical graphs, fractional (g,f,n)-critical graphs, all fractional (a,b,k)-critical graphs, fractional ID-k-factor-critical graphs, fractional ID-$[a,b]$-factor-critical graphs and fractional ID-(g,f)-factor-critical graphs. Since verifying graphic parameter conditions are often easier than that of characterizations, and as well the parameter conditions reflect the structures and properties of graphs from different perspectives, it is quite common in graph theory to investigate the links among the parameters. In this section, we show some graphic parameters, such as binding number, toughness, isolated toughness, independence number, connectivity and minimum degree.

The minimum degree of G is denoted by $\delta(G)$, i. e., $\delta(G)=\min\{d_G(x):x\in V(G)\}$. The maximum degree of G is denoted by $\Delta(G)$, i. e., $\Delta(G)=\max\{d_G(x):x\in V(G)\}$.

A vertex subset S of G is called independent if $G[S]$ has no edges. An independent set S of G is called a maximum independent set if G excludes a independent set S' with $|S'|>|S|$. The number of vertices in the maximum independent set S of G is called the independence number, and is denoted by $\alpha(G)$.

Let G be a connected graph. $\kappa(G)=\min\{|T|: T\subseteq V(G), G-T$ is disconnected or is a trivial graph$\}$ is called the connectivity of G. $\lambda(G)=\min\{|E_G(S,V(G)\setminus S)|: S\subseteq V(G)\}$ is called the edge-connectivity of G.

The binding number was introduced by Woodall in [26] and is defined as

$$\text{bind}(G)=\min\left\{\frac{|N_G(X)|}{|X|}:\varnothing\neq X\subseteq V(G), N_G(X)\neq V(G)\right\}.$$

Obviously, for any nonempty subset $S\subseteq V(G)$ with $N_G(S)\neq V(G)$, we have $|N_G(S)|\geqslant \text{bind}(G)|S|$.

The number of connected components in G is denoted by $\omega(G)$. The toughness $t(G)$ of a connected graph G was first defined by Chvátal in [7] as follows:

$$t(G)=\min\left\{\frac{|S|}{\omega(G-S)}:S\subseteq V(G), \omega(G-S)\geqslant 2\right\},$$

if G is not complete; otherwise, $t(G)=+\infty$.

Enomoto[8] introduced a new parameter $\tau(G)$ which is a slight variation of toughness, but seems better to fit the research of graph factors and fractional factors. For a connected graph G, we define

$$\tau(G)=\min\left\{\frac{|S|}{\omega(G-S)-1}:S\subseteq V(G),\omega(G-S)\geqslant 2\right\},$$

if G is not complete; otherwise, $\tau(G)=+\infty$.

We use $i(G)$ to denote the number of isolated vertices of G. The isolated toughness $I(G)$ of a graph G is defined by Yang, Ma and Liu[27] as follows:

$$I(G)=\min\left\{\frac{|S|}{i(G-S)}:S\subseteq V(G),i(G-S)\geqslant 2\right\},$$

if G is not complete; otherwise, $I(G)=+\infty$.

Ma and Liu[21] introduced a new parameter $I'(G)$ which is a slight variation of isolated toughness. For a graph G, we define

$$I'(G)=\min\left\{\frac{|S|}{i(G-S)-1}:S\subseteq V(G),i(G-S)\geqslant 2\right\},$$

if G is not complete; otherwise, $I'(G)=+\infty$.

Chapter 2 Graphic Parameter Conditions for Factor Critical Graphs

Let a and b be integers with $1 \leqslant a \leqslant b$. A graph G is called an (a,b,k)-critical graph if after deleting any k vertices of G the remaining graph of G has an $[a,b]$-factor. If G is an (a,b,k)-critical graph, then we also say that G is (a,b,k)-critical. In this chapter, we investigate (a,b,k)-critical graphs which are natural generalizations of $[a,b]$-factors in graphs. This chapter is divided into three parts. First, we show some neighborhood conditions for graphs to be (a,b,k)-critical graphs. Second, we study the relationship between binding number and (a,b,k)-critical graph, and obtain some results on (a,b,k)-critical graphs. Finally, we present some independence number conditions for (a,b,k)-critical graphs.

2.1 Neighborhood Conditions for (a,b,k)-Critical Graphs

Liu and Wang[18] first investigated (a,b,k)-critical graphs, and obtained a necessary and sufficient condition for a graph to be an (a,b,k)-critical graph which is shown in the following.

Theorem 2.1.1[18] Let a,b and k be nonnegative integers with $a<b$, and let G be a graph of order $n \geqslant a+k+1$. Then G is (a,b,k)-critical if and only if for any $S \subseteq V(G)$ with $|S| \geqslant k$

$$\sum_{j=0}^{a-1}(a-j)p_j(G-S) \leqslant b|S|-bk,$$

or

$$\delta_G(S,T)=b|S|+d_{G-S}(T)-a|T| \geqslant bk,$$

where $p_j(G-S)=|\{x:d_{G-S}(x)=j\}|$ and $T=\{x:x\in V(G)\setminus S, d_{G-S}(x)\leqslant a-1\}$.

By using Theorem 2.1.1, we presented some results on (a,b,k)-critical graphs depending on neighborhood conditions. These results are shown in the following.

Theorem 2.1.2[29] Let a, b and k be nonnegative integers such that $2 \leqslant a < b$, and let G be a graph of order n with

$$n \geqslant \frac{(a+b-1)(2a+b-5)+b+1}{b} + \frac{bk}{b-1}.$$

Suppose for any subset $X \subset V(G)$, we have

$$N_G(X) = V(G) \quad \text{if} \quad |X| \geqslant \left\lfloor \frac{[b(n-1)-bk]n}{(a+b-1)(n-1)} \right\rfloor;$$

or

$$|N_G(X)| \geqslant \frac{(a+b-1)(n-1)}{b(n-1)-bk}|X| \quad \text{if} \quad |X| < \left\lfloor \frac{[b(n-1)-bk]n}{(a+b-1)(n-1)} \right\rfloor.$$

Then G is an (a, b, k)-critical graph.

In Theorem 2.1.2, if $k=0$, then we get the following corollary.

Corollary 2.1.1 Let a and b be integers such that $2 \leqslant a < b$, and let G be a graph of order n with

$$n \geqslant \frac{(a+b-1)(2a+b-5)+b+1}{b}.$$

Suppose for any subset $X \subset V(G)$, we have

$$N_G(X) = V(G) \quad \text{if} \quad |X| \geqslant \left\lfloor \frac{bn}{a+b-1} \right\rfloor;$$

or

$$|N_G(X)| \geqslant \frac{a+b-1}{b}|X| \quad \text{if} \quad |X| < \left\lfloor \frac{bn}{a+b-1} \right\rfloor.$$

Then G has an $[a, b]$-factor.

Obviously,

$$\frac{(a+b-1)(2a+b-5)+b+1}{b} \leqslant 6a+b.$$

Then by Corollary 2.1.1, the following result holds. Hence, Theorem 2.1.2 is an improvement and generalization of the following theorem obtained by Kano[11].

Theorem 2.1.3[11] Let a and b be integers such that $2 \leqslant a < b$, and let G be a graph of order n with $n \geqslant 6a+b$. Suppose for any subset $X \subset V(G)$, we have

$$N_G(X) = V(G) \quad \text{if} \quad |X| \geqslant \left\lfloor \frac{bn}{a+b-1} \right\rfloor;$$

or

$$|N_G(X)|\geq\frac{a+b-1}{b}|X| \quad \text{if} \quad |X|<\left\lfloor\frac{bn}{a+b-1}\right\rfloor.$$

Then G has an $[a,b]$-factor.

Lemma 2.1.1 Let G be a graph of order n which satisfies the assumption of Theorem 2.1.2. Then

$$\delta(G)\geq\frac{(a-1)n+b+bk}{a+b-1}.$$

Proof Let u be a vertex of G with degree $\delta(G)$. Let $Y=V(G)\setminus N_G(u)$. Clearly, $u\notin N_G(Y)$, then we have

$$\begin{aligned}(a+b-1)(n-1)|Y|&\leq[b(n-1)-bk]|N_G(Y)|\\&\leq[b(n-1)-bk](n-1),\end{aligned}$$

that is,

$$(a+b-1)|Y|\leq b(n-1)-bk.$$

Since $|Y|=n-\delta(G)$, we get

$$(a+b-1)(n-\delta(G))\leq b(n-1)-bk.$$

Thus, we obtain

$$\begin{aligned}\delta(G)&\geq\frac{(a+b-1)n-[b(n-1)-bk]}{a+b-1}\\&=\frac{(a-1)n+b+bk}{a+b-1}.\end{aligned}$$

This completes the proof of Lemma 2.1.1.

Proof of Theorem 2.1.2 Suppose that a graph G satisfies the condition of Theorem 2.1.2, but is not an (a,b,k)-critical graph. Then by Theorem 2.1.1, there exists a subset S of $V(G)$ with $|S|\geq k$ such that

$$\delta_G(S,T)=b|S|+d_{G-S}(T)-a|T|\leq bk-1, \tag{2.1.1}$$

where $T=\{x:x\in V(G)\setminus S, d_{G-S}(x)\leq a-1\}$. Clearly, $T\neq\varnothing$ by (2.1.1). Define

$$h=\min\{d_{G-S}(t):t\in T\}.$$

In view of the definition of T, we get

$$0\leq h\leq a-1. \tag{2.1.2}$$

We shall consider various cases according to the value of h and derive contradictions.

Case 1 $h=0$.

At first, we prove the following claim.

Claim 1 $\frac{b(n-1)-bk}{n-1}>1$.

Proof Since

$$n\geqslant\frac{(a+b-1)(2a+b-5)+b+1}{b}+\frac{bk}{b-1},$$

then we have

$$\begin{aligned}b(n-1)-bk-(n-1)&=(b-1)n-b-bk+1\\&\geqslant(b-1)\left[\frac{(a+b-1)(2a+b-5)+b+1}{b}+\frac{bk}{b-1}\right]-b-bk+1\\&>(b-1)(2a+b-5)+(b-1)+bk-b-bk+1\\&=(b-1)(2a+b-5)\geqslant0\end{aligned}$$

Thus, we obtain

$$\frac{b(n-1)-bk}{n-1}>1.$$

Claim 1 is proved.

Put $m=|\{t:t\in T,d_{G-S}(t)=0\}|$. Obviously, $m\geqslant1$. Let $Y=V(G)\setminus S$. Then $N_G(Y)\neq V(G)$ since $h=0$. According to the condition of the theorem, we have

$$\begin{aligned}n-m&\geqslant|N_G(Y)|\\&\geqslant\frac{(a+b-1)(n-1)}{b(n-1)-bk}|Y|\\&=\frac{(a+b-1)(n-1)}{b(n-1)-bk}(n-|S|),\end{aligned}$$

that is,

$$|S|\geqslant n-\frac{[b(n-1)-bk](n-m)}{(a+b-1)(n-1)}.\tag{2.1.3}$$

According to $|S|+|T|\leqslant n$ and Claim 1 and (2.1.3), we obtain

$$\begin{aligned}\delta_G(S,T)&=b|S|+d_{G-S}(T)-a|T|\\&\geqslant b|S|-(a-1)|T|-m\\&\geqslant b|S|-(a-1)(n-|S|)-m\\&=(a+b-1)|S|-(a-1)n-m\\&\geqslant(a+b-1)\left\{n-\frac{[b(n-1)-bk](n-m)}{(a+b-1)(n-1)}\right\}-(a-1)n-m\\&=bn-\frac{[b(n-1)-bk](n-m)}{n-1}-m\\&\geqslant bn-\frac{[b(n-1)-bk](n-1)}{n-1}-1\end{aligned}$$

$$=bn-[b(n-1)-bk]-1$$
$$=bk+b-1$$
$$>bk,$$

which contradicts (2.1.1).

Case 2 $h=1$.

Subcase 2.1 $|T|>\left\lfloor\frac{[b(n-1)-bk]n}{(a+b-1)(n-1)}\right\rfloor$.

Obviously,

$$|T|\geqslant\left\lfloor\frac{[b(n-1)-bk]n}{(a+b-1)(n-1)}\right\rfloor+1. \quad (2.1.4)$$

Since $h=1$, then there exists $v\in T$ such that $d_{G-S}(v)=h=1$. Thus, we obtain

$$v\notin N_G(T\setminus N_G(v)). \quad (2.1.5)$$

In view of (2.1.4) and $d_{G-S}(v)=h=1$, we have

$$|T\setminus N_G(v)|\geqslant|T|-1$$
$$\geqslant\left\lfloor\frac{[b(n-1)-bk]n}{(a+b-1)(n-1)}\right\rfloor,$$

which implies that

$$N_G(T\setminus N_G(v))=V(G),$$

which contradicts (2.1.5).

Subcase 2.2 $|T|\leqslant\left\lfloor\frac{[b(n-1)-bk]n}{(a+b-1)(n-1)}\right\rfloor$.

Let $p=|\{t:t\in T,d_{G-S}(t)=1\}|$. Clearly, $p\geqslant1$ and $|T|\geqslant p$. According to $h=1$ and Lemma 2.1.1, we have

$$|S|\geqslant\delta(G)-1$$
$$\geqslant\frac{(a-1)n+b+bk}{a+b-1}-1$$
$$=\frac{(a-1)(n-1)+bk}{a+b-1}. \quad (2.1.6)$$

Suppose that

$$|T|\leqslant\frac{b(n-1)-bk}{a+b-1}.$$

Then we obtain

$$bk-1\geqslant\delta_G(S,T)$$
$$=b|S|+d_{G-S}(T)-a|T|$$
$$\geqslant b|S|+2|T|-p-a|T|$$
$$=b|S|-(a-2)|T|-p$$

$$\geqslant b\frac{(a-1)(n-1)+bk}{a+b-1}-(a-2)\frac{b(n-1)-bk}{a+b-1}-p$$

$$=\frac{b(n-1)-bk+bk(a+b-1)}{a+b-1}-p$$

$$\geqslant bk+|T|-p$$

$$\geqslant bk,$$

which is a contradiction. Hence we may assume

$$|T|>\frac{b(n-1)-bk}{a+b-1}.$$

Then by (2.1.6), we get

$$|S|+|T|>\frac{(a-1)(n-1)+bk}{a+b-1}+\frac{b(n-1)-bk}{a+b-1}$$

$$=n-1.$$

Complying this with $|S|+|T|\leqslant n$, we have

$$|S|+|T|=n. \tag{2.1.7}$$

In view of (2.1.1) and (2.1.7) and

$$|T|\leqslant\left\lfloor\frac{[b(n-1)-bk]n}{(a+b-1)(n-1)}\right\rfloor$$

$$\leqslant\frac{[b(n-1)-bk]n}{(a+b-1)(n-1)},$$

we obtain

$$bk-1\geqslant\delta_G(S,T)$$

$$=b|S|+d_{G-S}(T)-a|T|$$

$$\geqslant b|S|+|T|-a|T|$$

$$=b(n-|T|)-(a-1)|T|$$

$$=bn-(a+b-1)|T|$$

$$\geqslant bn-(a+b-1)\frac{[b(n-1)-bk]n}{(a+b-1)(n-1)}$$

$$=bn-\frac{[b(n-1)-bk]n}{n-1}$$

$$\geqslant bk,$$

which is a contradiction.

Case 3 $2\leqslant h\leqslant a-1$.

By Lemma 2.1.1 and the definition of h, we have

$$|S|\geqslant\delta(G)-h\geqslant\frac{(a-1)n+b+bk}{a+b-1}-h. \tag{2.1.8}$$