I. S. Gradsbteyn
L. M. Ryzbik

SIXTH EDITION

TABLE of INTEGRALS, SERIES, AND PRODUCTS

积分、级数和乘积表

第6版

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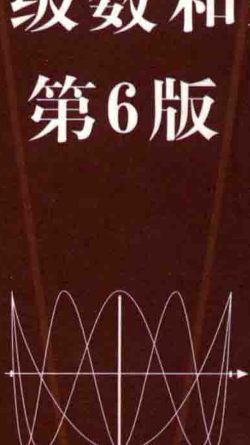

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Elsevier(Singapore)Pte.Ltd
世界图书出版公司

Table of Integrals, Series, and Products

Sixth Edition

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Translated from the Russian by Scripta Technica, Inc.

ACADEMIC PRESS

世界图书出版公司

书　　名：Table of Integrals, Series and Products 6th ed.

作　　者：I. S. Gradshteyn, I. M. Ryzhik

中 译 名：积分、级数和乘积表 第 6 版

出 版 者：世界图书出版公司北京公司

印 刷 者：北京世图印刷厂

发　　行：世界图书出版公司北京公司 (北京朝内大街 137 号　100010)

联系电话：010-64015659, 64038347

电子信箱：kjsk@vip.sina.com

开　　本：24 开　　　印　张：50.5

出版年代：2004 年 4 月

书　　号：7-5062-6546-X/ O・408

版权登记：图字:01-2004-2008

定　　价：198.00 元

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Sixth Edition

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ISBN: 0-12-294757-6

Authorized English language reprint edition published by the Proprietor.

Reprint ISBN: 981-2591-435

Elsevier (Singapore) Pte Ltd.

3 Killiney Road

#08-01 Winsland Hose I

Sinagpore 239519

Tel: (65) 6349-0200

Fax: (65) 6733-1817

Preface to the Sixth Edition

This completely reset sixth edition of Gradshteyn and Ryzhik is a corrected and expanded version of the previous edition. The book was completely reset in order to add more material and to enhance the visual appearance of the material. To preserve compatibility with the previous edition, the original numbering system for entries has been retained. New entries and sections have been inserted in a manner consistent with the original scheme. Whenever possible, new entries and corrections have been checked by means of symbolic computation.

The diverse ways in which corrections have been contributed have made it impossible to attribute them to reference sources that are accessible to users of this book. However, as in previous editions, our indebtedness to these contributors is shown in the form of an acknowledgment list on page xxiii. This list gives the names of those who have written to us directly sending corrections and suggestions for addenda, and added to it are the names of those who have published errata in *Mathematics of Computation.* Certain individuals must be singled out for special thanks due to their significant contributions: Professors H. van Haeringen and L. P. Kok of The Netherlands and Dr. K. S. Kölbig have contributed new material, corrections, and suggestions for new material.

Great care was taken to check the reset version of Gradshteyn and Ryzhik against the fifth edition, both by computer means and hand inspection, but it is inevitable that some transcription errors will remain. These will be rectified in subsequent reprinting of the book as and when they are identified. The authors would appreciate knowledge of any errors or deficiencies, their email addresses are listed below. If any errata are found, they will be posted on the web site `www.az-tec.com/gr/errata`.

As in the previous edition, a numerical superscript added to an entry number is used to indicate either a new addition to the book or a correction. When an entire section is new, an asterisk has been added only to the section heading. Continuing the previous convention, a superscript 10 has been added to entry numbers to indicate the most recent changes that have been made.

This latest version of Gradshteyn and Ryzhik also forms the source for a revised electronic version of Gradshteyn and Ryzhik.

Special thanks are extended to John Law and the Newcastle University Computing Service for their help in the preparation of this edition.

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Acknowledgments

The publisher and editors would like to take this opportunity to express their gratitude to the following users of the *Table of Integrals, Series, and Products* who either directly or through errata published in *Mathematics of Computation* have generously contributed corrections and addenda to the original printing.

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The order of presentation of the formulas

The question of the most expedient order in which to give the formulas, in particular, in what division to include particular formulas such as the definite integrals, turned out to be quite complicated. The thought naturally occurs to set up an order analogous to that of a dictionary. However, it is almost impossible to create such a system for the formulas of integral calculus. Indeed, in an arbitrary formula of the form

$$\int_a^b f(x)\,dx = A$$

one may make a large number of substitutions of the form $x = \varphi(t)$ and thus obtain a number of "synonyms" of the given formula. We must point out that the table of definite integrals by Bierens de Haan and the earlier editions of the present reference both sin in the plethora of such "synonyms" and formulas of complicated form. In the present edition, we have tried to keep only the simplest of the "synonym" formulas. Basically, we judged the simplicity of a formula from the standpoint of the simplicity of the arguments of the "outer" functions that appear in the integrand. Where possible, we have replaced a complicated formula with a simpler one. Sometimes, several complicated formulas were thereby reduced to a single simpler one. We then kept only the simplest formula. As a result of such substitutions, we sometimes obtained an integral that could be evaluated by use of the formulas of chapter two and the Newton–Leibniz formula, or to an integral of the form

$$\int_{-a}^{a} f(x)\,dx,$$

where $f(x)$ is an odd function. In such cases the complicated integrals have been omitted.

Let us give an example using the expression

$$\int_0^{\pi/4} \frac{(\cot x - 1)^{p-1}}{\sin^2 x} \ln \tan x\,dx = -\frac{\pi}{p}\operatorname{cosec} p\pi. \tag{1}$$

By making the natural substitution $u = \cot x - 1$, we obtain

$$\int_0^{\infty} u^{p-1} \ln(1+u)\,du = \frac{\pi}{p}\operatorname{cosec} p\pi. \tag{2}$$

Integrals similar to formula (1) are omitted in this new edition. Instead, we have formula (2).

As a second example, let us take

$$I = \int_0^{\pi/2} \ln\left(\tan^p x + \cot^p x\right) \ln \tan x \, dx = 0.$$

The substitution $u = \tan x$ yields

$$I = \int_0^{\infty} \frac{\ln\left(u^p + u^{-p}\right) \ln u}{1 + u^2} \, du.$$

If we now set $v = \ln u$, we obtain

$$I = \int_{-\infty}^{\infty} \frac{v e^v}{1 + e^{2v}} \ln\left(e^{pv} + e^{-pv}\right) \, dv = \int_{-\infty}^{\infty} v \, \frac{\ln 2 \cosh pv}{2 \cosh v} \, dv.$$

The integrand is odd and, consequently, the integral is equal to 0.

Thus, before looking for an integral in the tables, the user should simplify as much as possible the arguments (the "inner" functions) of the functions in the integrand.

The functions are ordered as follows: First we have the elementary functions:

1. The function $f(x) = x$.
2. The exponential function.
3. The hyperbolic functions.
4. The trigonometric functions.
5. The logarithmic function.
6. The inverse hyperbolic functions. (These are replaced with the corresponding logarithms in the formulas containing definite integrals.)
7. The inverse trigonometric functions.

Then follow the special functions:

8. Elliptic integrals.
9. Elliptic functions.
10. The logarithm integral, the exponential integral, the sine integral, and the cosine integral functions.
11. Probability integrals and Fresnel's integrals.
12. The gamma function and related functions.
13. Bessel functions.
14. Mathieu functions.
15. Legendre functions.
16. Orthogonal polynomials.
17. Hypergeometric functions.
18. Degenerate hypergeometric functions.
19. Parabolic cylinder functions.
20. Meijer's and MacRobert's functions.
21. Riemann's zeta function.

The integrals are arranged in order of outer function according to the above scheme: the farther down in the list a function occurs, (i.e., the more complex it is) the later will the corresponding formula appear in the tables. Suppose that several expressions have the same outer function. For example, consider $\sin e^x$, $\sin x$, $\sin \ln x$. Here, the outer function is the sine function in all three cases. Such expressions are then arranged in order of the inner function. In the present work, these functions are therefore arranged in the following order: $\sin x$, $\sin e^x$, $\sin \ln x$.

Our list does not include polynomials, rational functions, powers, or other algebraic functions. An algebraic function that is included in tables of definite integrals can usually be reduced to a finite combination of roots of rational power. Therefore, for classifying our formulas, we can conditionally treat a power function as a generalization of an algebraic and, consequently, of a rational function.* We shall distinguish between all these functions and those listed above and we shall treat them as operators. Thus, in the expression $\sin^2 e^x$, we shall think of the squaring operator as applied to the outer function, namely, the sine. In the expression $\frac{\sin x+\cos x}{\sin x-\cos x}$, we shall think of the rational operator as applied to the trigonometric functions sine and cosine. We shall arrange the operators according to the following order:

1. Polynomials (listed in order of their degree).
2. Rational operators.
3. Algebraic operators (expressions of the form $A^{p/q}$, where q and p are rational, and $q > 0$; these are listed according to the size of q).
4. Power operators.

Expressions with the same outer and inner functions are arranged in the order of complexity of the operators. For example, the following functions (whose outer functions are all trigonometric, and whose inner functions are all $f(x) = x$) are arranged in the order shown:

$$\sin x, \quad \sin x \cos x, \quad \frac{1}{\sin x} = \operatorname{cosec} x, \quad \frac{\sin x}{\cos x} = \tan x, \quad \frac{\sin x + \cos x}{\sin x - \cos x}, \quad \sin^m x, \quad \sin^m x \cos x.$$

Furthermore, if two outer functions $\varphi_1(x)$ and $\varphi_2(x)$, where $\varphi_1(x)$ is more complex than $\varphi_2(x)$, appear in an integrand and if any of the operations mentioned are performed on them, the corresponding integral will appear (in the order determined by the position of $\varphi_2(x)$ in the list) after all integrals containing only the function $\varphi_1(x)$. Thus, following the trigonometric functions are the trigonometric and power functions (that is, $\varphi_2(x) = x$). Then come

- combinations of trigonometric and exponential functions,
- combinations of trigonometric functions, exponential functions, and powers, etc.,
- combinations of trigonometric and hyperbolic functions, etc.

Integrals containing two functions $\varphi_1(x)$ and $\varphi_2(x)$ are located in the division and order corresponding to the more complicated function of the two. However, if the positions of several integrals coincide because they contain the same complicated function, these integrals are put in the position defined by the complexity of the second function.

To these rules of a general nature, we need to add certain particular considerations that will be easily understood from the tables. For example, according to the above remarks, the function $e^{\frac{1}{x}}$ comes after e^x as regards complexity, but $\ln x$ and $\ln \frac{1}{x}$ are equally complex since $\ln \frac{1}{x} = -\ln x$. In the section on "powers and algebraic functions", polynomials, rational functions, and powers of powers are formed from power functions of the form $(a + bx)^n$ and $(\alpha + \beta x)^\nu$.

*For any natural number n, the involution $(a + bx)^n$ of the binomial $a + bx$ is a polynomial. If n is a negative integer, $(a + bx)^n$ is a rational function. If n is irrational, the function $(a + bx)^n$ is not even an algebraic function.

Use of the tables*

For the effective use of the tables contained in this book it is necessary that the user should first become familiar with the classification system for integrals devised by the authors Ryzhik and Gradshteyn. This classification is described in detail in the section entitled *The Order of Presentation of the Formulas* (see page xxvii) and essentially involves the separation of the integrand into *inner* and *outer* functions. The principal function involved in the integrand is called the *outer* function and its argument, which is itself usually another function, is called the *inner* function. Thus, if the integrand comprised the expression $\ln \sin x$, the *outer* function would be the logarithmic function while its argument, the *inner* function, would be the trigonometric function $\sin x$. The desired integral would then be found in the section dealing with logarithmic functions, its position within that section being determined by the position of the *inner* function (here a trigonometric function) in Ryzhik and Gradshteyn's list of functional forms.

It is inevitable that some duplication of symbols will occur within such a large collection of integrals and this happens most frequently in the first part of the book dealing with algebraic and trigonometric integrands. The symbols most frequently involved are α, β, γ, δ, t, u, z, z_k, and Δ. The expressions associated with these symbols are used consistently within each section and are defined at the start of each new section in which they occur. Consequently, reference should be made to the beginning of the section being used in order to verify the meaning of the substitutions involved.

Integrals of algebraic functions are expressed as combinations of roots with rational power indices, and definite integrals of such functions are frequently expressed in terms of the Legendre elliptic integrals $F(\phi, k)$, $E(\phi, k)$ and $\Pi(\phi, n, k)$, respectively, of the first, second and third kinds.

The four inverse hyperbolic functions $\operatorname{arcsinh} z$, $\operatorname{arccosh} z$, $\operatorname{arctanh} z$ and $\operatorname{arccoth} z$ are introduced through the definitions

$$\arcsin z = \frac{1}{i} \operatorname{arcsinh}(iz)$$

$$\arccos z = \frac{1}{i} \operatorname{arccosh}(z)$$

$$\arctan z = \frac{1}{i} \operatorname{arctanh}(iz)$$

$$\operatorname{arccot} z = i \operatorname{arccoth}(iz)$$

*Prepared by Alan Jeffrey for the English language edition.

or

$$\begin{aligned}
\operatorname{arcsinh} z &= \frac{1}{i}\arcsin(iz) \\
\operatorname{arccosh} z &= i\arccos z \\
\operatorname{arctanh} z &= \frac{1}{i}\arctan(iz) \\
\operatorname{arccoth} z &= \frac{1}{i}\operatorname{arccot}(-iz)
\end{aligned}$$

The numerical constants C and G which often appear in the definite integrals denote Euler's constant and Catalan's constant, respectively. Euler's constant C is defined by the limit

$$C = \lim_{s\to\infty}\left(\sum_{m=1}^{s}\frac{1}{m} - \ln s\right) = 0.577215\ldots.$$

On occasions other writers denote Euler's constant by the symbol γ, but this is also often used instead to denote the constant

$$\gamma = e^{C} = 1.781072\ldots.$$

Catalan's constant G is related to the complete elliptic integral

$$\mathbf{K} \equiv \mathbf{K}(k) \equiv \int_0^{\pi/2}\frac{da}{\sqrt{1-k^2\sin^2 a}}$$

by the expression

$$G = \frac{1}{2}\int_0^1 \mathbf{K}\,dk = \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2} = 0.915965\ldots.$$

Since the notations and definitions for higher transcendental functions that are used by different authors are by no means uniform, it is advisable to check the definitions of the functions that occur in these tables. This can be done by identifying the required function by symbol and name in the *Index of Special Functions and Notation* on page xxxix, and by then referring to the defining formula or section number listed there. We now present a brief discussion of some of the most commonly used alternative notations and definitions for higher transcendental functions.

Bernoulli and Euler Polynomials and Numbers

Extensive use is made throughout the book of the Bernoulli and Euler numbers B_n and E_n that are defined in terms of the Bernoulli and Euler polynomials of order n, $B_n(x)$ and $E_n(x)$, respectively. These polynomials are defined by the generating functions

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \qquad \text{for } |t| < 2\pi$$

and

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \qquad \text{for } |t| < \pi.$$

The Bernoulli numbers are always denoted by B_n and are defined by the relation

$$B_n = B_n(0) \qquad \text{for } n = 0, 1, \ldots,$$

when

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \ldots.$$

The Euler numbers E_n are defined by setting

$$E_n = 2^n E_n\left(\frac{1}{2}\right) \qquad \text{for } n = 0, 1, \ldots.$$

The E_n are all integral and $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$, ...

An alternative definition of Bernoulli numbers, which we shall denote by the symbol B_n^*, uses the same generating function but identifies the B_n^* differently in the following manner:

$$\frac{t}{e^t - 1} = 1 - \frac{1}{2}t + B_1^* \frac{t^2}{2!} - B_2^* \frac{t^4}{4!} + \ldots.$$

This definition then gives rise to the alternative set of Bernoulli numbers

$$B_1^* = 1/6, \quad B_2^* = 1/30, \quad B_3^* = 1/42, \quad B_4^* = 1/30, \quad B_5^* = 5/66,$$
$$B_6^* = 691/2730, \quad B_7^* = 7/6, \quad B_8^* = 3617/510, \quad \ldots.$$

These differences in notation must also be taken into account when using the following relationships that exist between the Bernoulli and Euler polynomials:

$$B_n(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} B_{n-k}\, E_k(2x) \qquad n = 0, 1, \ldots$$

$$E_{n-1}(x) = \frac{2^n}{n}\left\{B_n\left(\frac{x+1}{2}\right) - B_n\left(\frac{x}{2}\right)\right\}$$

or

$$E_{n-1}(x) = \frac{2}{n}\left\{B_n(x) - 2^n B_n\left(\frac{x}{2}\right)\right\} \qquad n = 1, 2, \ldots$$

and

$$E_{n-2}(x) = 2\binom{n}{2}^{-1} \sum_{k=0}^{n-2} \binom{n}{k} \left(2^{n-k} - 1\right) B_{n-k}\, B_n(x) \qquad n = 2, 3, \ldots$$

There are also alternative definitions of the Euler polynomial of order n, and it should be noted that some authors, using a modification of the third expression above, call

$$\left(\frac{2}{n+1}\right)\left\{B_n(x) - 2^n B_n\left(\frac{x}{2}\right)\right\}$$

the Euler polynomial of order n.

Elliptic Functions and Elliptic Integrals

The following notations are often used in connection with the inverse elliptic functions $\operatorname{sn} u$, $\operatorname{cn} u$, and $\operatorname{dn} u$:

$$\operatorname{ns} u = \frac{1}{\operatorname{sn} u} \qquad \operatorname{nc} u = \frac{1}{\operatorname{cn} u} \qquad \operatorname{nd} u = \frac{1}{\operatorname{dn} u}$$

$$\operatorname{sc} u = \frac{\operatorname{sn} u}{\operatorname{cn} u} \qquad \operatorname{cs} u = \frac{\operatorname{cn} u}{\operatorname{sn} u} \qquad \operatorname{ds} u = \frac{\operatorname{dn} u}{\operatorname{sn} u}$$

$$\operatorname{sd} u = \frac{\operatorname{sn} u}{\operatorname{dn} u} \qquad \operatorname{cd} u = \frac{\operatorname{cn} u}{\operatorname{dn} u} \qquad \operatorname{dc} u = \frac{\operatorname{dn} u}{\operatorname{cn} u}$$

The elliptic integral of the third kind is defined by Ryzhik and Gradshteyn to be

$$\Pi\left(\varphi, n^2, k\right) = \int_0^{\varphi} \frac{da}{\left(1 - n^2 \sin^2 a\right)\sqrt{1 - k^2 \sin^2 a}} = \int_0^{\sin\varphi} \frac{dx}{\left(1 - n^2 x^2\right)\sqrt{\left(1 - x^2\right)\left(1 - k^2 x^2\right)}} \qquad \left(-\infty < n^2 < \infty\right)$$

The Jacobi Zeta Function and Theta Functions

The Jacobi zeta function $\operatorname{zn}(u,k)$, frequently written $Z(u)$, is defined by the relation

$$\operatorname{zn}(u,k) = Z(u) = \int_0^u \left\{\operatorname{dn}^2 v - \frac{E}{K}\right\} dv = E(u) - \frac{E}{K}u.$$

This is related to the theta functions by the relationship

$$\operatorname{zn}(u,k) = \frac{\partial}{\partial u}\ln\Theta(u)$$

giving

(i). $$\operatorname{zn}(u,k) = \frac{\pi}{2K}\frac{\vartheta_1'\left(\frac{\pi u}{2K}\right)}{\vartheta_1\left(\frac{\pi u}{2K}\right)} - \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u}$$

(ii). $$\operatorname{zn}(u,k) = \frac{\pi}{2K}\frac{\vartheta_2'\left(\frac{\pi u}{2K}\right)}{\vartheta_2\left(\frac{\pi u}{2K}\right)} - \frac{\operatorname{dn} u \operatorname{sn} u}{\operatorname{cn} u}$$

(iii). $$\operatorname{zn}(u,k) = \frac{\pi}{2K}\frac{\vartheta_3'\left(\frac{\pi u}{2K}\right)}{\vartheta_3\left(\frac{\pi u}{2K}\right)} - k^2\frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u}$$

(iv). $$\operatorname{zn}(u,k) = \frac{\pi}{2K}\frac{\vartheta_4'\left(\frac{\pi u}{2K}\right)}{\vartheta_4\left(\frac{\pi u}{2K}\right)}$$

Many different notations for the theta function are in current use. The most common variants are the replacement of the argument u by the argument u/π and, occasionally, a permutation of the identification of the functions ϑ_1 to ϑ_4 with the function ϑ_4 replaced by ϑ.

The Factorial (Gamma) Function

In older reference texts the gamma function $\Gamma(z)$, defined by the Euler integral

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt,$$

is sometimes expressed in the alternative notation

$$\Gamma(1+z) = z! = \Pi(z).$$

On occasions the related derivative of the logarithmic factorial function $\Psi(z)$ is used where

$$\frac{d\,(\ln z!)}{dz} = \frac{(z!)'}{z!} = \Psi(z).$$

This function satisfies the recurrence relation

$$\Psi(z) = \Psi(z-1) + \frac{1}{z-1}$$

and is defined by the series

$$\Psi(z) = -C + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{z+n}\right).$$

The derivative $\Psi'(z)$ satisfies the recurrence relation

$$-\Psi'(z-1) = -\Psi'(z) + \frac{1}{z^2}$$

and is defined by the series

$$\Psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}.$$

Exponential and Related Integrals

The exponential integrals $E_n(z)$ have been defined by Schloemilch using the integral

$$E_n(z) = \int_1^{\infty} e^{-zt} t^{-n}\, dt \qquad (n = 0, 1, \ldots, \quad \operatorname{Re} z > 0)$$

They should not be confused with the Euler polynomials already mentioned. The function $E_1(z)$ is related to the exponential integral $\operatorname{Ei}(z)$ through the expressions

$$E_1(z) = -\operatorname{Ei}(-z) = \int_z^{\infty} e^{-t} t^{-1}\, dt$$

and

$$\operatorname{li}(z) = \int_0^z \frac{dt}{\ln t} = \operatorname{Ei}(\ln z) \qquad (z > 1).$$

The functions $E_n(z)$ satisfy the recurrence relations

$$E_n(z) = \frac{1}{n-1}\left\{e^{-z} - z\, E_{n-1}(z)\right\} \qquad (n > 1)$$

and

$$E_n'(z) = -\, E_{n-1}(z)$$

with

$$E_0(z) = e^{-z}/z.$$

The function $E_n(z)$ has the asymptotic expansion

$$E_n(z) \sim \frac{e^{-z}}{z}\left\{1 - \frac{n}{z} + \frac{n(n+1)}{z^2} - \frac{n(n+1)(n+2)}{z^3} + \cdots\right\} \qquad \left(|\arg z| < \frac{3\pi}{2}\right)$$

while for large n,

$$E_n(x) = \frac{e^{-x}}{x+n}\left\{1 + \frac{n}{(x+n)^2} + \frac{n(n-2x)}{(x+n)^4} + \frac{n\left(6x^2 - 8nx + n^2\right)}{(x+n)^6} + R(n,x)\right\},$$

where

$$-0.36n^{-4} \le R(n,x) \le \left(1 + \frac{1}{x+n-1}\right) n^{-4} \qquad (x > 0)$$

The sine and cosine integrals $\operatorname{si}(x)$ and $\operatorname{ci}(x)$ are related to the functions $\operatorname{Si}(x)$ and $\operatorname{Ci}(x)$ by the integrals

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt = \operatorname{si}(x) + \frac{\pi}{2}$$

and

$$\operatorname{Ci}(x) = C + \ln x + \int_0^x \frac{(\cos t - 1)}{t}\, dt.$$

The hyperbolic sine and cosine integrals $\operatorname{shi}(x)$ and $\operatorname{chi}(x)$ are defined by the relations

$$\operatorname{shi}(x) = \int_0^x \frac{\sinh t}{t}\, dt$$

and