METHODS OF MODERN MATHEMATICAL PHYSICS

# III
# Scattering Theory

## 现代数学物理方法

第3卷

**Michael Reed / Barry Simon**

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# METHODS OF
# MODERN MATHEMATICAL PHYSICS

## III: SCATTERING THEORY

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*To Martha and Jackie*

# Preface

In the preparation of this volume we were fortunate to receive advice from C. Berning, P. Deift, V. Enss, G. Hagedorn, J. Holder, T. Ikebe, M. Klaus, S. Kuroda, J. Morgan III, S. Pinault, J. Rauch, S. Ruijsenaars, and L. Smith. We are grateful to these individuals and others whose comments made this book better.

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# Introduction

Scattering theory is the study of an interacting system on a scale of time and/or distance which is large compared to the scale of the interaction itself. As such, it is the most effective means, sometimes the only means, to study microscopic nature. To understand the importance of scattering theory, consider the variety of ways in which it arises. First, there are various phenomena in nature (like the blue of the sky) which are the result of scattering. In order to understand the phenomenon (and to identify it as the result of scattering) one must understand the underlying dynamics and its scattering theory. Second, one often wants to use the scattering of waves or particles whose dynamics one knows to determine the structure and position of small or inaccessible objects. For example, in x-ray crystallography (which led to the discovery of DNA), tomography, and the detection of underwater objects by sonar, the underlying dynamics is well understood. What one would like to construct are correspondences that link, via the dynamics, the position, shape, and internal structure of the object to the scattering data. Ideally, the correspondence should be an explicit formula which allows one to reconstruct, at least approximately, the object from the scattering data. A third use of scattering theory is as a probe of dynamics itself. In elementary particle physics, the underlying dynamics is not well understood and essentially all the experimental data are scattering data. The main test of any proposed particle dynamics is whether one can construct for the dynamics a scattering theory that predicts the observed experimental data. Scattering theory was not always so central to physics. Even though the Coulomb cross section could have been computed by Newton, had he bothered to ask the right question, its calculation is generally attributed to Rutherford more than two hundred years later. Of course, Rutherford's calculation was in connection with the first experiment in nuclear physics.

Scattering theory is so important for atomic, condensed matter, and high energy physics that an enormous physics literature has grown up. Unfortunately, the development of the associated mathematics has been much slower. This is partially because the mathematical problems are hard but also because lack of communication often made it difficult for mathematicians to appreciate the many beautiful and challenging problems in scattering theory. The physics literature, on the other hand, is not entirely satisfactory because of the many heuristic formulas and ad hoc methods. Much of the physics literature deals with the "time-independent" approach to scattering theory because the time-independent approach provides powerful calculational tools. We feel that to use the time-independent formulas one must understand them in terms of and derive them from the underlying dynamics. Therefore, in this book we emphasize scattering theory as a time-dependent phenomenon, in particular, as a comparison between the interacting and free dynamics. This approach leads to a certain imbalance in our presentation since we therefore emphasize large times rather than large distances. However, as the reader will see, there is considerable geometry lurking in the background.

The scattering theories in branches of physics as different as classical mechanics, continuum mechanics, and quantum mechanics, have in common the two foundational questions of the existence and completeness of the wave operators. These two questions are, therefore, our main object of study in individual systems and are the unifying theme that runs throughout the book. Because we treat so many different systems, we do not carry the analysis much beyond the construction and completeness of the wave operators, except in two-body quantum scattering, which we develop in some detail. However, even there, we have not been able to include such important topics as Regge theory, inverse scattering, and double dispersion relations.

Since quantum mechanics is a linear theory, it is not surprising that the heart of the mathematical techniques is the spectral analysis of Hamiltonians. Bound states (corresponding to point spectra) of the interaction Hamiltonian do not scatter, while states from the absolutely continuous spectrum do. The mathematical property that distinguishes these two cases (and that connects the physical intuition with the mathematical formulation) is the decay of the Fourier transform of the corresponding spectral measures. The case of singular continuous spectrum lies between and the crucial (and often hardest) step in most proofs of asymptotic completeness is the proof that the interacting Hamiltonian has no singular continuous spectrum. Conversely, one of the best ways of showing that a self-adjoint operator has no singular continuous spectrum is to show that it is the interaction Hamiltonian of a quantum system with complete wave operators. This deep connection between scattering theory and spectral analysis shows the artificiality of the division of material into Volumes III and IV. We have, therefore, preprinted at the end of this volume three sections on the absence of continuous singular spectrum from Volume IV.

While we were reading the galley proofs for this volume, V. Enss introduced new and beautiful methods into the study of quantum-mechanical scattering. Enss's paper is not only of interest for what it proves, but also for the future direction that it suggests. In particular, it seems likely that the methods will provide strong results in the theory of multiparticle scattering. We have added a section at the end of this Chapter (Section XI.17) to describe Enss's method in the two-body case. We would like to thank Professor Enss for his generous attitude, which helped us to include this material.

The general remarks about notes and problems made in earlier introductions are applicable here with one addition: the bulk of the material presented in this volume is from advanced research literature, so many of the problems are quite substantial. Some of the starred problems summarize the contents of research papers!

# Contents of Other Volumes

**Volume I: Functional Analysis**

I *Preliminaries*
II *Hilbert Spaces*
III *Banach Spaces*
IV *Topological Spaces*
V *Locally Convex Spaces*
VI *Bounded Operators*
VII *The Spectral Theorem*
VIII *Unbounded Operators*

**Volume II: Fourier Analysis, Self-Adjointness**

IX *The Fourier Transform*
X *Self-Adjointness and the Existence of Dynamics*

**Volume IV: Analysis of Operators**

XII *Perturbation of Point Spectra*
XIII *Spectral Analysis*

**Contents of Future Volumes:** *Convex Sets and Functions, Commutative Banach Algebras, Introduction to Group Representations, Operator Algebras, Applications of Operator Algebras to Quantum Field Theory and Statistical Mechanics, Probabilistic Methods*

# Contents


| | |
|---|---|
| *Preface* | *vii* |
| *Introduction* | *ix* |
| *Contents of Other Volumes* | *xv* |

**XI: SCATTERING THEORY**

| | |
|---|---|
| 1. *An overview of scattering phenomena* | *1* |
| 2. *Classical particle scattering* | *5* |
| 3. *The basic principles of scattering in Hilbert space* | *16* |
| *Appendix 1 Stationary phase methods* | *37* |
| *Appendix 2 Trace ideal properties of* $f(x)g(-i\nabla)$ | *47* |
| *Appendix 3 A general invariance principle for wave operators* | *49* |
| 4. *Quantum scattering I: Two-body case* | *54* |
| 5. *Quantum scattering II: N-body case* | *75* |
| 6. *Quantum scattering III: Eigenfunction expansions* | *96* |
| *Appendix Introduction to eigenfunction expansions by the auxiliary space method* | *112* |
| 7. *Quantum scattering IV: Dispersion relations* | *116* |
| 8. *Quantum scattering V: Central potentials* | *121* |
| A. *Reduction of the S-matrix by symmetries* | *121* |
| B. *The partial wave expansion and its convergence* | *127* |
| C. *Phase shifts and their connection to the Schrödinger equation* | *129* |
| D. *The variable phase equation* | *133* |
| E. *Jost functions and Levinson's theorem* | *136* |
| F. *Analyticity of the partial wave amplitude for generalized Yukawa potentials* | *143* |
| G. *The Kohn variational principle* | *147* |

*Appendix 1* *Legendre polynomials and spherical Bessel functions* 149
*Appendix 2* *Jost solutions for oscillatory potentials* 155
*Appendix 3* *Jost solutions and the fundamental problems of scattering theory* 164
9. *Long-range potentials* 169
10. *Optical and acoustical scattering I: Schrödinger operator methods* 184
*Appendix* *Trace class properties of Green's functions* 203
11. *Optical and acoustical scattering II: The Lax–Phillips method* 210
*Appendix* *The twisting trick* 241
12. *The linear Boltzmann equation* 243
13. *Nonlinear wave equations* 252
*Appendix* *Conserved currents* 278
14. *Spin wave scattering* 285
15. *Quantum field scattering I: The external field* 293
16. *Quantum field scattering II: The Haag–Ruelle theory* 317
17. *Phase space analysis of scattering and spectral theory* 331
*Appendix* *The RAGE theorem* 340
*Notes* 344
*Notes on scattering theory on C*-algebras* 382
*Problems* 385


## MATERIAL PREPRINTED FROM VOLUME IV


*XIII.6* *The absence of singular continuous spectrum I: General theory* 406
*XIII.7* *The absence of singular continuous spectrum II: Smooth perturbations* 411
A. *Weakly coupled quantum systems* 421
B. *Positive commutators and repulsive potentials* 427
C. *Local smoothness and wave operators for repulsive potentials* 433
*XIII.8* *The absence of singular continuous spectrum III: Weighted $L^2$ spaces* 438
*Notes* 447
*Problems* 450

*List of Symbols* 455
*Index* 457

# XI: Scattering Theory

*It is notoriously difficult to obtain reliable results for quantum mechanical scattering problems. Since they involve complicated interference phenomena of waves, any simple uncontrolled approximation is not worth more than the weather forecast. However, for two body problems with central forces the computer can be used to calculate the phase shifts . . . .* *W. Thirring*

## XI.1 An overview of scattering phenomena

In this chapter we shall discuss scattering in a variety of physical situations. Our main goal is to illustrate the underlying similarities between the large time behavior of many kinds of dynamical systems. We study the case of nonrelativistic quantum scattering in great detail. Other systems we treat to a lesser extent, emphasizing simple examples.

Scattering normally involves a comparison of two different dynamics for the same system: the given dynamics and a "free" dynamics. It is hard to give a precise definition of "free dynamics" which will cover all the cases we consider, although we shall give explicit definitions in each individual case. The characteristics that these free dynamical systems have in common are that they are simpler than the given dynamics and generally they conserve the momentum of the "individual constituents" of the physical system. It is important to bear in mind that scattering involves more than just the interacting dynamics since certain features of the results will seem strange otherwise. Because two dynamics are involved, scattering theory can be viewed as a branch of perturbation theory. In the quantum-mechanical case we shall see that the perturbation theory of the absolutely continuous spectrum is involved rather than the perturbation theory of the discrete spectrum discussed in Chapter XII.

Scattering as a perturbative phenomenon emphasizes temporal asymptotics, and this is the approach we shall generally follow. But all the *concrete* examples we discuss will also have a geometric structure present and there is clearly lurking in the background a theory of scattering as correlations between spatial and temporal asymptotics. This is an approach we shall not explicitly develop, in part because it has been discussed to a much lesser degree. We do note that all the "free" dynamics we discuss have "straight-line motion" in the sense that solutions of the free equations which are concentrated as $t \to -\infty$ in some neighborhood of the direction $\mathbf{n}$ are concentrated as $t \to +\infty$ in a neighborhood of the direction $-\mathbf{n}$. These geometric ideas are useful for understanding the choice of free dynamics in Sections 14 and 16 where a piece of the interacting dynamics generates the free dynamics. And clearly, the geometric ideas are brought to the fore in the Lax–Phillips theory (Section 11) and in Enss's method (Section 17).

Scattering theory involves studying certain states of an interacting system, namely those states that appear to be "asymptotically free" in the distant past and/or the distant future. To be explicit, suppose that we can view the dynamics as transformations acting on the states. Let $T_t$ and $T_t^{(0)}$ stand for the interacting and free dynamical transformations on the "set of states" $\Sigma$. $\Sigma$ may be points in a phase space (classical mechanics), vectors in a Hilbert space (quantum mechanics), or Cauchy data for some partial differential equation (acoustics, optics). One is interested in pairs $\langle \rho_-, \rho \rangle \in \Sigma$ so that

$$\lim_{t \to -\infty} (T_t \rho - T_t^{(0)} \rho_-) = 0$$

for some appropriate sense of limit, and similarly for pairs that approach each other as $t \to +\infty$. One requirement that one must make on the notion of limit is that for each $\rho$ there should be at most one $\rho_-$.

The basic questions of scattering theory are the following:

(1) *Existence of scattering states* Physically, one prepares the interacting system in such a way that some of the constituents are so far from one another that the interaction between them is negligible. One then "lets go," that is, allows the interacting dynamics to act for a long time and then looks at what has happened. One usually describes the initial state in terms of the variables natural to describe free states, often momenta. One expects that any free state "can be prepared," that is, for any $\rho_- \in \Sigma$, there is a $\rho \in \Sigma$ with $\lim_{t \to -\infty} T_t \rho - T_t^{(0)} \rho_- = 0$. Proving this is the basic existence question of scattering.

(2) *Uniqueness of scattering states* In order to describe the prepared state in terms of free states, one must know that each free state is associated with a unique interacting state; that is, given $\rho_-$ there is at most one $\rho$ such that $T_t^{(0)}\rho_- - T_t\rho \to 0$ as $t \to -\infty$. Notice that this is distinct from the requirement on the limit above that there should be at most one $\rho_-$ for each $\rho$.

(3) *Weak asymptotic completeness* Suppose that one has an interacting state $\rho$ that looked like a free state in the distant past in the sense that $\lim_{t\to-\infty} T_t^{(0)}\rho_- - T_t\rho = 0$ for some state $\rho_-$. One hopes that for large positive times, the interacting state will again look like a free state in the sense that there exists a state $\rho_+$ so that $\lim_{t\to+\infty} T_t^{(0)}\rho_+ - T_t\rho = 0$. In order to prove this, one needs to show that the two subsets of $\Sigma$

$$\Sigma_{\text{in}} = \left\{\rho \in \Sigma \mid \exists\rho_- \in \Sigma \text{ with } \lim_{t\to-\infty} T_t^{(0)}\rho_- - T_t\rho = 0\right\}$$

and

$$\Sigma_{\text{out}} = \left\{\rho \in \Sigma \mid \exists\rho_+ \in \Sigma \text{ with } \lim_{t\to+\infty} T_t^{(0)}\rho_+ - T_t\rho = 0\right\}$$

are equal. If in fact $\Sigma_{\text{in}} = \Sigma_{\text{out}}$, then the system is said to have **weak asymptotic completeness.**

(4) *Definition of the S-transformation* If one has a pair of dynamical systems $\langle T_t^{(0)}, T_t\rangle$ for which one can prove existence and uniqueness of scattering states (both as $t \to -\infty$ and as $t \to \infty$) and for which weak asymptotic completeness holds, then one can define a natural bijection of $\Sigma$ onto itself. Given $\rho \in \Sigma$, existence and uniqueness of scattering states assures us that there exists a state $\Omega^+\rho \in \Sigma_{\text{in}}$ with $\lim_{t\to-\infty} (T_t(\Omega^+\rho) - T_t^{(0)}\rho) = 0$. Similarly, $\Omega^-$ is defined by $\lim_{t\to+\infty} (T_t(\Omega^-\rho) - T_t^{(0)}\rho) = 0$. $\Omega^+$ (respectively, $\Omega^-$) is a bijection from $\Sigma$ onto $\Sigma_{\text{in}}$ (respectively, $\Sigma_{\text{out}}$). Weak asymptotic completeness assures us that $\Sigma_{\text{in}} = \Sigma_{\text{out}}$, so one can define the *bijection*

$$S = (\Omega^-)^{-1}\Omega^+ : \Sigma \to \Sigma$$

$S$ is called the **scattering transformation.** Thus, $T_t^{(0)}(S\rho)$ and $T_t^{(0)}\rho$ are related by the condition that there exists a state $\psi$ ($\psi = \Omega^+\rho = \Omega^-(S\rho)$) so that $T_t\psi$ "interpolates" between them. That is, $T_t\psi$ looks like $T_t^{(0)}\rho$ in the past and $T_t^{(0)}S\rho$ in the future. Thus $S$ correlates the past and future asymptotics of interacting histories. The reader should be warned that the maps $S' = \Omega^+(\Omega^-)^{-1}: \Sigma_{\text{in}} \to \Sigma_{\text{out}}$ and also the maps $(\Omega^+)^{-1}\Omega^-$ and $\Omega^-(\Omega^+)^{-1}$ occasionally appear in the literature. When weak asymptotic completeness holds, $S' = \Omega^- S(\Omega^-)^{-1}$, so $S$ and $S'$ are "similar." For this reason, the choice between $S$ and $S'$ is to some extent a matter of personal preference. We use $S$, the so-called EBFM $S$-matrix, throughout this book. We discuss the reasons for the $\pm$ convention in Sections 3 and 6.

In classical particle mechanics $S$ is a bijection on phase space. In a quantum theory with weak asymptotic completeness $S$ is a *linear* unitary transformation and is called the $S$-operator or occasionally the $S$-matrix.

(5) *Reduction of $S$ due to symmetries* In many problems there is an underlying symmetry of *both* the free and interacting dynamics. This allows one to conclude a priori, without detailed dynamical calculations, that $S$ has a special form. See Sections 2 and 8 for explicit details.

(6) *Analyticity and the $S$-transformation* A common refinement of scattering theory for wave phenomena (quantum theory, optics, acoustics) is the realization of $S$ or the kernel of some associated integral operator as the boundary value of an analytic function. In a heuristic sense this analyticity is connected with Theorem IX.16. For schematically, $S$ describes the response $R$ of a system to some input $I$ in the following form:

$$R(t) = \int_{-\infty}^{t} f(t-t')I(t')\,dt'$$

This formula has two features built in: (i) time translation invariance, that is, $f$ is a function of only $t - t'$; (ii) causality: $R(t)$ depends only on $I(t')$ for $t' \leq t$. Thus $f$ is a function on $[0, \infty)$. Its Fourier transform is thus the boundary value of an analytic function. It is this causality argument that is intuitively in the back of physicists' minds when discussing analytic properties. Unfortunately, the proofs of these properties do not go along such simple lines. We shall restrict our detailed discussion of analyticity to the two-body quantum-mechanical case (Section 7) and to the Lax–Phillips theory (Section 11).

(7) *Asymptotic completeness* Consider a system with forces between its components that fall off as the components are moved apart. Physically, one expects a state of such a system to "decay" into freely moving clusters or to remain "bound." In many situations, there is a natural set of bound states, $\Sigma_{\text{bound}} \subset \Sigma$. One can usually prove that $\Sigma_{\text{bound}} \cap \Sigma_{\text{in}} = \emptyset$. The above physical expectation is

$$\Sigma_{\text{bound}} \text{ “}+\text{” } \Sigma_{\text{in}} = \Sigma = \Sigma_{\text{bound}} \text{ “}+\text{” } \Sigma_{\text{out}} \tag{1}$$

" + " is different in classical and quantum-mechanical systems. In classical particle mechanics " + " indicates set theoretic union; in quantum theory it indicates a direct sum of Hilbert spaces. Establishing that (1) holds is the problem of proving **asymptotic completeness**. Notice that asymptotic completeness implies weak asymptotic completeness. We remark that implicit in