

Graduate Texts in Mathematics

M. Loève

Probability Theory II

4th Edition

概率论

第2卷 第4版

Springer-Verlag

世界图书出版公司

M. Loève

Probability Theory II

4th Edition



Springer-Verlag

New York Heidelberg Berlin

书 名: Probability Theory II 4th ed.
作 者: M.Loève
中 译 名: 概率论 第2卷 第4版
出 版 者: 世界图书出版公司北京公司
印 刷 者: 北京世图印刷厂
发 行: 世界图书出版公司北京公司 (北京朝内大街 137 号 100010)
开 本: 1/24 711×1245 印 张: 18
出版年代: 2000 年 12 月
书 号: ISBN 7-5062-0076-7
定 价: 66.00 元

世界图书出版公司北京公司已获得 Springer-Verlag 授权在中国大陆独家重印发行。

M. Loève

Departments of Mathematics and Statistics
University of California at Berkeley
Berkeley, California 94720

Editorial Board

P. R. Halmos

Managing Editor
University of California
Department of Mathematics
Santa Barbara, California 93106

F. W. Gehring

University of Michigan
Department of Mathematics
Ann Arbor, Michigan 48104

C. C. Moore

University of California at Berkeley
Department of Mathematics
Berkeley, California 94720

AMS Subject Classifications
28-01, 60A05, 60Bxx, 60E05, 60Fxx

Library of Congress Cataloging in Publication Data

Loève, Michel, 1907-

Probability theory.

(Graduate texts in mathematics; 45-46)

Bibliography p.

Includes index.

1. Probabilities. I. Title. II. Series.

QA273.L63 1977 519.2 76-28332

All rights reserved.

No part of this book may be translated or reproduced in any form without written permission from Springer-Verlag.

© 1963 by M. Loève

© 1978 by Springer-Verlag New York Inc.

Originally published in the University Series in Higher Mathematics (D. Van Nostrand Company); edited by M. H. Stone, L. Nirenberg, and S. S. Chern.

This reprint has been authorized by Springer-Verlag (Berlin/Heidelberg/New York) for sale in the People's Republic of China only and not for export therefrom.
Reprinted in China by Beijing World Publishing Corporation, 2001

ISBN 0-387-90262-7 Springer-Verlag New York

ISBN 3-540-90262-7 Springer-Verlag Berlin Heidelberg

PREFACE TO THE FOURTH EDITION

This fourth edition contains several additions. The main ones concern three closely related topics: Brownian motion, functional limit distributions, and random walks. Besides the power and ingenuity of their methods and the depth and beauty of their results, their importance is fast growing in Analysis as well as in theoretical and applied Probability.

These additions increased the book to an unwieldy size and it had to be split into two volumes.

About half of the first volume is devoted to an elementary introduction, then to mathematical foundations and basic probability concepts and tools. The second half is devoted to a detailed study of Independence which played and continues to play a central role both by itself and as a catalyst.

The main additions consist of a section on convergence of probabilities on metric spaces and a chapter whose first section on domains of attraction completes the study of the Central limit problem, while the second one is devoted to random walks.

About a third of the second volume is devoted to conditioning and properties of sequences of various types of dependence. The other two thirds are devoted to random functions; the last Part on Elements of random analysis is more sophisticated.

The main addition consists of a chapter on Brownian motion and limit distributions.

It is strongly recommended that the reader begin with less involved portions. In particular, the starred ones ought to be left out until they are needed or unless the reader is especially interested in them.

I take this opportunity to thank Mrs. Rubalcava for her beautiful typing of all the editions since the inception of the book. I also wish to thank the editors of Springer-Verlag, New York, for their patience and care.

M. L.

January, 1977
Berkeley, California

PREFACE TO THE THIRD EDITION

This book is intended as a text for graduate students and as a reference for workers in Probability and Statistics. The prerequisite is honest calculus. The material covered in Parts Two to Five inclusive requires about three to four semesters of graduate study. The introductory part may serve as a text for an undergraduate course in elementary probability theory.

The Foundations are presented in:

- the Introductory Part on the background of the concepts and problems, treated without advanced mathematical tools;
- Part One on the Notions of Measure Theory that every probabilist and statistician requires;
- Part Two on General Concepts and Tools of Probability Theory.

Random sequences whose general properties are given in the Foundations are studied in:

- Part Three on Independence devoted essentially to sums of independent random variables and their limit properties;
- Part Four on Dependence devoted to the operation of conditioning and limit properties of sums of dependent random variables. The last section introduces random functions of second order.

Random functions and processes are discussed in:

- Part Five on Elements of random analysis devoted to the basic concepts of random analysis and to the martingale, decomposable, and Markov types of random functions.

Since the primary purpose of the book is didactic, methods are emphasized and the book is subdivided into:

- unstarred portions, independent of the remainder; starred portions, which are more involved or more abstract;
- complements and details, including illustrations and applications of the material in the text, which consist of propositions with fre-

PREFACE TO THE THIRD EDITION

quent hints; most of these propositions can be found in the articles and books referred to in the Bibliography.

Also, for teaching and reference purposes, it has proved useful to name most of the results.

Numerous historical remarks about results, methods, and the evolution of various fields are an intrinsic part of the text. The purpose is purely didactic: to attract attention to the basic contributions while introducing the ideas explored. Books and memoirs of authors whose contributions are referred to and discussed are cited in the Bibliography, which parallels the text in that it is organized by parts and, within parts, by chapters. Thus the interested student can pursue his study in the original literature.

This work owes much to the reactions of the students on whom it has been tried year after year. However, the book is definitely more concise than the lectures, and the reader will have to be armed permanently with patience, pen, and calculus. Besides, in mathematics, as in any form of poetry, the reader has to be a poet *in posse*.

This third edition differs from the second (1960) in a number of places. Modifications vary all the way from a prefix ("sub" martingale in lieu of "semi"-martingale) to an entire subsection (§36.2). To preserve pagination, some additions to the text proper (especially 9, p. 656) had to be put in the Complements and Details. It is hoped that moreover most of the errors have been eliminated and that readers will be kind enough to inform the author of those which remain.

I take this opportunity to thank those whose comments and criticisms led to corrections and improvements: for the first edition, E. Barankin, S. Bochner, E. Parzen, and H. Robbins; for the second edition, Y. S. Chow, R. Cogburn, J. L. Doob, J. Feldman, B. Jamison, J. Karush, P. A. Meyer, J. W. Pratt, B. A. Sevastianov, J. W. Woll; for the third edition, S. Dharmadhikari, J. Fabius, D. Freedman, A. Maitra, U. V. Prokhorov. My warm thanks go to Cogburn, whose constant help throughout the preparation of the second edition has been invaluable. This edition has been prepared with the partial support of the Office of Naval Research and of the National Science Foundation.

M. L.

April, 1962
Berkeley, California

CONTENTS OF VOLUME I

GRADUATE TEXTS IN MATHEMATICS VOL. 45

ELEMENTARY PROBABILITY THEORY

Intuitive Background
Axioms; Independence and the Bernoulli Case
Dependence and Chains

NOTIONS OF MEASURE THEORY

SETS, SPACES, AND MEASURES

Sets, Classes, and Functions
Topological Spaces
Additive Set Functions
Construction of Measures on σ -Fields

MEASURABLE FUNCTIONS AND INTEGRATION

Measurable Functions
Measure and Convergences
Integration
Indefinite Integrals; Iterated Integrals

GENERAL CONCEPTS AND TOOLS OF PROBABILITY THEORY

PROBABILITY CONCEPTS

Probability Spaces and Random Variables
Probability Distributions

DISTRIBUTION FUNCTIONS AND CHARACTERISTIC FUNCTIONS

Distribution Functions
Convergence of Probabilities on Metric Spaces

Characteristic Functions and Distribution Functions
Probability Laws and Types of Laws
Nonnegative-definiteness; Regularity

INDEPENDENCE

SUMS OF INDEPENDENT RANDOM VARIABLES

Concept of Independence
Convergence and Stability of Sums; Centering at Expectations and Truncation
Convergence and Stability of Sums; Centering at Medians and Symmetrization
Exponential Bounds and Normed Sums

CENTRAL LIMIT PROBLEM

Degenerate, Normal, and Poisson Types
Evolution of the Problem
Central Limit Problem; the Case of Bounded Variances
Solution of the Central Limit Problem
Normed Sums

INDEPENDENT IDENTICALLY DISTRIBUTED SUMMANDS

Regular Variation and Domains of Attraction
Random Walk

BIBLIOGRAPHY

INDEX

CONTENTS OF VOLUME II

GRADUATE TEXT IN MATHEMATICS VOL. 46

PART FOUR: DEPENDENCE

CHAPTER VIII: CONDITIONING

SECTION	PAGE
27. CONCEPT OF CONDITIONING	3
27.1 Elementary case	3
27.2 General case	7
27.3 Conditional expectation given a function	8
*27.4 Relative conditional expectations and sufficient σ -fields	10
28. PROPERTIES OF CONDITIONING	13
28.1 Expectation properties	13
28.2 Smoothing properties	15
*28.3 Concepts of conditional independence and of chains	17
29. REGULAR PR. FUNCTIONS	19
29.1 Regularity and integration	19
*29.2 Decomposition of regular c.pr.'s given separable σ -fields	21
30. CONDITIONAL DISTRIBUTIONS	24
30.1 Definitions and restricted integration	24
30.2 Existence	26
30.3 Chains; the elementary case	31
COMPLEMENTS AND DETAILS	36

CHAPTER IX: FROM INDEPENDENCE TO DEPENDENCE

31. CENTRAL ASYMPTOTIC PROBLEM	37
31.1 Comparison of laws	38
31.2 Comparison of summands	41
*31.3 Weighted prob. laws	44
32. CENTERINGS, MARTINGALES, AND A.S. CONVERGENCE	51
32.1 Centerings	51
32.3 Martingales: generalities	54

SECTION	PAGE
32.3 Martingales: convergence and closure	57
32.4 Applications	63
*32.5 Indefinite expectations and a.s. convergence	67
COMPLEMENTS AND DETAILS	73

CHAPTER X: ERGODIC THEOREMS

33. TRANSLATION OF SEQUENCES; BASIC ERGODIC THEOREM AND STATIONARITY	77
*33.1 Phenomenological origin	77
33.2 Basic ergodic inequality	79
33.3 Stationarity	83
33.4 Applications; ergodic hypothesis and independence	89
*33.5 Applications; stationary chains	90
*34. ERGODIC THEOREMS AND L_r -SPACES	96
*34.1 Translations and their extensions	96
*34.2 A.s. ergodic theorem	98
*34.3 Ergodic theorems on spaces L_r	101
*35. ERGODIC THEOREMS ON BANACH SPACES	106
*35.1 Norms ergodic theorem	106
*35.2 Uniform norms ergodic theorems	110
*35.3 Application to constant chains	114
COMPLEMENTS AND DETAILS	118

CHAPTER XI: SECOND ORDER PROPERTIES

36. ORTHOGONALITY	121
36.1 Orthogonal r.v.'s; convergence and stability	122
36.2 Elementary orthogonal decomposition	125
36.3 Projection, conditioning, and normality	128
37. SECOND ORDER RANDOM FUNCTIONS	130
37.1 Covariances	131
37.2 Calculus in q.m.; continuity and differentiation	135
37.3 Calculus in q.m.; integration	137
37.4 Fourier-Stieltjes transforms in q.m.	140
37.5 Orthogonal decompositions	143
37.6 Normality and almost-sure properties	151
37.7 A.s. stability	152
COMPLEMENTS AND DETAILS	156

PART FIVE: ELEMENTS OF RANDOM ANALYSIS

CHAPTER XII: FOUNDATIONS; MARTINGALES AND DECOMPOSABILITY

SECTION	PAGE
38. FOUNDATIONS	163
38.1 Generalities	164
38.2 Separability	170
38.3 Sample continuity	179
38.4 Random times	188
39. MARTINGALES	193
39.1 Closure and limits	194
39.2 Martingale times and stopping	207
40. DECOMPOSABILITY	212
40.1 Generalities	212
40.2 Three parts decomposition	216
40.3 Infinite decomposability; normal and Poisson cases	221
COMPLEMENTS AND DETAILS	231

CHAPTER XIII: BROWNIAN MOTION AND LIMIT DISTRIBUTIONS

41. BROWNIAN MOTION	235
41.1 Origins	235
41.2 Definitions and relevant properties	237
41.3 Brownian sample oscillations	246
41.4 Brownian times and functionals	254
42. LIMIT DISTRIBUTIONS	263
42.1 Pr.'s on \mathcal{C}	264
42.2 Limit distributions on \mathcal{C}	268
42.3 Limit distributions; Brownian embedding	271
42.4 Some specific functionals	278
Complements and Details	281

CHAPTER XIV: MARKOV PROCESSES

43. MARKOV DEPENDENCE	289
43.1 Markov property	289
43.2 Regular Markov processes	294
43.4 Stationarity	301
43.4 Strong Markov property	304

SECTION	PAGE
44. TIME-CONTINUOUS TRANSITION PROBABILITIES	310
44.1 Differentiation of tr. pr.'s	312
44.2 Sample functions behavior	321
45. MARKOV SEMI-GROUPS	331
45.1 Generalities	331
45.2 Analysis of semi-groups	336
45.3 Markov processes and semi-groups	346
46. SAMPLE CONTINUITY AND DIFFUSION OPERATORS	357
46.1 Strong Markov property and sample rightcontinuity	357
46.2 Extended infinitesimal operator	366
46.3 One-dimensional diffusion operator	374
COMPLEMENTS AND DETAILS	381
BIBLIOGRAPHY	384
INDEX	391

Part Four

DEPENDENCE

For about two centuries probability theory has been concerned almost exclusively with independence. Yet, very particular forms of dependence appear already in the theory of games of chance. But a first general type of dependence—chains—was introduced only at the beginning of this century by Markov. Another type of dependence—stationarity—appears in ergodic theory, and a related type—second order stationarity—is then introduced in probability theory by Khintchine (1932). Centering at conditional expectations by P. Lévy (1935) gives rise to a new type of dependence—martingales.

At the very core of the study of dependence lies the concept of conditioning—with respect to a function—put in an abstract and rigorous form by Kolmogorov. In this part, the concept of conditioning is introduced in a more general form—with respect to a σ -field—and, as much as possible, the properties of various types of dependence are related to more general results, with emphasis given to the methods.

Chapter VIII

CONDITIONING

§ 27. CONCEPT OF CONDITIONING

The concept of "conditioning" can be expressed in terms of sub σ -fields of events. Conditional probabilities of events and conditional expectations of r.v.'s "given a σ -field \mathcal{B} ," to be introduced and investigated in this chapter, are \mathcal{B} -measurable functions defined up to an equivalence. If \mathcal{B} is determined by a countable partition of the sure event, then these functions are elementary. In this "elementary case," a constructive approach with a definite intuitive appeal is possible and there are no technical difficulties. In the general case, there is no suitable and rigorous constructive approach, and a descriptive one, requiring more powerful tools, especially the Radon-Nikodym theorem, has to be used.

The R.-N. theorem was obtained in its abstract form in 1930 and the concept of conditional probabilities and of conditional expectations of integrable r.v.'s "given" a measurable function, finite or not, numerical or not, was then put on a rigorous basis by Kolmogorov in 1933.

27.1. Elementary case. Investigation of the elementary case will give us an insight into the ideas involved in the intuitive notion of conditioning and will lead "naturally" to the notions and problems which appear in the general case.

The notion of conditional probability of an event A "given an event B " corresponds to that of frequencies of A in the repeated trials where B occurs; it is one of the oldest probability notions. For every event A , the relation

$$PB \cdot P_B A = PAB$$

defines the *conditional probability (c.pr.)* $P_B A$ of A given B as the ratio PAB/PB , provided B is a nonnull event; if B is null, so is AB , and the

foregoing relation leaves $P_B A$ undetermined. In what follows, we assume that, unless otherwise stated, B is nonnull.

The function P_B on the σ -field \mathcal{A} of events, whose values are $P_B A$, $A \in \mathcal{A}$, is called *conditional pr. given B* . The defining relation shows at once that since P on \mathcal{A} is normed, nonnegative, and σ -additive, so is P_B on \mathcal{A} :

$$P_B \Omega = 1, \quad P_B \geq 0, \quad P_B \sum A_j = \sum P_B A_j.$$

Thus, the conditioning expressed by "given B " means that the initial pr. space (Ω, \mathcal{A}, P) is replaced by the pr. space $(\Omega, \mathcal{A}, P_B)$. The expectation, if it exists, of a r.v. X on this new pr. space is called *conditional expectation (c.exp.) given B* and is denoted by $E_B X$; in symbols

$$E_B X = \int X dP_B.$$

Since $P_B = 0$ on $\{AB^c, A \in \mathcal{A}\}$, the right-hand side reduces to $\int_B X dP_B$

and, since $P_B = \frac{1}{P_B} P$ on $\{AB, A \in \mathcal{A}\}$, it becomes $\frac{1}{P_B} \int_B X dP$.

Therefore, the c.exp. of X given B can be defined directly by

$$P_B E_B X = \int_B X dP$$

and is determined if B is a nonnull event. In particular,

$$P_B E_B I_A = \int_B I_A dP = PAB$$

so that the c.pr. $P_B A$ can be defined, thereafter, by

$$P_B A = E_B I_A.$$

Thus, if E_B is the *c.exp. given B* , with values $E_B X$ on the family \mathcal{E}_B of all r.v.'s X whose integral on B exists, the c.pr. P_B becomes the restriction of E_B to the family $\mathcal{I}_\mathcal{A}$ of indicators of events. Furthermore, properties of P_B become particular cases of the immediate properties of E_B below.

If $X \geq 0$ then $E_B X \geq 0$, and if c is a constant then $E_B c = c$. If the X_j are nonnegative, or if the X_j are integrable and their consecutive sums are uniformly bounded by an integrable r.v., then $E_B \sum X_j = \sum E_B X_j$.

C.exp.'s (hence c.pr.'s) acquire their full meaning when reinterpreted as values of functions, as follows. The number $E_B X$ is no longer assigned