D.S.Mitrinović

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Band 165

Analytic Inequalities

解析不等式

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Die Grundlehren der mathematischen Wissenschaften

in Einzeldarstellungen
mit besonderer Berücksichtigung
der Anwendungsgebiete

Band 165

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Analytic Inequalities
by Dragoslav S. Mitrinović

Preface

The Theory of Inequalities began its development from the time when C. F. GAUSS, A. L. CAUCHY and P. L. ČEBYŠEV, to mention only the most important, laid the theoretical foundation for approximative methods. Around the end of the 19th and the beginning of the 20th century, numerous inequalities were proved, some of which became classic, while most remained as isolated and unconnected results.

It is almost generally acknowledged that the classic work "Inequalities" by G. H. HARDY, J. E. LITTLEWOOD and G. PÓLYA, which appeared in 1934, transformed the field of inequalities from a collection of isolated formulas into a systematic discipline. The modern Theory of Inequalities, as well as the continuing and growing interest in this field, undoubtedly stem from this work. The second English edition of this book, published in 1952, was unchanged except for three appendices, totalling 10 pages, added at the end of the book.

Today inequalities play a significant role in all fields of mathematics, and they present a very active and attractive field of research.

J. DIEUDONNÉ, in his book "Calcul Infinitésimal" (Paris 1968), attributed special significance to inequalities, adopting the method of exposition characterized by "majorer, minorer, approcher".

Since 1934 a multitude of papers devoted to inequalities have been published: in some of them new inequalities were discovered, in others classical inequalities were sharpened or extended, various inequalities were linked by finding their common source, while some other papers gave a large number of miscellaneous applications.

The book "Inequalities" by E. F. BECKENBACH and R. BELLMAN, which appeared in 1961 and the second revised printing in 1965, contains an account of some results on inequalities obtained in the period 1934—1960.

The present book — "Analytic Inequalities" — is devoted for the most part to topics which are not included in the two mentioned above. However, even in the exposition of classical inequalities new facts have been added.

We have done our best to be as accurate as possible and have given all the relevant references we could. A systematic bibliographical search was undertaken for a large number of inequalities, and we believe the results included are up to date.

In writing this book we have consulted a very extensive literature. It is enough to mention that "Analytic Inequalities" cites over 750 names, some several times. As a rule, we have studied the original papers and only exceptionally have we leaned on the reviews published in Jahrbuch über die Fortschritte der Mathematik (1868—1944), Zentralblatt für Mathematik (since 1931), Mathematical Reviews (since 1940) and Referativnyĭ Žurnal Matematika (since 1953). Nevertheless, it was impossible to scan every relevant source and, for various reasons, some omissions were inevitable; we apologize in advance to anyone whose work may not have been given proper credit through oversight. Besides, our selection from the enormous material considered expresses our preference for simple and attractive results.

The greater part of the results included have been checked, although this could not, of course, be done for all the results which appear in the book. We hope, however, that there are not many errors, but the very nature of this book is such that it seems impossible to expect it to be entirely free of them. It is perhaps unnecessary to point out the advisability of checking an inequality before use. It is also worthwhile to turn to the original papers whenever possible, since the reader will frequently find the problem which first motivated the search for the inequality in question.

Though we have emphasized only in a relatively few places that there are unsolved problems, it can be seen from the text itself that there are many results which can be improved or developed in various directions.

This book is, in fact, a considerably extended and improved version of the author's book "Nejednakosti" which appeared in Serbian in 1965. Although following the idea and the outline of that work, "Analytic Inequalities" is on a higher level, and contains very little of the same material. It also contains many inequalities which are now published for the first time, owing to the fact that many mathematicians generously offered us their unpublished results.

The material of this book is divided into three parts. In the first part — "Introduction" — an approach to inequalities is given, while the main attention is devoted to the Section on Convex Functions.

The second and probably main part — "General Inequalities" — consists of 27 sections, each of which is dedicated to a class of inequalities of importance in Analysis. Special attention was paid to some sections, for instance to Sections 2.11, 2.14, 2.16, 2.23, 2.25, and we believe that they will be of benefit for further research.

Finally, the third part — "Particular Inequalities" — is aimed at providing a collection of various inequalities, more or less closely interconnected, some of which are of considerable theoretical interest. They are classified in a certain manner, although we must admit that this has not been done perfectly. Part 3 is, in fact, a collection of over 450 special inequalities, and with a few exceptions we were able to add bibliographical references for each one. Owing to lack of space, only a few inequalities are supplied with a complete proof.

As may be inferred from the title — "Analytic Inequalities" — various topics such as geometric inequalities, isoperimetric inequalities, as well as inequalities arising in Probability Theory have not been included. We have also omitted inequalities for univalent and multivalent functions, inequalities arising in Number Theory, inequalities which belong to the Theory of Forms, inequalities such as BESSEL's inequality, which belong to the Theory of Orthogonal Series, as well as inequalities arising in the Theory of Special Functions.

This book could be used as a postgraduate reference book, but undergraduate students may also successfully consult individual sections of it. Naturally "Analytic Inequalities" will be of use to those researching in the Theory of Inequalities, but we believe it will also prove useful to mathematicians, engineers, statisticians, physicists and all who come across inequalities in their work.

If it is true that "all analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove", we may expect that "Analytic Inequalities" will be of some help to them.

A large number of inequalities also hold under weaker conditions than those given here. This is especially true of inequalities involving integrals or positive integers.

It is a shortcoming of the book that the conditions under which strict inequalities hold are not specified everywhere. The form of exposition is not uniform throughout Part 2, in which the majority of results are stated as theorems with proofs, while the others are given more descriptively without emphasizing the theorem.

In the final phase of composing this book, Assistant Professor P.M. VASIĆ gave a considerable help in the classification of various topics, in writing some individual sections or subsections, as well as in the critical review of almost the whole text, and for these reasons his name appears on the title page.

Professor P.S. BULLEN of Vancouver University (Canada) wrote the Section 2.15 on Symmetric Means and Functions which is included here with some minor changes and additions.

Professor P.R. BEESACK of Ottawa University (Canada) was good enough to read the whole manuscript as well as the proofs. His remarks, suggestions and comments were very helpful.

Individual sections or subsections of "Analytic Inequalities" were kindly read by Professors J. ACZÉL, P.S. BULLEN and D. Ž. DJOKOVIĆ from Canada, Professors ROY O. DAVIES, H. KOBER, L. MIRSKY and R.A.

RANKIN from England, Professor M. JANET from France, Professor E. MAKAI from Hungary, Professors J. MUSIELAK and Z. OPIAL from Poland, Professors S. MARCUS and T. POPOVICIU from Romania, Professor P. H. DIANANDA from Singapore, Professor A. V. BOYD from South Africa, Professors R. P. BOAS, K. FAN, D. C. B. MARSH, J. RYFF and O. TAUSSKY-TODD from the U.S.A., and Professors S. KUREPA, M. MARJANOVIĆ and S.B. PREŠIĆ from Yugoslavia. The author greatly profited by their criticism, comments and suggestions.

Without their assistance many misprints and even errors would have probably remained unnoticed. In addition, some sections or subsections have been largely rewritten as a result of BEESACK's and KUREPA's suggestions.

The young Yugoslav mathematician J. D. KEČKIĆ not only helped with the translation of the manuscript into English, but also gave valuable comments on the text itself. He also compiled the subject index.

Dr. R. R. JANIĆ assisted in collecting documentational material. Dr. D. DJ. TOŠIĆ, D. V. SLAVIĆ and M. D. MITRINOVIĆ helped in the technical preparation of the manuscript for print.

The author feels indebted to all those mentioned above for the help which they have, in one way or another, given him.

The author is also indebted to a number of mathematicians and institutions for their extremely valuable assistance in furnishing the necessary literature and regrets his inability to quote all of them.

The author will be obliged to readers for further bibliographical data and also for any comments on the content and form of this book. The author believes that it can be improved in various directions. Such comments would be especially valuable as the author, with several associates, is preparing a series of books treating individual classes of inequalities as, for example, integral inequalities, inequalities involving polynomials, trigonometric inequalities, inequalities involving special functions, etc.

The author wishes to thank Springer-Verlag for publishing his book in their distinguished series "Grundlehren der mathematischen Wissenschaften" and for their readiness to meet all requests.

Finally, we list the main books and sources related to inequalities in various directions:

1. HARDY, G. H., J. E. LITTLEWOOD and G. PÓLYA: Inequalities. Cambridge 1934, 314 pp.

Russian translation by V. I. LEVIN, with Supplements by V. I. LEVIN and S. B. STEČKIN, Moscow 1948, 456 pp. (Supplements: pp. 361–441).

A selection of the above Supplements by R. P. BOAS appeared under the title: LEVIN, V. I., and S. B. STEČKIN: Inequalities. Amer. Math. Soc. Transl. (2) 14, 1–29 (1960).

Second English edition 1952, 324 pp.

Chinese translation of the second English edition. Peking 1965, X + 352 pp.

2. PETROVIĆ, M.: Računanje sa brojnim razmacima. Beograd 1932, 193 pp.; 2nd ed. 1969, 169 pp.
3. PÓLYA, G., and G. SZEGÖ: Isoperimetric Inequalities in Mathematical Physics. Princeton 1951, XVI + 279 pp.
4. NATANSON, I. P.: Konstruktive Funktionentheorie. Berlin 1955, XIV + 515 pp.
5. AHIEZER, N. I.: Theory of Approximation. New York 1956, X + 307 pp.
6. TIMAN, A. F.: Theory of Approximation of Functions of a Real Variable (Russian). Moscow 1960, 624 pp.
7. BECKENBACH, E. F., and R. BELLMAN: Inequalities. Berlin-Heidelberg-NewYork 1961; 2nd ed. 1965, 198 pp.
8. Studies in Mathematical Analysis and Related Topics. Stanford 1962, 447 pp.
9. MARCUS, M., and H. MINC: A Survey of Matrix Theory and Matrix Inequalities. Boston 1964, 180 pp.
10. MITRINOVIĆ, D. S.: Elementary Inequalities. Groningen 1964, 159 pp.
11. WALTER, W.: Differential- und Integral-Ungleichungen. Berlin-Göttingen-Heidelberg-New York 1964, XIII + 269 pp.
12. MITRINOVIĆ, D. S.: Nejednakosti. Beograd 1965, 240 pp.
13. HARDY, G. H.: Collected Papers, vol. 2. Oxford 1967, pp. 379–682.
14. Inequalities (Proceedings of a Symposium held at Wright-Patterson Air Force Base, Ohio 1965), edited by O. SHISHA. New York-London 1967, 360 pp.
15. SZARSKI, J.: Differential Inequalities. Warszawa 1967, 256 pp.
16. MITRINOVIĆ, D. S., and P. M. VASIĆ: Sredine. Beograd 1969, 122 pp.
17. BOTTEMA, O., R. Ž. DJORDJEVIĆ, R. R. JANIĆ, D. S. MITRINOVIĆ and P. M. VASIĆ: Geometric Inequalities. Groningen 1969, 151 pp.

Belgrade, May 1970 D. S. MITRINOVIĆ

Organization of the Book

Besides the Preface, Notations and Definitions, and the Indexes, the book contains three parts, each of which is divided into a number of sections, and some of these into subsections. The numeration of theorems, definitions, remarks and formulas is continuous throughout a subsection, or a section which does not contain subsections. If the theorem referred to belongs to the same subsection, only its number is given, while if it belongs to another, the numbers of the part, section, subsection and of the theorem are given. Similar notations are used if a section is not divided into subsections.

There are many cross-references in the book. So, for example, 2.1.4 means Part 2, Section 1, Subsection 4.

As a rule, bibliographical references are quoted after each subsection if it exists, or after each section, if it does not. Sections 1.1, 1.4, 2.15 and 2.25 present exceptions to this rule.

The abbreviations of the cited journals are given according to Mathematical Reviews.

On Notations and Definitions

The notations and concepts used throughout the book are more or less specified. The reader is assumed to be familiar with the elements of Mathematical Analysis and with the basic concepts of General Algebra and Topology, and since the standard notations were used, it was believed unnecessary to define all of them. We shall, therefore, list only a few of them.

$[x]$ denotes the integral part of the real number x.

If $a > 0$, and if p/q is any rational number, with p and q both integers and $q > 0$, then $a^{p/q}$ means the unique positive q-th root of a^p.

If r is a real number, then $(f(x))^r$ is often denoted by $f(x)^r$, and $(f^{(k)}(x))^r$ by $f^{(k)}(x)^r$.

If A and B are two arbitrary sets, then the set $A \times B$ is defined by

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

R^n denotes the n-dimensional vector space of points x with coordinates $x_1, \ldots, x_n$. According to whether the coordinates are real or complex, R^n is called the real or the complex n-dimensional space.

A vector or a sequence is called positive (negative) if all its coordinates are positive (negative).

The scalar, or the inner, product of two vectors $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ is the number $(a, b) = a \cdot b = a_1 \bar{b}_1 + \cdots + a_n \bar{b}_n$, where $\bar{b}_1, \ldots, \bar{b}_n$ denotes the complex conjugates of $b_1, \ldots, b_n$.

For two sequences $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ we define the r sum and product as follows:

$$a + b = (a_1 + b_1, \ldots, a_n + b_n), \quad a \cdot b = (a_1 b_1, \ldots, a_n b_n).$$

Their difference and quotient are defined analogously, provided in the latter case that $b_i \neq 0$ for $i = 1, \ldots, n$.

$C[a, b]$ denotes the set of all real or complex functions continuous on the interval $[a, b]$.

$L^p[a, b]$ denotes the set of all real or complex functions f such that f^p is integrable on $[a, b]$.

$\|f\|$ denotes the norm of f with respect to a certain space.

If $A = (a_{ij})$ is an $n \times n$ matrix, then $\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}$.

Contents

1. Introduction

1.1 Real Number System

1.1.1 Axioms of the Set of Real Numbers

A systematic and a detailed construction of the real number system can be found, for example, in the book [1] of E. LANDAU, or in the book [2] of L. W. COHEN and G. EHRLICH.

We shall, in a manner similar to that of J. DIEUDONNÉ [3], give definitions and the system of axioms of the set of real numbers together with a number of theorems which follow directly from these axioms. Their proofs are more or less simple and will be omitted.

The set of real numbers is a nonempty set R together with two mappings

$$(x, y) \mapsto x + y \quad \text{and} \quad (x, y) \mapsto xy$$

from $R \times R$ into R, called addition and multiplication respectively, and an order relation $x \leq y$ (also written $y \geq x$) between elements of R so that:

1° R is a field,

2° R is an ordered field,

3° R is an Archimedean ordered field,

4° R is complete, i.e., R satisfies the axiom of nested intervals.

Since this monograph deals with inequalities, we shall consider in some detail only the order properties.

1.1.2 Order Properties of Real Numbers

In all relations below x, y, z are arbitrary elements of R.

By "R is an ordered field" we mean that the following axioms are satisfied:

2.1° $x \leq y$ and $y \leq z$ imply $x \leq z$;

2.2° $x \leq y$ and $y \leq x$ is equivalent to $x = y$;

2.3° for any x and y, either $x \leq y$ or $y \leq x$;

2.4° $x \leq y$ implies $x + z \leq y + z$;

2.5° $0 \leq x$ and $0 \leq y$ imply $0 \leq xy$.

The relation "$x \leq y$ and $x \neq y$" is written $x < y$, or $y > x$. The relation $x \leq y$ is equivalent to "$x < y$ or $x = y$".

Let $a < b$. The set $\{x \mid a < x < b\}$ is called the open interval with end-points a and b, and written (a, b). The set $\{x \mid a \leq x \leq b\}$ is called the closed interval or the segment with end-points a and b, and written $[a, b]$. For $a = b$ the notation $[a, a]$ denotes the one-point set $\{a\}$. By $[a, b)$ and $(a, b]$ we denote the sets $\{x \mid a \leq x < b\}$ and $\{x \mid a < x \leq b\}$ respectively, and these are called semi-open intervals.

Using the axioms 1°—4° in 1.1.1, and in particular, using the axioms given under 2.1°—2.5° we can prove the following important theorems.

Theorem 1. *For any $x, y \in R$ one and only one of the three relations $x < y$, $x = y$, $x > y$ holds.*

Theorem 2. *If "$x \leq y$ and $y < z$", or "$x < y$ and $y \leq z$", then $x < z$.*

Theorem 3. *Any finite subset A of R has the greatest element b and the smallest element a, thus $a \leq x \leq b$ for every $x \in A$.*

Theorem 4. *If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are two finite sequences both of n real numbers, such that $x_k \leq y_k$ for $k = 1, \ldots, n$, then*

$$x_1 + \cdots + x_n \leq y_1 + \cdots + y_n.$$

If, in addition, $x_k < y_k$ for at least one index k, then

$$x_1 + \cdots + x_n < y_1 + \cdots + y_n.$$

A real number x is called positive if $x > 0$; negative if $x < 0$. A real number x is called nonnegative if $x \geq 0$, and nonpositive if $x \leq 0$. If $x > 0$ and $y > 0$, or if $x < 0$ and $y < 0$, we say that x and y are of the same sign. If $x > 0$ and $y < 0$, or if $x < 0$ and $y > 0$, we say that x and y have opposite signs.

Theorem 5. *If $x_1, \ldots, x_n$ is a sequence of n real numbers and if $y_1, \ldots, y_n$ is a sequence of n nonnegative real numbers such that $y_k \leq x_k$ for $k = 1, \ldots, n$, then*

$$y_1 \cdots y_n \leq x_1 \cdots x_n.$$

Theorem 6. *If $x + z \leq y + z$ $(x + z < y + z)$ for at least one $z \in R$, then $x \leq y$ $(x < y)$.*

Theorem 7. *The relations* $x \leq y$, $0 \leq y - x$, $x - y \leq 0$, $-y \leq -x$ *are equivalent. Same results hold if* $\leq$ *is replaced by* $<$.

For an interval (a, b), with $a < b$, the positive number $b - a$ is called the length of the interval.

Theorem 8. *Let* $J_1, \ldots, J_n$ *be n disjoint intervals, and let I be an interval containing* $\bigcup_{k=1}^{n} J_k$. *Then, if* l_k *is the length of* J_k, *for* $k = 1, \ldots, n$, *and if l is the length of I,*

$$l_1 + \cdots + l_n \leq l.$$

For any real number x, we define

$$|x| = x \text{ for } x \geq 0, \text{ and } |x| = -x \text{ for } x \leq 0.$$

Hence $|x| = \max(x, -x)$ and $|-x| = |x|$.
$|x|$ is called the absolute value of x.
$|x| = 0$ is equivalent to $x = 0$.
For $x \neq 0$ we write

$$x^+ = \frac{1}{2}(|x| + x) \quad \text{(positive part of } x\text{)},$$

$$x^- = \frac{1}{2}(|x| - x) \quad \text{(negative part of } x\text{)}.$$

We also write $0^+ = 0^- = 0$.
Using the above notations we have

$$x^+ = x \text{ if } x \geq 0, \quad x^+ = 0 \text{ if } x \leq 0;$$

$$x^- = 0 \text{ if } x \geq 0, \quad x^- = -x \text{ if } x \leq 0;$$

$$x = x^+ - x^-, \quad |x| = x^+ + x^-.$$

Theorem 9. *If* $a > 0$, *then* $|x| \leq a$ *is equivalent to* $-a \leq x \leq a$, *and* $|x| < a$ *to* $-a < x < a$.

Theorem 10. *For any pair x, y of real numbers,*

(1) $$|x + y| \leq |x| + |y|,$$

(2) $$\big||x| - |y|\big| \leq |x + y|,$$

and, by induction,

(3) $$|x_1 + \cdots + x_n| \leq |x_1| + \cdots + |x_n|.$$

Equality in (1) holds if and only if $x = 0$, or $y = 0$, or if x and y have the same sign.

Equality in (2) holds if and only if $x = 0$, or $y = 0$, or if x and y have opposite signs.

Equality in (3) holds if and only if all the numbers $x_1, \ldots, x_n$ not equal to zero have the same sign.

Theorem 11. *For any real numbers x, y*

$$(|x| - |y|)^2 \leq |x^2 - y^2|,$$

with equality if and only if $x = 0$, or $y = 0$, or if the absolute values of x and y are equal;

$$\left|\sqrt{|x|} - \sqrt{|y|}\right| \leq \sqrt{|x - y|},$$

with equality if and only if $x = 0$, or $y = 0$, or $x = y$.

Theorem 12. *If $z \geq 0$, then $x \leq y$ implies $xz \leq yz$.*

Theorem 13. *The relations $x \leq 0$ and $y \geq 0$ imply $xy \leq 0$. The relations $x \leq 0$ and $y \leq 0$ imply $xy \geq 0$. Same results hold with $\leq$ replaced by $<$. In particular, $x^2 \geq 0$ for any real number and $x^2 > 0$ unless $x = 0$.*

Theorem 14. *If $x > 0$, then $1/x > 0$. If $z > 0$, then $x \leq y$ is equivalent to $xz \leq yz$. The relation $0 < x < y$ is equivalent to $0 < 1/y < 1/x$.*

A real number b is said to be a majorant (resp. minorant) of a subset X of the set R if $x \leq b$ (resp. $b \leq x$) for every $x \in X$. A set $X \subset R$ is said to be majorized, or bounded from above (resp. minorized, or bounded from below) if the set of majorants (resp. minorants) of X is not empty. If X is majorized, then $-X = \{-x \mid x \in X\}$ is minorized, and for every majorant b of X, $-b$ is a minorant of $-X$, and vice versa. A set which is both majorized and minorized is said to be bounded.

Theorem 15. *For any real number $a > 0$ and $a \neq 1$, there exists the function $x \mapsto a^x$ from R onto the set of all positive numbers such that:*

1° $a^{x+y} = a^x \cdot a^y$ *for any* $x, y \in R$,

2° $a^0 = 1$,

3° $x < y \Leftrightarrow a^x < a^y$ *if* $a > 1$, *and* $x < y \Leftrightarrow a^x > a^y$ *if* $0 < a < 1$.

The inverse of the function $x \mapsto a^x$ is denoted by $x \mapsto \log_a x$ and called the logarithmic function.

Remark. There is the unique number $e > 1$ such that $(e^x)' = e^x$ for any $x \in R$. This number e is transcendental; $e = 2.7182818284\ldots$ The function $x \mapsto e^x$ is called the exponential function and sometimes denoted by $\exp x$. In fact $e = \lim_{x \to 0} (1 + x)^{1/x}$.

Theorem 16. *If $a \geq b > 0$ and $x \geq 0$, then*

$$a^x \geq b^x.$$

Equality holds if and only if $a = b$ or $x = 0$.