海外优秀数学类教材系列丛书

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影印版

Thomas' CALCULUS

(Tenth Edition)

托马斯微积分（第10版）

（下册）

□ FINNEY
□ WEIR
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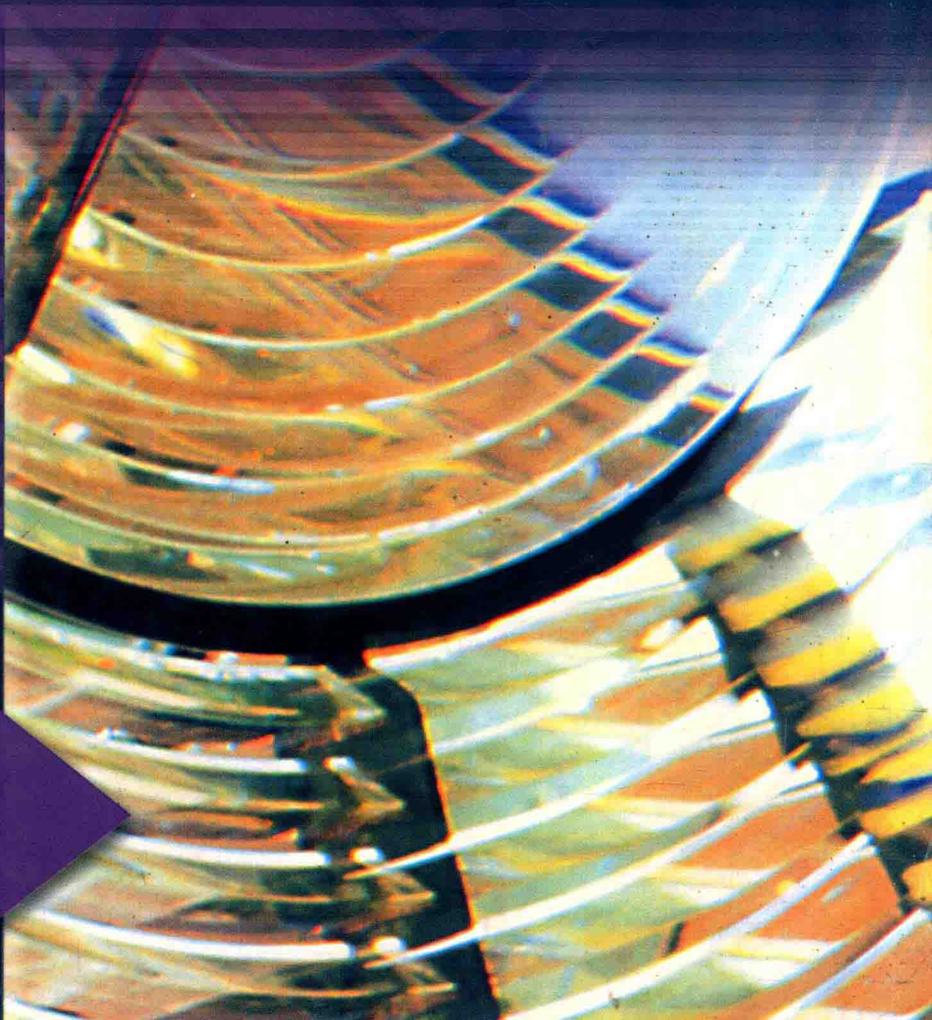

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托马斯微积分（第10版）（下册）

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出版者的话

在我国已经加入 WTO、经济全球化的今天，为适应当前我国高校各类创新人才培养的需要，大力推进教育部倡导的双语教学，配合教育部实施的“高等学校教学质量与教学改革工程”和“精品课程”建设的需要，高等教育出版社有计划、大规模地开展了海外优秀数学类系列教材的引进工作。

高等教育出版社和 Pearson Education，John Wiley & Sons，McGraw-Hill，Thomson Learning 等国外出版公司进行了广泛接触，经国外出版公司的推荐并在国内专家的协助下，提交引进版权总数 100 余种。收到样书后，我们聘请了国内高校一线教师、专家、学者参与这些原版教材的评介工作，并参考国内相关专业的课程设置和教学实际情况，从中遴选出了这套优秀教材组织出版。

这批教材普遍具有以下特点：（1）基本上是近 3 年出版的，在国际上被广泛使用，在同类教材中具有相当的权威性；（2）高版次，历经多年教学实践检验，内容翔实准确、反映时代要求；（3）各种教学资源配套整齐，为师生提供了极大的便利；（4）插图精美、丰富，图文并茂，与正文相辅相成；（5）语言简练、流畅、可读性强，比较适合非英语国家的学生阅读。

本系列丛书中，有 Finney、Weir 等编的《托马斯微积分》（第 10 版，Pearson），其特色可用“呈传统特色、富革新精神”概括，本书自 20 世纪 50 年代第 1 版以来，平均每四五年就有一个新版面世，长在 50 余年始终盛行于西方教坛，作者既有相当高的学术水平，又热爱教学，长期工作在教学第一线，其中，年近 90 的 G.B.Thomas 教授长年在 MIT 工作，具有丰富的教学经验；Finney 教授也在 MIT 工作达 10 年；Weir 是美国数学建模竞赛委员会主任。Stewart 编的立体化教材《微积分》（第 5 版，Thomson Learning）配备了丰富的教学资源，是国际上最畅销的微积分原版教材，2003 年全球销量约 40 余万册，在美国，占据了约 50%~60%的微积分教材市场，其用户包括耶鲁等名牌院校及众多一般院校 600 余所。本系列丛书还包括 Anton 编的经典教材《线性代数及其应用》（第 8 版，Wiley）；Jay L. Devore 编的优秀教材《概率论与数理统计》（第 5 版，Thomson Learning）等。在努力降低引进教材售价方面，高等教育出版社做了大量和细致的工作，这套引进的教材体现了一定的权威性、系统性、先进性和经济性等特点。

通过影印、翻译、编译这批优秀教材，我们一方面要不断地分析、学习、消化吸收国外优秀教材的长处，吸取国外出版公司的制作经验，提升我们自编教材的立体化配套标准，使我国高校教材建设水平上一个新的台阶；与此同时，我们还将尝试组织海外作者和国内作者合编外文版基础课数学教材，并约请国内专家改编部分国外优秀教材，以适应我国实际教学环境。

这套教材出版后，我们将结合各高校的双语教学计划，开展大规模的宣传、培训工作，及时地将本套丛书推荐给高校使用。在使用过程中，我们衷心希望广大高校教师和同学提出宝贵的意见和建议，如有好的教材值得引进，请与高等教育出版社高等理科分社联系。

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高等教育出版社

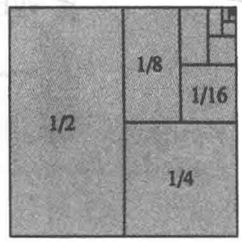
2004 年 4 月 20 日

Contents

8 Infinite Series

OVERVIEW One infinite process that had puzzled mathematicians for centuries was the summing of infinite series. Sometimes an infinite series of terms added to a number, as in

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1.$$



(You can see this by adding the areas in the "infinitely halved" unit square at the right.) Sometimes the infinite sum was infinite, however, as in

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \infty$$

(although this is far from obvious), and sometimes the infinite sum was impossible to pin down, as in

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots.$$

(Is it 0? Is it 1? Is it neither?)

Nonetheless, mathematicians like Gauss and Euler successfully used infinite series to derive previously inaccessible results. Laplace used infinite series to prove the stability of the solar system (although that does not stop some people from worrying about it today when they feel that "too many" planets have swung to the same side of the sun). It was years later that careful analysts like Cauchy developed the theoretical foundation for series computations, sending many mathematicians (including Laplace) back to their desks to verify their results.

Infinite series form the basis for a remarkable formula that enables us to express many functions as "infinite polynomials" and at the same time tells how much error we incur if we truncate those polynomials to make them finite. In addition to providing effective polynomial approximations of differentiable functions, these infinite polynomials (called power series) have many other uses. We also see how to use infinite sums of trigonometric terms, called Fourier series, to represent important functions used in science and engineering applications. Infinite series provide an efficient way to evaluate nonelementary integrals, and they solve differential equations that give insight into heat flow, vibration, chemical diffusion, and signal transmission. What you learn here sets the stage for the roles played by series of functions of all kinds in science and mathematics.

8.1 Limits of Sequences of Numbers

Definitions and Notation • Convergence and Divergence • Calculating Limits of Sequences • Using L'Hôpital's Rule • Limits That Arise Frequently

Informally, a sequence is an ordered list of things, but in this chapter, the things will usually be numbers. We have seen sequences before, such as the sequence $x_0, x_1, \ldots, x_n, \ldots$ of numbers generated by Newton's method. Later we consider sequences involving powers of x and others involving trigonometric terms like $\sin x$, $\cos x$, $\sin 2x$, $\cos 2x$, $\ldots$, $\sin nx$, $\cos nx$, $\ldots$. A central question is whether a sequence has a limit or not.

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Historical Essay
Sequences and Series

Definitions and Notation

We can list the integer multiples of 3 by assigning each multiple a position:

$$\begin{array}{lcccc} \text{Domain:} & 1 & 2 & 3 \ldots & n \ldots \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ \text{Range:} & 3 & 6 & 9 & 3n \end{array}$$

The first number is 3, the second 6, the third 9, and so on. The assignment is a function that assigns $3n$ to the nth place. That is the basic idea for constructing sequences. There is a function placing each number in the range in its correct ordered position.

Definition Sequence
An infinite sequence of numbers is a function whose domain is the set of integers greater than or equal to some integer n_0.

Usually, n_0 is 1 and the domain of the sequence is the set of positive integers. Sometimes, however, we want to start sequences elsewhere. We take $n_0 = 0$ when we begin Newton's method. We might take $n_0 = 3$ if we were defining a sequence of n-sided polygons.

Sequences are defined the same way as other functions, some typical rules being

$$a(n) = \sqrt{n}, \qquad a(n) = (-1)^{n+1}\frac{1}{n}, \qquad a(n) = \frac{n-1}{n}$$

(Example 1 and Figure 8.1). To indicate that the domains are sets of integers, we use a letter like n from the middle of the alphabet for the independent variable, instead of the x, y, z, and t used widely in other contexts. The formulas in the defining rules, however, like those above, are often valid for domains larger than the set of positive integers. This can be an advantage, as we will see. The number

(a) The terms $a_n = \sqrt{n}$ eventually surpass every integer, so the sequence $\{a_n\}$ diverges, . . .

(b) . . . but the terms $a_n = 1/n$ decrease steadily and get arbitrarily close to 0 as n increases, so the sequence $\{a_n\}$ converges to 0.

(c) The terms $a_n = (-1)^{n+1}(1/n)$ alternate in sign but still converge to 0.

(d) The terms $a_n = (n - 1)/n$ approach 1 steadily and get arbitrarily close as n increases, so the sequence $\{a_n\}$ converges to 1.

(e) The terms $a_n = (-1)^{n+1}[(n - 1)/n]$ alternate in sign. The positive terms approach 1. But the negative terms approach -1 as n increases, so the sequence $\{a_n\}$ diverges.

(f) The terms in the sequence of constants $a_n = 3$ have the same value regardless of n, so the sequence $\{a_n\}$ converges to 3.

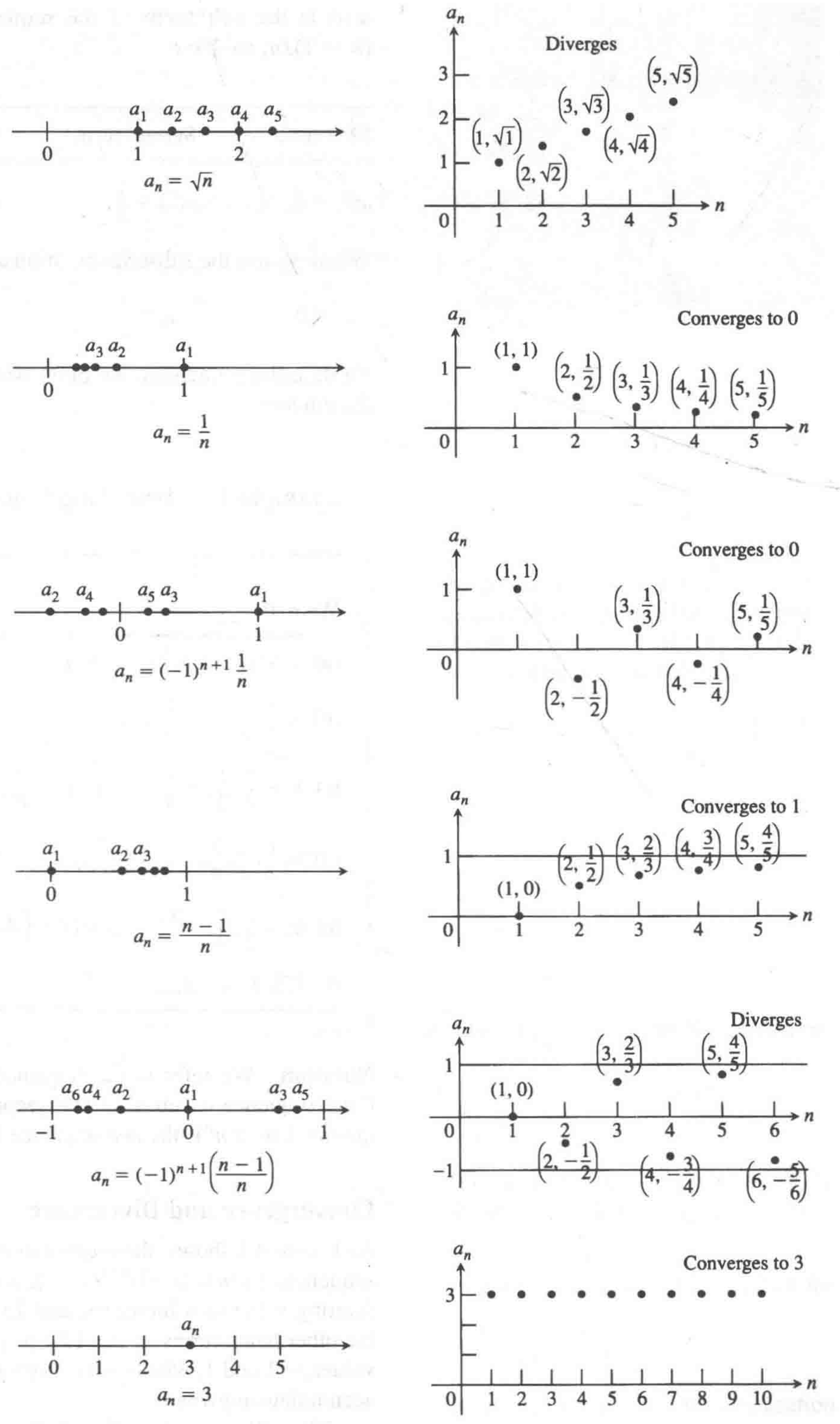

FIGURE 8.1 The sequences of Example 1 are graphed here in two different ways: by plotting the numbers a_n on a horizontal axis and by plotting the points (n, a_n) in the coordinate plane.

$a(n)$ is the **nth term** of the sequence, or the **term with index *n*.** If $a(n) = (n-1)/n$, we have

First term	Second term	Third term		nth term
$a(1) = 0$	$a(2) = \frac{1}{2}$,	$a(3) = \frac{2}{3}$,	$\ldots,$	$a(n) = \frac{n-1}{n}$.

When we use the subscript notation a_n for $a(n)$, the sequence is written

$$a_1 = 0, \qquad a_2 = \frac{1}{2}, \qquad a_3 = \frac{2}{3}, \qquad \ldots, \qquad a_n = \frac{n-1}{n}.$$

To describe sequences, we often write the first few terms as well as a formula for the nth term.

Example 1 Describing Sequences

We write	For the sequence whose defining rule is
(a) $1, \sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots, \sqrt{n}, \ldots$	$a_n = \sqrt{n}$
(b) $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$	$a_n = \frac{1}{n}$
(c) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots$	$a_n = (-1)^{n+1}\frac{1}{n}$
(d) $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n-1}{n}, \ldots$	$a_n = \frac{n-1}{n}$
(e) $0, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \ldots, (-1)^{n+1}\left(\frac{n-1}{n}\right), \ldots$	$a_n = (-1)^{n+1}\left(\frac{n-1}{n}\right)$
(f) $3, 3, 3, \ldots, 3, \ldots$	$a_n = 3$

Notation We refer to the sequence whose nth term is a_n with the notation $\{a_n\}$ ("the sequence a sub n"). The second sequence in Example 1 is $\{1/n\}$ ("the sequence 1 over n"); the last sequence is $\{3\}$ ("the constant sequence 3").

Convergence and Divergence

As Figure 8.1 shows, the sequences of Example 1 do not behave the same way. The sequences $\{1/n\}$, $\{(-1)^{n+1}(1/n)\}$, and $\{(n-1)/n\}$ each seem to approach a single limiting value as n increases, and $\{3\}$ is at a limiting value from the very first. On the other hand, terms of $\{(-1)^{n+1}(n-1)/n\}$ seem to accumulate near two different values, -1 and 1, whereas the terms of $\{\sqrt{n}\}$ become increasingly large and do not accumulate anywhere.

The following definition distinguishes those sequences that approach a unique limiting value L, as n increases, from those that do not.

Definitions Converges, Diverges, Limit

The sequence $\{a_n\}$ converges to the number L if to every positive number ϵ there corresponds an integer N such that for all n,

$$n > N \Rightarrow |a_n - L| < \epsilon.$$

If no such number L exists, we say that $\{a_n\}$ **diverges.**

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence (Figure 8.2).

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Historical Biography
Nicole Oresme
(ca. 1320 — 1382)

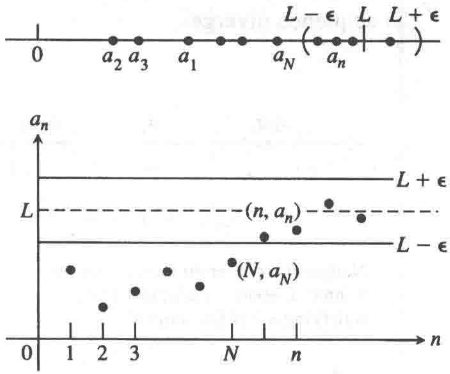


FIGURE 8.2 $a_n \to L$ if $y = L$ is a horizontal asymptote of the sequence of points $\{(n, a_n)\}$. In this figure, all the a_n's after a_N lie within ϵ of L.

Example 2 Testing the Definition

Show that

(a) $\lim_{n\to\infty} \frac{1}{n} = 0$

(b) $\lim_{n\to\infty} k = k$ (any constant k).

Solution

(a) Let $\epsilon > 0$ be given. We must show that there exists an integer N such that for all n,

$$n > N \Rightarrow \left|\frac{1}{n} - 0\right| < \epsilon.$$

This implication will hold if $(1/n) < \epsilon$ or $n > 1/\epsilon$. If N is any integer greater than $1/\epsilon$, the implication will hold for all $n > N$. This proves $\lim_{n\to\infty} (1/n) = 0$.

(b) Let $\epsilon > 0$ be given. We must show that there exists an integer N such that for all n,

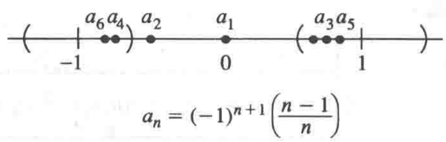
$$n > N \Rightarrow |k - k| < \epsilon.$$

Since $k - k = 0$, we can use any positive integer for N and the implication will hold. This proves that $\lim_{n\to\infty} k = k$ for any constant k.

Example 3 A Divergent Sequence

Show that $\{(-1)^{n+1}[(n-1)/n]\}$ diverges.

Solution Take a positive ϵ smaller than 1 so that the bands shown in Figure 8.3 about the lines $y = 1$ and $y = -1$ do not overlap. Any $\epsilon < 1$ will do. Convergence to 1 would require every point of the graph beyond a certain index N to lie inside the upper band, but this will never happen. As soon as a point (n, a_n) lies in the upper band, every alternate point starting with $(n+1, a_{n+1})$ will lie in the lower band. Hence, the sequence cannot converge to 1. Likewise, it cannot converge to -1. On the other hand, because the terms of the sequence get alternately closer to 1 and -1, they never accumulate near any other value. Therefore, the sequence diverges.



Neither the ϵ-interval about 1 nor the ϵ-interval about -1 contains all a_n satisfying $n \geq N$ for some N.

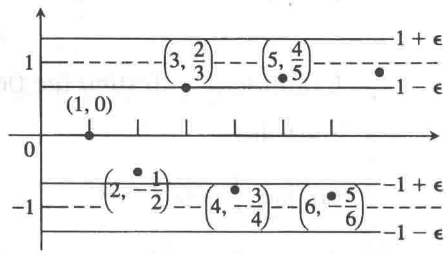


FIGURE 8.3 The sequence $\{(-1)^{n+1}[(n-1)/n]\}$ diverges.

The behavior of $\{(-1)^{n+1}[(n-1)/n]\}$ is qualitatively different from that of $\{\sqrt{n}\}$, which diverges because it outgrows every real number L. To describe the behavior of $\{\sqrt{n}\}$, we write

$$\lim_{n\to\infty} (\sqrt{n}) = \infty.$$

In speaking of infinity as a limit of a sequence $\{a_n\}$, we do not mean that the difference between a_n and infinity becomes small as n increases. We mean that a_n becomes numerically large as n increases.

Calculating Limits of Sequences

The study of limits would be cumbersome if we had to answer every question about convergence by applying the definition. Fortunately, three theorems make this largely unnecessary. The first theorem is not surprising, based on our previous work with limits. We omit the proofs.

Theorem 1 Limit Laws for Sequences

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule*: $\lim_{n\to\infty} (a_n + b_n) = A + B$
2. *Difference Rule*: $\lim_{n\to\infty} (a_n - b_n) = A - B$
3. *Product Rule*: $\lim_{n\to\infty} (a_n \cdot b_n) = A \cdot B$
4. *Constant Multiple Rule*: $\lim_{n\to\infty} (k \cdot b_n) = k \cdot B$ (any number k)
5. *Quotient Rule*: $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

Example 4 Applying the Limit Laws

By combining Theorem 1 with the limit results in Example 2, we have

(a) $\lim_{n\to\infty} \left(-\frac{1}{n}\right) = -1 \cdot \lim_{n\to\infty} \frac{1}{n} = -1 \cdot 0 = 0$

(b) $\lim_{n\to\infty} \left(\frac{n-1}{n}\right) = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n} = 1 - 0 = 1$

(c) $\lim_{n\to\infty} \frac{5}{n^2} = 5 \cdot \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{1}{n} = 5 \cdot 0 \cdot 0 = 0$

(d) $\lim_{n\to\infty} \frac{4 - 7n^6}{n^6 + 3} = \lim_{n\to\infty} \frac{(4/n^6) - 7}{1 + (3/n^6)} = \frac{0 - 7}{1 + 0} = -7.$

Example 5 Constant Multiples of Divergent Sequences Diverge

Every nonzero multiple of a divergent sequence $\{a_n\}$ diverges. Suppose, to the contrary, that $\{ca_n\}$ converges for some number $c \neq 0$. Then, by taking $k = 1/c$ in the Constant Multiple Rule in Theorem 1, we see that the sequence

$$\left\{\frac{1}{c} \cdot ca_n\right\} = \{a_n\}$$

converges. Thus, $\{ca_n\}$ cannot converge unless $\{a_n\}$ also converges. If $\{a_n\}$ does not converge, then $\{ca_n\}$ does not converge.

You are asked to prove the next Theorem in Exercise 69.

Theorem 2 The Sandwich Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \le b_n \le c_n$ holds for all n beyond some index N and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.