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RELATIONS BETWEEN THE SOLUTIONS OF A LINEAR DIFFERENTIAL EQUATION OF SECOND ORDER WITH FOUR REGULAR SINGULAR POINTS

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I. INTRODUCTION.

The most general type of linear differential equation of the second order with three regular singular points is the well-known hypergeometric equation. This classical equation is familiar to all mathematicians and the relations between its functions are of remarkable interest.

It is the purpose of this paper to study the relations between the solutions of an analogous equation with four regular singular points; the results of this study we can generalize to Klein's equation with n regular singular points, of which the hypergeometric function becomes a special case. This differential equation has been studied by both Karl Heun[1] and C. Franz[2] who obtained some special interesting properties about its series solutions. Here we manage to give a single way to derive the differential equation from that of Klein and a general and idoneous method to get the 192 integrals, and we may also generalize to obtain the

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1 Heun, Zur Theioie der Riemann'schen Functionen zweiter Ordnung mit vier Verzweigungspunkten, Mathem Ann Bd. 33 pg. 161 (1889).

2 Franz, Untersuchungen udie Lineare Homogene Differential gleichung 2. Ordnung der Fuchs'schen Lkasse mit drei im Endlichen gelegenen singularer Stellen, (1898).

$n!\,2^{n-1}$ integrals of Klein's equation. Since the equation has an arbitrary undetermined constant, this may be chosen with special value zero and supposing that a solution of this particular differential equation is known we thus get its relation with the general solution and consequently we can also apply to the most general Klein's equation.

II. Derivation of the Differential Equation.

The most general linear homogeneous equation of the second order and of Fuchsian type, having n singularities in the finite part of the plane, say $a_1, a_2, \ldots a_n$, with exponents α_i, β_i respectively, and ∞ being an ordinary point, was given by Klein to be[1]

$$\frac{d^2y}{dx^2}+\left\{\sum_{i=1}^{n}\frac{1-\alpha_i-\beta_i}{x-a_i}\right\}\frac{dy}{dx}+\left\{\sum_{i=1}^{n}\frac{\alpha_i\beta_i}{(x-a_i)^2}+\sum_{i=1}^{n}\frac{D_i}{x-a_i}\right\}y=0 \quad \cdots\cdots (1)$$

with the relations,

$$\sum_{i=1}(\alpha_i+\beta_i)=n-2, \quad \sum_{i=1}D_i=0, \quad \sum_{i=1}(a_iD_i+\alpha_i\beta_i)=0, \quad \sum_{i=1}^{n}(a_i^2D_i+2a_i\alpha_i\beta_i)$$

Taking $n=4$ and letting $\quad a_1\to 0,\ a_2\to 1,\ a_3=a,\ a_4\to\infty$

we have

$$\frac{d^2y}{dx^2}+\left(\frac{1-\alpha_1-\beta_1}{x}+\frac{1-\alpha_2-\beta_2}{x-1}+\frac{1-\alpha_3-\beta_3}{x-a}\right)\frac{dy}{dx}+\left[\frac{\alpha_1\beta_1}{x^2}+\frac{\alpha_2\beta_2}{(x-1)^2}+\frac{\alpha_3\beta_3}{(x-a)^2}+\frac{D_1}{x}+\frac{D_2}{x-1}+\frac{D_3}{x-a}\right]y=0 \cdots (2)$$

with the relations,

$$\alpha_1+\beta_1+\alpha_2+\beta_2+\alpha_3+\beta_3+\alpha_4+\beta_4=2, \quad D_1+D_2+D_3=0, \quad D_2+aD_3+\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3=\pm\alpha_4\beta_4$$

Though the last relation has double signs, we can choose one the other being obtainable by simple transformation. Thus choosing the + sign, and letting $aD_1=-\alpha_4\beta_4 q$, we obtain

$$\frac{d^2y}{dx^2}+\left(\frac{1-\alpha_1-\beta_1}{x}+\frac{1-\alpha_2-\beta_2}{x-1}+\frac{1-\alpha_3-\beta_3}{x-a}\right)\frac{dy}{dx}+\left[\frac{\alpha_1\beta_1}{x^2}+\frac{\alpha_2\beta_2}{(x-1)^2}+\frac{\alpha_3\beta_3}{(x-a)^2}+\frac{(\alpha_4\beta_4-\alpha_1\beta_1-\alpha_2\beta_2-\alpha_3\beta_3)x-\alpha_4\beta_4 q}{x(x-1)(x-a)}\right]y=0 \;\cdots\cdots (I)$$

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Whittaker, Modern Analysis, (1920) p.209, and also, Forsyth, Theory of Differential Equations, (1902), Vol. IV. pp. 154-5.

Following the Rieman P-function (I) is defined by the scheme

$$P\left\{\begin{matrix} 0 & 1 & a & \infty & \\ \alpha_1 & \alpha & \alpha_3 & \alpha_4 & x \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \end{matrix}\right\}$$

For $\alpha_1=\alpha_2=\alpha_3=0,\ \beta_1=1-\gamma,\ \beta_2=1-\delta,\ \beta_3=\gamma+\delta-\alpha-\beta,\ \alpha_4=\alpha,\ \beta_4=\beta,$

we have Heun's equation

$$x(x-1)(x-a)\frac{d^2y}{dx^2}+\left[(\alpha+\beta+1)x^2-\{\alpha+\beta-\delta+1+(\gamma+\delta)a\}x+a\gamma\right]\frac{dy}{dx}+\alpha\beta(x-q)y=0 \quad \ldots\ldots\ldots\ldots (I')$$

The scheme of Heun is

$$P\left\{\begin{matrix} 0 & 1 & a & \infty & \\ 0 & 0 & 0 & \alpha & x \\ 1-\gamma & 1-\delta & \gamma+\delta-\alpha-\beta & \beta & \end{matrix}\right\}$$

If a=1, q=1, the equation (I′) becomes:

$$x(x-1)^2\frac{d^2y}{dx^2}+\left[(\alpha+\beta+1)x^2-\{\alpha+\beta+\gamma+1\}x+\gamma\right]\frac{dy}{dx}+\alpha\beta(x-1)y=0$$

Which simplified reduces to

$$x(x-1)\frac{d^2y}{dx^2}+\left[(\alpha+\beta+1)x-\gamma\right]\frac{dy}{dx}+\alpha\beta y=0$$

which is the so-called hypergeometric equation and satisfied by

$$y=F(\alpha,\beta,\gamma,x).$$

and if a=o, q=o, The equation (I′) degenerates into

$$x(x-1)\frac{d^2y}{dx^2}+\left[(\alpha+\beta+1)x-(\alpha+\beta-\delta+1)\right]\frac{dy}{dx}+\alpha\beta y=0$$

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1 Heun; Franz; Whittaker, pp. 576-7 Forsyth, pp. 158-9; Ince, Ordinary Differential Equations (1927) p. 394.

Hence it is satisfied by

$$y = F(\alpha, \beta, \alpha+\beta-\delta+1 \quad x).$$

If $a \to \infty$, the equation degenerates also into the hypergeometric equation, a fact not pointed out by Heun and Franz. For from (2), we have

$$\frac{d^2y}{dx^2}+\left(\frac{\gamma}{x}+\frac{\delta}{x-1}+\frac{\varepsilon}{x-a}\right)\frac{dy}{dx}+\left[\frac{D_1}{x}+\frac{D_2}{x-1}+\frac{D_3}{x-a}\right]y=0$$

with the relations

$$\alpha+\beta+1=\gamma+\delta+\varepsilon, \quad D_1+D_2+D_3=0, \quad D_2+aD_3-\alpha\beta=0$$

Let $a \to \infty$ then $D_3=0$ and $D_1+D_2=0$. Hence we have

$$x(x-1)\frac{d^2y}{dx^2}+[(\gamma+\delta)x-\gamma]\frac{dy}{dx}-D_1 y=0$$

Let $D_1=-(\delta-1)\gamma$ we have

$$x(x-1)\frac{d^2y}{dx^2}+[(\gamma+\delta)x-\gamma]\frac{dy}{dx}+(\delta-1)\gamma y=0$$

which is satisfied by

$$y = F(\gamma, \delta-1, \gamma, x)$$

III. 192 solutions of the differential equation.

By a homographic transformation of the variable, the four points o, l, a, ∞, are interchanged, except that a may go into another a', among themselves. As is known 24 such substitutions are possible, namely

$$
\begin{array}{cccccc}
x & 1-x & \frac{}{x} & \frac{x-1}{x} & \frac{1}{1-x} & \frac{}{x-1} \\[1ex]
\frac{a}{x} & \frac{x-a}{x} & \frac{x}{a} & \frac{a-x}{a} & \frac{x}{x-a} & \frac{a}{a-x} \\[1ex]
\frac{x-a}{x-1} & \frac{a-1}{x-1} & \frac{x-1}{x-a} & \frac{1-a}{x-a} & \frac{x-1}{a-1} & \frac{x-a}{1-a} \\[1ex]
\frac{a(x-1)}{x-a} & \frac{(1-a)x}{x-a} & \frac{x-a}{a(x-1)} & \frac{(a-1)x}{a(x-1)} & \frac{x-a}{(1-a)x} & \frac{a(x-1)}{a-1)x}
\end{array}
$$

The following table illustrates the results of the aforementioned substitutions :

$$\{0\ \ 1\ \ a\ \ \infty\ \ x\}\{\infty\ \ a\ \ 1\ \ 0\ \ \tfrac{a}{x}\}\{a\ \ \infty\ \ 0\ \ 1\ \ \tfrac{x-a}{x-1}\}\{1\ \ 0\ \ \infty\ \ a\ \ \tfrac{a(x-1)}{x-a}\}$$

$$\{1\ \ 0\ \ 1-a\ \ \infty\ \ 1-x\}\{\infty\ \ 1-a\ \ 0\ \ 1\ \ \tfrac{x-a}{x}\}\{1-a\ \ \infty\ \ 1\ \ 0\ \ \tfrac{a-1}{x-1}\}\{0\ \ 1\ \ \infty\ \ 1-a\ \ \tfrac{(1-a)x}{x-a}\}$$

$$\{0\ \ \tfrac{1}{a}\ \ 1\ \ \infty\ \ \tfrac{x}{a}\}\{\infty\ \ 1\ \ \tfrac{1}{a}\ \ 0\ \ \tfrac{1}{x}\}\{1\ \ \infty\ \ 0\ \ \tfrac{1}{a}\ \ \tfrac{x-a}{a(x-1)}\}\{\tfrac{1}{a}\ \ 0\ \ \infty\ \ 1\ \ \tfrac{x-1}{x-a}\}$$

$$\{1\ \ \tfrac{a-1}{a}\ \ 0\ \ \infty\ \ \tfrac{a-x}{a}\}\{\infty\ \ 0\ \ \tfrac{a-1}{a}\ \ 1\ \ \tfrac{x-1}{x}\}\{0\ \ \infty\ \ 1\ \ \tfrac{a-1}{a}\ \ \tfrac{(a-1)x}{a(x-1)}\}\{\tfrac{a-1}{a}\ \ 1\ \ \infty\ \ c\ \ \tfrac{1-a}{x-a}\}$$

$$\{\tfrac{1}{1-a}\ \ 0\ \ 1\ \ \infty\ \ \tfrac{x-1}{a-1}\}\{\infty\ \ 1\ \ 0\ \ \tfrac{1}{1-a}\ \ \tfrac{x-a}{(1-a)x}\}\{1\ \ \infty\ \ \tfrac{}{1-a}\ \ 0\ \ \tfrac{1}{1-x}\}\{0\ \ \tfrac{1}{1-a}\ \ \infty\ \ 1\ \ \tfrac{x}{x-a}\}$$

$$\{\tfrac{a}{a-1}\ \ 1\ \ 0\ \ \infty\ \ \tfrac{x-a}{1-a}\}\{\infty\ \ 0\ \ 1\ \ \tfrac{a}{a-1}\ \ \tfrac{a(x-1)}{(a-1)x}\}\{0\ \ \infty\ \ \tfrac{}{a-1}\ \ 1\ \ \tfrac{x}{x-1}\}\{1\ \ \tfrac{a}{a-1}\ \ \infty\ \ 0\ \ \tfrac{a}{a-x}\}$$

We proceed to find the 192 solutions. The solutions of the equation (I'), which is regular in the vicinity of x = o, and belongs to the exponent o is given by[1]

$$F(a,q;\alpha,\beta,\gamma,\delta,x) = 1 + \alpha\beta \sum_{n=0}^{\infty} \frac{G_{n+1}(q)}{|\underline{n+1}\ \gamma(\gamma+1)\cdots(\gamma+n)} \left(\frac{x}{a}\right)^{n+1},$$

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1 Heun Math. Ann XXXIII Beitrage zur Theorie der Lame' schen Functionen, and Franz.

Where $G_1(q) = q$, $G_2(q) = \alpha\beta q^2 + \{(\alpha+\beta-\delta+1) + (\gamma+\delta)a\}q - a\gamma$,

$$G_{n+1}(q) = [n\{(\alpha+\beta-\delta+n) + (\gamma+\delta+n-1)a\} + \alpha\beta q]G_n(q) - (\alpha+n-1)(\beta+n-1)(\gamma+n-1)na\,G_{n-1}(q).$$

The series is absolutely convergent for $|x| < 1$ if $(a) > 1$;

and fonr $|x| < |a|$ if $(a) < 1$. And when $(x) = 1$ and $(a) > 1$

a sufficient condition[1] for absolute convergenee is that the real part of

$(\delta - 2)$ shall be less than -1 ; when $(x) = a$, $|a| < 1$,

the real part of $(\alpha+\beta-\gamma-\delta-1)$ shall be less than -1.

Moreover $F(a, q; \alpha, \beta, \gamma, \delta, 1)$ has a definite value if real part of

$(\delta - 2) < 0$ and $|a| > 1$; and also $F(a, q; \alpha, \beta, \gamma, \delta, a)$,

if real part of $(\alpha+\beta-\gamma-\delta-1) < 0$ and $|a| < 1$.[2]

1. Making the Fuchsian substitutions[3]

$$y = x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3}u$$

in equation (I) we have

$$\frac{d^2u}{dx^2} + \left(\frac{1+\alpha_1-\beta_1}{x} + \frac{1+\alpha_2-\beta_2}{x-1} + \frac{1+\alpha_3-\beta_3}{x-a}\right)\frac{du}{dx} + \frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\beta_4)x - (\alpha_1+\alpha_2-\alpha_4\beta_2-\alpha_4\beta_1)a - (\alpha_1+\alpha_3-\alpha_4\beta_1-\alpha_4\beta_3) - \alpha_4\beta_4 q}{x(x-1)(x-a)}u = 0$$

Let $q_1 = + \dfrac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4) + (\alpha_1+\alpha_2+\alpha_3+\beta_4) + \alpha_4\beta_4 q}{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\beta_3)}$

Hence the above equation becomes

$$\frac{d^2u}{dx^2} + \left(\frac{1+\alpha_1-\beta_1}{x} + \frac{1+\alpha_2-\beta_2}{x-1} + \frac{1+\alpha_3-\beta_3}{x-a}\right)\frac{du}{dx} + \frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\beta_4)(x-q_1)}{x(x-1)(x-a)}u = 0 \ldots\ldots \quad (\mathrm{II}_1')$$

whose scheme is

$$P\left\{\begin{matrix} 0 & 1 & a & \infty & \\ 0 & & 0 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & x \\ \beta_1-\alpha_1 & \beta_2-\alpha_2 & \beta_3-\alpha_3 & \alpha_1+\alpha_2+\alpha_3+\beta_4 & \end{matrix}\right\}$$

and a particular solution of (I) we can easily see is

$$y = x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3}F(a, q_1; \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\beta_4, 1+\alpha_1-\beta_1, 1+\alpha_2-\beta_2, x),$$

provided $\beta_1 - \alpha_1$ is not a negative integer. For simplicity, we shall, throughout this discussion

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1 Weierstrass, Abhandlungen aus der Functionenlehre p. 220. The condition is also necessary. Cf. Bromwich, Infinite Series, pp. 202-4.

2 Heun and Franz.

3 A formula given in L. Heffter Linearen differential gleichungen (1894), pp. 224-6, also T. Craig, Linear differential equations (1889), Vol. 1, pp. 154-6.

assume none of the exponent differences $\beta_i - \alpha_i$, $(i = 1, 2, 3, 4)$ is zero or an integer, as in this exceptional case the general solution of the differential equation may involve logarithmic terms. The formulae in the exceptional case can be found in Franz's work. Now if in the above expression α_j be interchanged with β_j, $(j = 1, 2, 3)$ singly, doubly, or triply while α_4 and β_4 remain fixed it must still satisfy the differential equation (I), since the latter is unaffected by this change. We thus obtain altogether eight expressions. Moreover, if, in equation (II'_1), we set $t = 1 - x$ then we have

$$\frac{d^2u}{dt^2} + \left(\frac{1+\alpha_2-\beta_2}{t} + \frac{1+\alpha_1-\beta_1}{t-1} + \frac{1+\alpha_3-\beta_3}{t-\overline{1-a}}\right)\frac{du}{dx} + \frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\beta_4)\left[t-\{1-q_1\}\right]}{t(t-1)(t-\overline{1-a})}u = 0 \cdots (II'_2)$$

with scheme

$$P\left\{\begin{array}{ccccc} & 0 & 1-a & \infty & \\ & & & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & 1-x \\ \beta_1-\alpha_1 & \beta_2-\alpha_2 & \beta_3-\alpha_3 & \alpha_1+\alpha_2+\alpha_3+\beta_4 & \end{array}\right\}$$

and a particular solution of (I) is

$$y = x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3} F(1-a, 1-q_1)\ \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\beta_4, 1+\alpha_2-\beta_2\ 1+\alpha_1-\beta_1, 1-x).$$

We thus obtain eight new expressions. Similarly, we set

$$t = \frac{x}{a},\quad \frac{a-x}{a},\quad \frac{x-1}{a-1},\quad \frac{x-a}{1-a}$$

respectively, we have the differential equations with the corresponding schemes and solutions as follows :

$$\frac{d^2u}{dt^2} + \left(\frac{1+\alpha_1-\beta_1}{t} + \frac{1+\alpha_3-\beta_3}{t-1} + \frac{1+\alpha_2-\beta_2}{t-\frac{1}{a}}\right)\frac{du}{dt} + \frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\beta_4)(t-\frac{q_1}{a})}{t(t-1)(t-\frac{1}{a})}u = 0 \cdots\cdots (II'_3)$$

$$P\left\{\begin{array}{ccccc} 0 & \frac{1}{a} & 1 & \infty & \\ 0 & 0 & 0 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & \frac{x}{a} \\ \beta_1-\alpha_1 & \beta_2-\alpha_2 & \beta_3-\alpha_3 & \alpha_1+\alpha_2+\alpha_3+\beta_4 & \end{array}\right\}$$

$$y = x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3} F\left(\frac{1}{a}, \frac{q_1}{a}, \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\beta_4, 1+\alpha_1-\beta_1, 1+\alpha_3-\beta_3, \frac{x}{a}\right);$$

$$\frac{d^2u}{dt^2}+\left(\frac{1+\alpha_3-\beta_3}{t}+\frac{1+\alpha_1-\beta_1}{t-1}+\frac{1+\alpha_2-\beta_2}{t-\frac{a-1}{a}}\right)\frac{du}{dx}+\frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\beta_4)\left(t-\frac{a-q_1}{a}\right)u}{t(t-1)\left(t-\frac{a-1}{a}\right)}=0 \quad \cdots\cdots (II_4')$$

$$P\left\{\begin{array}{ccccc} 1 & \frac{a-1}{a} & & \infty & \\ 0 & & 0 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & \frac{a-x}{a} \\ \beta_1-\alpha_1 & \beta_2-\alpha_2 & \beta_3-\alpha_3 & \alpha_1+\alpha_2+\alpha_3+\beta_4 & \end{array}\right\}$$

$$y = x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3} F\left(\frac{a-1}{a}, \frac{a-q_1}{a}; \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\beta_4, 1+\alpha_3-\beta_3\ 1+\alpha_1-\beta_1, \frac{a-x}{a}\right);$$

$$\frac{d^2u}{dt^2}+\left(\frac{1+\alpha_2-\beta_2}{t}+\frac{1+\alpha_3-\beta_3}{t-1}+\frac{1+\alpha_1-\beta_1}{t-\frac{1}{1-a}}\right)\frac{du}{dt}+\frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\beta_4)\left(t-\frac{q_1-1}{a-1}\right)u}{t(t-1)\left(t-\frac{1}{1-a}\right)}=0 \cdots\cdots (II_5')$$

$$P\left\{\begin{array}{ccccc} \frac{1}{1-a} & 0 & 1 & \infty & \\ 0 & & 0 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & \frac{x-1}{a-1} \\ \beta_1-\alpha & \beta_2-\alpha_2 & \beta_3-\alpha_3 & \alpha_1+\alpha_2+\alpha_3+\beta_4 & \end{array}\right\}$$

$$y = x'(x-1)^{\alpha_2}(x-a)^{\alpha_3} F\left(\frac{1}{1-a}, \frac{q_1-1}{a-1}, \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\beta_4, 1+\alpha_2-\beta_2, 1+\alpha_3-\beta_3, \frac{x-1}{a-1}\right);$$

$$\frac{d^2u}{dt^2}+\left(\frac{1+\alpha_3-\beta_3}{t}+\frac{1+\alpha_2-\beta_2}{t-1}+\frac{1+\alpha_1-\beta_1}{t-\frac{a}{a-1}}\right)\frac{du}{dt}+\frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\beta_4)\left(t-\frac{q_1-a}{1-a}\right)u}{t(t-1)\left(t-\frac{a}{a-1}\right)}=0 \cdots\cdots\cdots (II_6')$$

$$P\left\{\begin{array}{ccccc} \overline{a-1} & 1 & 0 & \infty & \\ 0 & 0 & 0 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & \frac{x-a}{1-a} \\ \beta_1-\alpha_1 & \beta_2-\alpha_2 & \beta_3-\alpha_3 & \alpha_1+\alpha_2+\alpha_3+\beta_4 & \end{array}\right\}$$

$$y = x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3} F\left(\frac{a}{a-1}, \frac{q_1-a}{1-a}, \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\beta_4, 1+\alpha_3-\beta_3, 1+\alpha_2-\beta_2, \frac{x-a}{1-a}\right)$$

We thus obtain $5 \times 8 = 40$ new expressions, which together with the original eight mkae forty-eight particular solutions of equation (I). The first set of forty-eight solutions may be written down as follows :

$$\left.\begin{aligned}
y_1 &= x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3} F(a, q_1; \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\beta_4, 1+\alpha_1-\beta_1, 1+\alpha_2-\beta_2, x) \\
y_2 &= x^{\beta_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3} F(a, q_1'; \beta_1+\alpha_2+\alpha_3+\alpha_4, \beta_1+\alpha_2+\alpha_3+\beta_4, 1+\beta_1-\alpha_1, 1+\alpha_2-\beta_2, x) \\
y_3 &= x^{\alpha_1}(x-1)^{\beta_2}(x-a)^{\alpha_3} F(a, q_1''; \alpha_1+\beta_2+\alpha_3+\alpha_4, \alpha_1+\beta_2+\alpha_3+\beta_4, 1+\alpha_1-\beta_1, 1+\beta_2-\alpha_2, x) \\
y_4 &= x^{\beta_1}(x-1)^{\beta_2}(x-a)^{\alpha_3} F(a, q_1'''; \beta_1+\beta_2+\alpha_3+\alpha_4, \beta_1+\beta_2+\alpha_3+\beta_4, 1+\beta_1-\alpha_1, 1+\beta_2-\alpha_2, x) \\
y_5 &= x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\beta_3} F(a, q_1^{iv}; \alpha_1+\alpha_2+\beta_3+\alpha_4, \alpha_1+\alpha_2+\beta_3+\beta_4, 1+\alpha_1-\beta_1, 1+\alpha_2-\beta_2, x) \\
y_6 &= x^{\alpha_1}(x-1)^{\beta_2}(x-a)^{\beta_3} F(a, q_1^{v}; \alpha_1+\beta_2+\beta_3+\alpha_4, \alpha_1+\beta_2+\beta_3+\beta_4, 1+\alpha_1-\beta_1, 1+\beta_2-\alpha_2, x) \\
y_7 &= x^{\beta_1}(x-1)^{\alpha_2}(x-a)^{\beta_3} F(a, q_1^{vi}; \beta_1+\alpha_2+\beta_3+\alpha_4, \beta_1+\alpha_2+\beta_3+\beta_4, 1+\beta_1-\alpha_1, 1+\alpha_2-\beta_2, x) \\
y_8 &= x^{\beta_1}(x-1)^{\beta_2}(x-a)^{\beta_3} F(a, q_1^{vii}; \beta_1+\beta_2+\beta_3+\alpha_4, \beta_1+\beta_2+\beta_3+\beta_4, 1+\beta_1-\alpha_1, 1+\beta_2-\alpha_2, x)
\end{aligned}\right\} (\mathrm{II}_1)$$

$$\left.\begin{aligned}
y_9 &= x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3} F(1-a, 1-q_1; \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\beta_4, 1+\alpha_2-\beta_2, 1+\alpha_1-\beta_1, 1-x) \\
y_{10} &= x^{\beta_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3} F(1-a, 1-q_1'; \beta_1+\alpha_2+\alpha_3+\alpha_4, \beta_1+\alpha_2+\alpha_3+\beta_4, 1+\alpha_2-\beta_2, 1+\beta_1-\alpha_1, 1-x) \\
y_{11} &= x^{\alpha_1}(x-1)^{\beta_2}(x-a)^{\alpha_3} F(1-a, 1-q_1''; \alpha_1+\beta_2+\alpha_3+\alpha_4, \alpha_1+\beta_2+\alpha_3+\beta_4, 1+\beta_2-\alpha_2, 1+\alpha_1-\beta_1, 1-x) \\
y_{12} &= x^{\beta_1}(x-1)^{\beta_2}(x-a)^{\alpha_3} F(1-a, 1-q_1'''; \beta_1+\beta_2+\alpha_3+\alpha_4, \beta_1+\beta_2+\alpha_3+\beta_4, 1+\beta_2-\alpha_2, 1+\beta_1-\alpha_1, 1-x) \\
y_{13} &= x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\beta_3} F(1-a, 1-q_1^{iv}; \alpha_1+\alpha_2+\beta_3+\alpha_4, \alpha_1+\alpha_2+\beta_3+\beta_4, 1+\alpha_2-\beta_2, 1+\alpha_1-\beta_1, 1-x) \\
y_{14} &= x^{\alpha_1}(x-1)^{\beta_2}(x-a)^{\beta_3} F(1-a, 1-q_1^{v}; \alpha_1+\beta_2+\beta_3+\alpha_4, \alpha_1+\beta_2+\beta_3+\beta_4, 1+\beta_2-\alpha_2, 1+\alpha_1-\beta_1, 1-x) \\
y_{15} &= x^{\beta_1}(x-1)^{\alpha_2}(x-a)^{\beta_3} F(1-a, 1-q_1^{vi}; \beta_1+\alpha_2+\beta_3+\alpha_4, \beta_1+\alpha_2+\beta_3+\beta_4, 1+\alpha_2-\beta_2, 1+\beta_1-\alpha_1, 1-x) \\
y_{16} &= x^{\beta_1}(x-1)^{\beta_2}(x-a)^{\beta_3} F(1-a, 1-q_1^{vii}; \beta_1+\beta_2+\beta_3+\alpha_4, \beta_1+\beta_2+\beta_3+\beta_4, 1+\beta_2-\alpha_2, 1+\beta_1-\alpha_1, 1-x)
\end{aligned}\right\} (\mathrm{II}'_2)$$

$$\left.\begin{aligned}
y_{17} &= x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3}\, F\left(\tfrac{1}{a}, \tfrac{q_1}{a};\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\alpha_2+\alpha_3+\beta_4,\ 1+\alpha_1-\beta_1,\ 1+\alpha_3-\beta_3,\ \tfrac{x}{a}\right)\\
y_{18} &= x^{\beta_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3}\, F\left(\tfrac{1}{a}, \tfrac{q_1'}{a};\ \beta_1+\alpha_2+\alpha_3+\alpha_4,\ \beta_1+\alpha_2+\alpha_3+\beta_4,\ 1+\beta_1-\alpha_1,\ 1+\alpha_3-\beta_3,\ \tfrac{x}{a}\right)\\
y_{19} &= x^{\alpha_1}(x-1)^{\beta_2}(x-a)^{\alpha_3}\, F\left(\tfrac{1}{a}, \tfrac{q_1''}{a};\ \alpha_1+\beta_2+\alpha_3+\alpha_4,\ \alpha_1+\beta_2+\alpha_3+\beta_4,\ 1+\alpha_1-\beta_1,\ 1+\alpha_3-\beta_3,\ \tfrac{x}{a}\right)\\
y_{20} &= x^{\beta_1}(x-1)^{\beta_2}(x-a)^{\alpha_3}\, F\left(\tfrac{1}{a}, \tfrac{q_1'''}{a};\ \beta_1+\beta_2+\alpha_3+\alpha_4,\ \beta_1+\beta_2+\alpha_3+\beta_4,\ 1+\beta_1-\alpha_1,\ 1+\alpha_3-\beta_3,\ \tfrac{x}{a}\right)\\
y_{21} &= x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\beta_3}\, F\left(\tfrac{1}{a}, \tfrac{q_1^{iv}}{a};\ \alpha_1+\alpha_2+\beta_3+\alpha_4,\ \alpha_1+\alpha_2+\beta_3+\beta_4,\ 1+\alpha_1-\beta_1,\ 1+\beta_3-\alpha_3,\ \tfrac{x}{a}\right)\\
y_{22} &= x^{\alpha_1}(x-1)^{\beta_2}(x-a)^{\beta_3}\, F\left(\tfrac{1}{a}, \tfrac{q_1^{v}}{a};\ \alpha_1+\beta_2+\beta_3+\alpha_4,\ \alpha_1+\beta_2+\beta_3+\beta_4,\ 1+\alpha_1-\beta_1,\ 1+\beta_3-\alpha_3,\ \tfrac{x}{a}\right)\\
y_{23} &= x^{\beta_1}(x-1)^{\alpha_2}(x-a)^{\beta_3}\, F\left(\tfrac{1}{a}, \tfrac{q_1^{vi}}{a};\ \beta_1+\alpha_2+\beta_3+\alpha_4,\ \beta_1+\alpha_2+\beta_3+\beta_4,\ 1+\beta_1-\alpha_1,\ 1+\beta_3-\alpha_3,\ \tfrac{x}{a}\right)\\
y_{24} &= x^{\beta_1}(x-1)^{\beta_2}(x-a)^{\beta_3}\, F\left(\tfrac{1}{a}, \tfrac{q_1^{vii}}{a};\ \beta_1+\beta_2+\beta_3+\alpha_4,\ \beta_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_1-\alpha_1,\ 1+\beta_3-\alpha_3,\ \tfrac{x}{a}\right)
\end{aligned}\right\}\ (\mathrm{II}_3')$$

$$\left.\begin{aligned}
y_{25} &= x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3}\, F\left(\tfrac{a-1}{a}, \tfrac{a-q_1}{a};\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\alpha_2+\alpha_3+\beta_4,\ 1+\alpha_3-\beta_3,\ 1+\alpha_1-\beta_1,\ \tfrac{a-x}{a}\right)\\
y_{26} &= x^{\beta_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3}\, F\left(\tfrac{a-1}{a}, \tfrac{a-q_1'}{a};\ \beta_1+\alpha_2+\alpha_3+\alpha_4,\ \beta_1+\alpha_2+\alpha_3+\beta_4,\ 1+\alpha_3-\beta_3,\ 1+\beta_1-\alpha_1,\ \tfrac{a-x}{a}\right)\\
y_{27} &= x^{\alpha_1}(x-1)^{\beta_2}(x-a)^{\alpha_3}\, F\left(\tfrac{a-1}{a}, \tfrac{a-q_1''}{a};\ \alpha_1+\beta_2+\alpha_3+\alpha_4,\ \alpha_1+\beta_2+\alpha_3+\beta_4,\ 1+\alpha_3-\beta_3,\ 1+\alpha_1-\beta_1,\ \tfrac{a-x}{a}\right)\\
y_{28} &= x^{\beta_1}(x-1)^{\beta_2}(x-a)^{\alpha_3}\, F\left(\tfrac{a-1}{a}, \tfrac{a-q_1'''}{a};\ \beta_1+\beta_2+\alpha_3+\alpha_4,\ \beta_1+\beta_2+\alpha_3+\beta_4,\ 1+\alpha_3-\beta_3,\ 1+\beta_1-\alpha_1,\ \tfrac{a-x}{a}\right)\\
y_{29} &= x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\beta_3}\, F\left(\tfrac{a-1}{a}, \tfrac{a-q_1^{iv}}{a};\ \alpha_1+\alpha_2+\beta_3+\alpha_4,\ \alpha_1+\alpha_2+\beta_3+\beta_4,\ 1+\beta_3-\alpha_3,\ 1+\alpha_1-\beta_1,\ \tfrac{a-x}{a}\right)\\
y_{30} &= x^{\alpha_1}(x-1)^{\beta_2}(x-a)^{\beta_3}\, F\left(\tfrac{a-1}{a}, \tfrac{a-q_1^{v}}{a};\ \alpha_1+\beta_2+\beta_3+\alpha_4,\ \alpha_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_3-\alpha_3,\ 1+\alpha_1-\beta_1,\ \tfrac{a-x}{a}\right)\\
y_{31} &= x^{\beta_1}(x-1)^{\alpha_2}(x-a)^{\beta_3}\, F\left(\tfrac{a-1}{a}, \tfrac{a-q_1^{vi}}{a};\ \beta_1+\alpha_2+\beta_3+\alpha_4,\ \beta_1+\alpha_2+\beta_3+\beta_4,\ 1+\beta_3-\alpha_3,\ 1+\beta_1-\alpha_1,\ \tfrac{a-x}{a}\right)\\
y_{32} &= x^{\beta_1}(x-1)^{\beta_2}(x-a)^{\beta_3}\, F\left(\tfrac{a-1}{a}, \tfrac{a-q_1^{vii}}{a};\ \beta_1+\beta_2+\beta_3+\alpha_4,\ \beta_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_3-\alpha_3,\ 1+\beta_1-\alpha_1,\ \tfrac{a-x}{a}\right)
\end{aligned}\right\}\ (\mathrm{II}_4')$$

$$\left.\begin{aligned}
y_{33} &= x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3} F\left(\tfrac{1}{1-a}, \tfrac{q_1-1}{a-1};\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\alpha_2+\alpha_3+\beta_4,\ 1+\alpha_2-\beta_2,\ 1+\alpha_3-\beta_3,\ \tfrac{x-1}{a-1}\right)\\
y_{34} &= x^{\beta_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3} F\left(\tfrac{1}{1-a}, \tfrac{q_1'-1}{a-1};\ \beta_1+\alpha_2+\alpha_3+\alpha_4,\ \beta_1+\alpha_2+\alpha_3+\beta_4,\ 1+\alpha_2-\beta_2,\ 1+\alpha_3-\beta_3,\ \tfrac{x-1}{a-1}\right)\\
y_{35} &= x^{\alpha_1}(x-1)^{\beta_2}(x-a)^{\alpha_3} F\left(\tfrac{1}{1-a}, \tfrac{q_1''-1}{a-1};\ \alpha_1+\beta_2+\alpha_3+\alpha_4,\ \alpha_1+\beta_2+\alpha_3+\beta_4,\ 1+\beta_2-\alpha_2,\ 1+\alpha_3-\beta_3,\ \tfrac{x-1}{a-1}\right)\\
y_{36} &= x^{\beta_1}(x-1)^{\beta_2}(x-a)^{\alpha_3} F\left(\tfrac{1}{1-a}, \tfrac{q_1'''-1}{a-1};\ \beta_1+\beta_2+\alpha_3+\alpha_4,\ \beta_1+\beta_2+\alpha_3+\beta_4,\ 1+\beta_2-\alpha_2,\ 1+\alpha_3-\beta_3,\ \tfrac{x-1}{a-1}\right)\\
y_{37} &= x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\beta_3} F\left(\tfrac{1}{1-a}, \tfrac{q_1^{iv}-1}{a-1};\ \alpha_1+\alpha_2+\beta_3+\alpha_4,\ \alpha_1+\alpha_2+\beta_3+\beta_4,\ 1+\alpha_2-\beta_2,\ 1+\beta_3-\alpha_3,\ \tfrac{x-1}{a-1}\right)\\
y_{38} &= x^{\alpha_1}(x-1)^{\beta_2}(x-a)^{\beta_3} F\left(\tfrac{1}{1-a}, \tfrac{q_1^{v}-1}{a-1};\ \alpha_1+\beta_2+\beta_3+\alpha_4,\ \alpha_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_2-\alpha_2,\ 1+\beta_3-\alpha_3,\ \tfrac{x-1}{a-1}\right)\\
y_{39} &= x^{\beta_1}(x-1)^{\alpha_2}(x-a)^{\beta_3} F\left(\tfrac{1}{1-a}, \tfrac{q_1^{vi}-1}{a-1};\ \beta_1+\alpha_2+\beta_3+\alpha_4,\ \beta_1+\alpha_2+\beta_3+\beta_4,\ 1+\alpha_2-\beta_2,\ 1+\beta_3-\alpha_3,\ \tfrac{x-1}{a-1}\right)\\
y_{40} &= x^{\beta_1}(x-1)^{\beta_2}(x-a)^{\beta_3} F\left(\tfrac{1}{1-a}, \tfrac{q_1^{vii}-1}{a-1};\ \beta_1+\beta_2+\beta_3+\alpha_4,\ \beta_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_2-\alpha_2,\ 1+\beta_3-\alpha_3,\ \tfrac{x-1}{a-1}\right)
\end{aligned}\right\}\ (\mathrm{II}'_5)$$

$$\left.\begin{aligned}
y_{41} &= x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3} F\left(\tfrac{a}{a-1}, \tfrac{q_1-a}{1-a};\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\alpha_2+\alpha_3+\beta_4,\ 1+\alpha_3-\beta_3,\ 1+\alpha_2-\beta_2,\ \tfrac{x-a}{a-1}\right)\\
y_{42} &= x^{\beta_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3} F\left(\tfrac{a}{a-1}, \tfrac{q_1'-a}{1-a};\ \beta_1+\alpha_2+\alpha_3+\alpha_4,\ \beta_1+\alpha_2+\alpha_3+\beta_4,\ 1+\alpha_3-\beta_3,\ 1+\alpha_2-\beta_2,\ \tfrac{x-a}{a-1}\right)\\
y_{43} &= x^{\alpha_1}(x-1)^{\beta_2}(x-a)^{\alpha_3} F\left(\tfrac{a}{a-1}, \tfrac{q_1''-a}{1-a};\ \alpha_1+\beta_2+\alpha_3+\alpha_4,\ \alpha_1+\beta_2+\alpha_3+\beta_4,\ 1+\alpha_3-\beta_3,\ 1+\beta_2-\alpha_2,\ \tfrac{x-a}{a-1}\right)\\
y_{44} &= x^{\beta_1}(x-1)^{\beta_2}(x-a)^{\alpha_3} F\left(\tfrac{a}{a-1}, \tfrac{q_1'''-a}{1-a};\ \beta_1+\beta_2+\alpha_3+\alpha_4,\ \beta_1+\beta_2+\alpha_3+\beta_4,\ 1+\alpha_3-\beta_3,\ 1+\beta_2-\alpha_2,\ \tfrac{x-a}{a-1}\right)\\
y_{45} &= x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\beta_3} F\left(\tfrac{a}{a-1}, \tfrac{q_1^{iv}-a}{1-a};\ \alpha_1+\alpha_2+\beta_3+\alpha_4,\ \alpha_1+\alpha_2+\beta_3+\beta_4,\ 1+\beta_3-\alpha_3,\ 1+\alpha_2-\beta_2,\ \tfrac{x-a}{a-1}\right)\\
y_{46} &= x^{\alpha_1}(x-1)^{\beta_2}(x-a)^{\beta_3} F\left(\tfrac{a}{a-1}, \tfrac{q_1^{v}-a}{1-a};\ \alpha_1+\beta_2+\beta_3+\alpha_4,\ \alpha_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_3-\alpha_3,\ 1+\beta_2-\alpha_2,\ \tfrac{x-a}{a-1}\right)\\
y_{47} &= x^{\beta_1}(x-1)^{\alpha_2}(x-a)^{\beta_3} F\left(\tfrac{a}{a-1}, \tfrac{q_1^{vi}-a}{1-a};\ \beta_1+\alpha_2+\beta_3+\alpha_4,\ \beta_1+\alpha_2+\beta_3+\beta_4,\ 1+\beta_3-\alpha_3,\ 1+\alpha_2-\beta_2,\ \tfrac{x-a}{a-1}\right)\\
y_{48} &= x^{\beta_1}(x-1)^{\beta_2}(x-a)^{\beta_3} F\left(\tfrac{a}{a-1}, \tfrac{q_1^{vii}-a}{1-a};\ \beta_1+\beta_2+\beta_3+\alpha_4,\ \beta_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_3-\alpha_3,\ 1+\beta_2-\alpha_2,\ \tfrac{x-a}{a-1}\right)
\end{aligned}\right\}\ (\mathrm{II}'_6)$$

Where

$$q_1 = \frac{(\alpha_1+\alpha_2-\alpha_1\beta_2-\alpha_2\beta_1)a+(\alpha_1+\alpha_3-\alpha_1\beta_3-\alpha_3\beta_1)+\alpha_4\beta_4 q}{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\beta_4)}$$

$$q_1 = \frac{(\beta_1+\alpha_2-\beta_1\beta_2-\alpha_2\alpha_1)a+(\beta_1+\alpha_3-\beta_1\beta_3-\alpha_3\beta_1)+\alpha_4\beta_4 q}{(\beta_1+\alpha_2+\alpha_3+\alpha_4)(\beta_1+\alpha_2+\alpha_3+\beta_4)}$$

$$q_1'' = \frac{(\alpha_1+\beta_2-\alpha_1\alpha_2-\beta_2\beta_1)a+(\alpha_1+\alpha_3-\alpha_1\beta_3-\alpha_3\beta_1)+\alpha_4\beta_4 q}{(\alpha_1+\beta_2+\alpha_3+\alpha_4)(\alpha_1+\beta_2+\alpha_3+\beta_4)}$$

$$q_1''' = \frac{(\beta_1+\beta_2-\beta_1\alpha_2-\beta_2\alpha_1)a+(\beta_1+\alpha_3-\beta_1\beta_3-\alpha_3\alpha_1)+\alpha_4\beta_4 q}{(\beta_1+\beta_2+\alpha_3+\alpha_4)(\beta_1+\beta_2+\alpha_3+\beta_4)}$$

$$q_1^{iv} = \frac{(\alpha_1+\alpha_2-\alpha_1\beta_2-\alpha_2\beta_1)a+(\alpha_1+\beta_3-\alpha_1\alpha_3-\beta_3\beta_1)+\alpha_4\beta_4 q}{(\alpha_1+\alpha_2+\beta_3+\alpha_4)(\alpha_1+\alpha_2+\beta_3+\beta_4)}$$

$$q_1^{v} = \frac{(\alpha_1+\beta_2-\alpha_1\alpha_2-\beta_1\beta_2)a+(\alpha_1+\beta_3-\alpha_1\alpha_3-\beta_1\beta_3)+\alpha_4\beta_4 q}{(\alpha_1+\beta_2+\beta_3+\alpha_4)(\alpha_1+\beta_2+\beta_3+\beta_4)}$$

$$q_1^{vi} = \frac{(\beta_1+\alpha_2-\beta_1\beta_2-\alpha_1\alpha_2)a+(\beta_1+\beta_3-\beta_1\alpha_3-\alpha_1\beta_3)+\alpha_4\beta_4 q}{(\beta_1+\alpha_2+\beta_2+\alpha_4)(\beta_1+\alpha_2+\beta_3+\beta_4)}$$

$$q_1^{vii} = \frac{(\beta_1+\beta_2-\beta_1\alpha_2-\alpha_1\beta_2)a+(\beta_1+\beta_3-\beta_1\alpha_3-\alpha_1\beta_3)+\alpha_4\beta_4 q}{(\beta_1+\beta_2+\beta_3+\alpha_4)(\beta_1+\beta_2+\beta_3+\beta_4)}$$