运筹与管理科学丛书 24

A First Course in Graph Theory

(图论基础教程)

Xu Junming

(徐俊明)

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《运筹与管理科学丛书》序

运筹学是运用数学方法来刻画、分析以及求解决策问题的科学. 运筹学的例子在我国古已有之, 春秋战国时期著名军事家孙膑为田忌赛马所设计的排序就是一个很好的代表. 运筹学的重要性同样在很早就被人们所认识, 汉高祖刘邦在称赞张良时就说道:"运筹帷幄之中, 决胜千里之外."

运筹学作为一门学科兴起于第二次世界大战期间, 源于对军事行动的研究. 运筹学的英文名字 Operational Research 诞生于 1937 年. 运筹学发展迅速, 目前已有众多的分支, 如线性规划、非线性规划、整数规划、网络规划、图论、组合优化、非光滑优化、锥优化、多目标规划、动态规划、随机规划、决策分析、排队论、对策论、物流、风险管理等.

我国的运筹学研究始于 20 世纪 50 年代, 经过半个世纪的发展, 运筹学队伍已具相当大的规模. 运筹学的理论和方法在国防、经济、金融、工程、管理等许多重要领域有着广泛应用, 运筹学成果的应用也常常能带来巨大的经济和社会效益. 由于在我国经济快速增长的过程中涌现出了大量迫切需要解决的运筹学问题, 因而进一步提高我国运筹学的研究水平、促进运筹学成果的应用和转化、加快运筹学领域优秀青年人才的培养是我们当今面临的十分重要、光荣、同时也是十分艰巨的任务. 我相信,《运筹与管理科学丛书》能在这些方面有所作为.

《运筹与管理科学丛书》可作为运筹学、管理科学、应用数学、系统科学、计算机科学等有关专业的高校师生、科研人员、工程技术人员的参考书, 同时也可作为相关专业的高年级本科生和研究生的教材或教学参考书. 希望该丛书能越办越好, 为我国运筹学和管理科学的发展做出贡献.

袁亚湘

2007 年 9 月

Preface

In the spectrum of mathematics, graph theory, as a recognized discipline, is a relative newcomer. In recent five decades, the exciting and rapidly growing area of the subject abounds with new mathematical developments and significant applications to real-world problems. More and more colleges and universities have made it a required course for the senior or the beginning postgraduate students who are majoring in mathematics, computer science, electronics, scientific management and others. This book provides a first course in graph theory for these students.

Graphs are mathematical structures used to model pairwise relations between objects. The richness of theory and the wideness of applications of graphs make it impossible to include all topics on graphs in a book. All materials presented in this book, I think, are the most classical, fundamental, interesting and important, and some of which are new. The method dealt with the materials is to particularly lay stress on digraphs, regarding undirected graphs as their special cases. My own experience from teaching out of the subject more than twenty years at University of Science and Technology of China (USTC) shows that this treatment makes hardly the course difficult, but much more accords with the essence and the development trend of the subject.

The book consists of eight chapters. The first two chapters introduce the most basic concepts and related results. From the third chapter to the eighth chapter, each chapter focuses on a special topic, including trees and graphic spaces, plane and planar graphs, flows and connectivity, matchings and independent sets, colorings and integer flows, graphs and groups. These topics are treated in some depth, both theoretical and applied, with some suggestions for further reading. Every effort will be made to strengthen the mutual connections among these topics, with an aim to make the materials more systematic and cohesive. All theorems will be clearly stated, together with full and concise proofs, some of them are new. A number of examples and figures are given to help the reader to understand the given materials. To explore the mathematical nature and perfection of graph theory better, this book will specially stress the equivalence of some classical results, such as the max-flow min-cut theorem of Ford and Fulkerson, Menger's theorem, Hall's theorem, Tutte's theorem and König's theorem.

To expand the reader's scope of knowledge, some further reading materials, including self-contained proofs of some theorems, new concepts, problems and conjectures, are added to the back of some sections, separated by the stars $*\ldots*$, at the first reading some readers may wish to skip them.

Throughout this book, the reader will see that graph theory has closed connection with other branches of mathematics, including linear algebra, matrix theory, group theory, combinatorics, combinatorial optimization and operation research, and wide applications to other subjects, including computer science, electronics, scientific management and so on. Thus, the reader who will read this book is supposed to familiarize himself with some basic concepts and methods of linear algebra and group theory. The applications carefully selected are arranged in the latter sections of the chapter with some classical and fundamental algorithms.

Exercises of each section, from routine practice to challenging, are supplements to the text. Some of them are very important results in graph theory. The harder ones are indicated by bold type.

In the development process of graph theory, people found many important results. With the loss of time, some findings are gradually being forgotten. So to be able to indicate the provenance of results is vital. To this end, the book lists related references and provides brief biographical notes on major scholars mentioned in this book.

The style of writing and presentation of this book have been, to a great extent, influenced by *Graph Theory with Applications*, a popular textbook written by J. A. Bondy and U. S. R. Murty whom I am grateful to, from which some typical materials have been directly selected in this book.

The book is developed from the text for a senior and first-year postgraduate course in one semester at USTC. I would like to thank Graduate School and School of Mathematical Sciences at USTC for their support and encouragement, and "211 Project" for its financial support.

Many people have contributed, directly or indirectly, to this book. I avail myself of this opportunity to particularly express my heartfelt gratitude to Li Qiao, Tian Feng, Liu Yanpei, Shao Jiayu, Chen Yongchuan, Yuan Yaxiang, Zhang Cunquan, Zhang Shenggui, et al. for their continuous help and valuable suggestions, also to my students Huang Jia, Yang Chao, Hu Futao, Hong Zhenmu and Li Xiangjun for drawing elegant diagrams.

Finally, I would like to express my appreciation to my wife, Qiu Jingxia, for her support, understanding and love, without which this work would have been impossible.

Xu Junming
(xujm@ustc.edu.cn)
October 2014, USTC, Hefei

Contents

Chapter 1

Basic Concepts of Graphs

In many real-world situations, it is particularly convenient to describe the specified relationship between pairs of certain given objects by means of a diagram, in which points represent the objects and (directed or undirected) lines represent the relationship between pairs of the objects. For example, a national traffic map describes a condition of the communication lines among cities in the country, where the points represent cities and the lines represent the highways or the railways joining pairs of cities. Notice that in such diagrams one is mainly interested in whether or not two given points are joined by a line; the manner in which they are joined is immaterial. A mathematical abstraction of situations of this type gives rise to the concept of a graph.

In fact, a graph provides the natural structures from which to construct mathematical models that are appropriate to almost all fields of scientific (natural and social) inquiry. The underlying subject of study in these fields is some set of "objects" and one or more "relations" between the objects.

In this chapter, we will introduce the concept and the geometric representation of a graph, terminology and natation, basic operations used in the remaining parts of the book. It should, for the beginner specially, be worth noting that most graph theorists use personalized terminology in their books, papers and lectures. Even the meaning of the word "graph" varies with different authors. We will adopt the most standard terminology and notation extensively used by most authors, such as Bondy and Murty[1] [42], with a subject index and a list of notations in the end of the book.

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[1] **J. A. Bondy** (John Adrian Bondy) is a professor of University of Waterloo and Université Lyon 1, received his Ph.D. from University of Oxford in 1969. **U. S. R. Murty** (Uppaluri Siva Ramachandra Murty) is a professor of University of Waterloo, received his Ph.D. from Indian Statistical Institute in 1967. Bondy and Murty served as editors-in-chief of *Journal of Combinatorial Theory*, Series B (1985-2004, see this journal, 2004, 90(1):1). They are well known and respected for many contributions to graph theory. Particularly, their joint textbook *Graph Theory with Applications* [42] is acclaimed by readers. The book's clear exposition and careful choice of topics made it widely influential, and for many years it was used as the principal reference for graph theory courses around the world. It is this textbook that plays an important role to standardize the terminology and notation of graphs. In 2008, they published the new book *Graph Theory* [43].

1.1 Graph and Graphical Representation

Let V be a non-empty set. An ordered pair (x, y) or an unorder pair xy is often used to denote a binary relation between two elements in V, where (x, y) denotes a unilateral relation from x to y and xy denotes a bilateral relation between x and y. A set of binary relations on V can be denoted as a subset of $V \times V$, the Cartesian product of V with itself. Mathematically, a *graph*[1] G is a mathematical structure (V, E), denoted by $G = (V, E)$, where $E \subseteq V \times V$.

Example 1.1.1 $D = (V(D), E(D))$ is a graph, where

$$V(D) = \{x_1, x_2, x_3, x_4, x_5\} \text{ and}$$
$$E(D) = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\},$$

and for each $i = 1, 2, \ldots, 8$, a_i is a unilateral relation defined by

$$a_1 = (x_1, x_2), \quad a_2 = (x_3, x_2), \quad a_3 = (x_3, x_3), \quad a_4 = (x_4, x_3),$$
$$a_5 = (x_4, x_2), \quad a_6 = (x_5, x_2), \quad a_7 = (x_2, x_5), \quad a_8 = (x_3, x_5).$$

Example 1.1.2 $H = (V(H), E(H))$ is a graph, where

$$V(H) = \{y_1, y_2, y_3, y_4, y_5\} \text{ and}$$
$$E(H) = \{b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8\},$$

and for each $i = 1, 2, \ldots, 8$, b_i is a unilateral relation defined by

$$b_1 = (y_1, y_2), \quad b_2 = (y_3, y_2), \quad b_3 = (y_3, y_3), \quad b_4 = (y_4, y_3),$$
$$b_5 = (y_4, y_2), \quad b_6 = (y_5, y_2), \quad b_7 = (y_2, y_5), \quad b_8 = (y_3, y_5).$$

Example 1.1.3 $G = (V(G), E(G))$ is a graph, where

$$V(G) = \{z_1, z_2, z_3, z_4, z_5, z_6\} \text{ and}$$
$$E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\},$$

and for each $i = 1, 2, \ldots, 9$, e_i is a bilateral relation defined by

$$e_1 = z_1 z_2, \quad e_2 = z_1 z_4, \quad e_3 = z_1 z_6, \quad e_4 = z_2 z_3, \quad e_5 = z_3 z_4,$$
$$e_6 = z_3 z_6, \quad e_7 = z_2 z_5, \quad e_8 = z_4 z_5, \quad e_9 = z_5 z_6.$$

A graph $G = (V, E)$ can be drawn on the plane. Each element in V is indicated by a point. For clarity, such a point is often depicted as a small circle. For an

[1] The word "**graph**" was first used in this sense by **J. J. Sylvester** (James Joseph Sylvester, 1814-1897) in 1878 (Chemistry and algebra. *Nature*, 1877-8, 17: 284). Sylvester was an English mathematician, played a leadership role in American mathematics in the later half of the 19th century as a professor at the Johns Hopkins University and as founder of the *American Journal of Mathematics*.

element e in E, if $e = (x, y)$, we draw a directed line segment or curve joining two points from x to y; if $e = xy$, we draw an undirected line segment or curve joining two points x and y. Such a geometric diagram is called a *graphical representation* or *geometric representation* of the graph, which intuitively shows the configuration of the graph. Clearly, graphical representations of a graph are not unique, strongly depending on its drawing.

For instance, the diagrams shown in Figure 1.1 are two graphical representations of the graph D defined in Example 1.1.1, which show that a graph may have different graphical representations depending on position of points and drawing of lines. The diagrams shown in Figure 1.2 are graphical representations of the graph H and the graph G defined in Example 1.1.2 and Example 1.1.3, respectively.

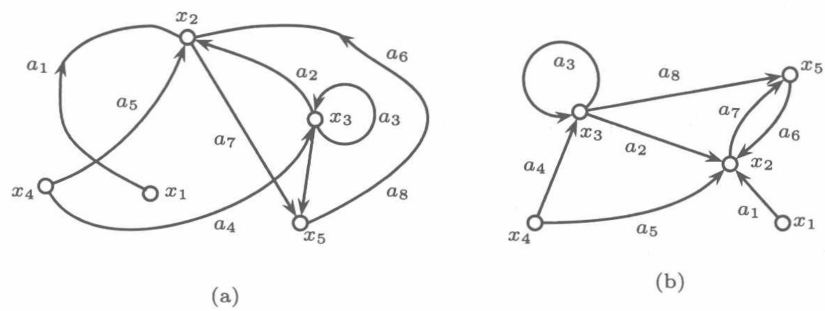

Figure 1.1 Two graphical representations of the digraph D

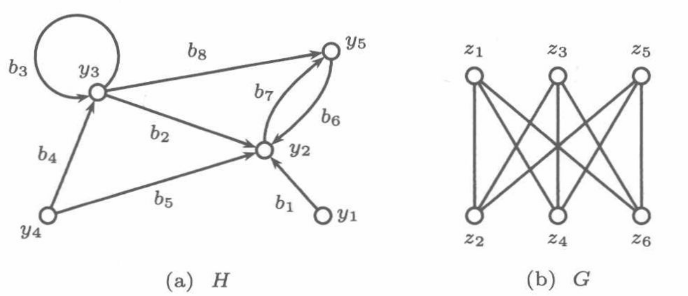

Figure 1.2 Graphical representations of graphs H and G

It is this representation that gives graph its name and much of its appeal. For instance, for a graph $G = (V, E)$, the set V is called the *vertex-set* of G, $x \in V$ called a *vertex*; the set E is called the *edge-set* of G, $e \in E$ called an *edge*[1]. Two vertices linked by an edge e is called *end-vertices* of the edge e. The end-vertices of an edge are said to be *incident* with the edge, and vice versa. Two vertices which

[1] Some authors prefer to call a vertex as a *point*, an edge as a *line* (see for example [170]) or call an edge as an *arc* if the edge is directed (see for example [21]). In some of the older papers we may find "branch" used for "edge", and "node" for "vertex".

are incident with a common edge are *adjacent*, as are two edges which are incident with a common vertex.

If $V \times V$ is considered as a set of ordered pairs (x, y), then the graph G is called a *directed graph*, or *digraph* for short. For instance, the two graphs in Example 1.1.1 and Example 1.1.2 are both digraphs. For an edge e of a digraph, sometimes, called a *directed edge* or *arc*, if $e = (x, y)$, then the end-vertices x and y are called the *tail* and the *head* of the edge e, respectively; and the edge e is sometimes called an *out-going edge* of x or an *in-coming edge* of y.

If $V \times V$ is considered as a set of unordered pairs xy, then the graph G is called an *undirected graph*. For instance, the graph in Example 1.1.3 is an undirected graph. Edges of an undirected graph are sometimes called *undirected edges*.

It should be emphasized that a graph is a mathematical structure (V, E), a geometric diagram is only one of its several representations, and besides, graphical representations have some restrictions since it is unable to draw a geometric diagram of a graph if it has a large order or a complex structure.

In addition, in general, for a graph (V, E), its vertex-set V and edge-set E are not always visible expressions as the above examples, while only give the rules to form the vertex-set and the edge-set. A simple example is as follows.

Example 1.1.4 Let Ω_n^k be the family of sets of k distinct elements on n letters. For given integers n, k and i with $n \geqslant k \geqslant i \geqslant 0$, a graph, denoted by $J(n, k, i)$ and called the *$J(n, k, i)$-graph*, can be defined as (V, E), where the vertex-set $V = \Omega_n^k$ and the edge-set $E = \{XY : X, Y \in V, |X \cap Y| = i\}$. Clearly, the $J(n, k, i)$-graph is an undirected graph and $|V| = |\Omega_n^k| = \binom{n}{k}$.

From definition, as a mathematical structure, such a graph exists indeed. However, it is quite difficult to see what appearance of the structure of such a graph is for general n, k and i. For $n = 5$, $k = 2$ and $i = 0$, Figure 1.3 shows two graphical representations of the $J(5, 2, 0)$-graph, from which its structure is open-and-shut. It is an undirected graph with 10 vertices and 15 edges.

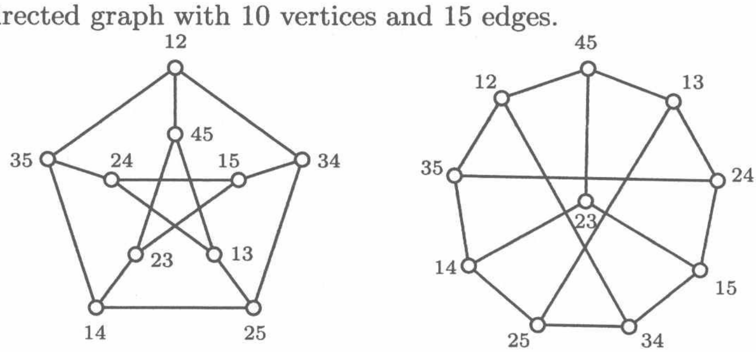

Figure 1.3 Two drawings of the $J(5, 2, 0)$-graph

For $n \geqslant 2k$, the $J(n,k,k-1)$-graph is known as the *Johnsom graph*, and the $J(n,k,0)$-graph is known as the *Kneser graph* (see [143]). The $J(5,2,0)$-graph is often called *Petersen graph*[1], a very useful and interesting graph, which often occurs in the literature and any textbook on graph theory, that serves as a counterexample for many problems in graph theory (see [189]).

Example 1.1.5 The n-dimensional *cube* or *hypercube*[2] Q_n is the best-known class of graphs, also an important topological structure of interconnection networks (see [360] for details). Note that letters in Ω_n^k defined in Example 1.1.4 is different from each other. If letters in Ω_n^k may be identical, then Q_n can be defined as the $J(2,n,n-1)$-graph. In other words, Q_n is an undirected graph (V,E), where

$$V = \{x_1x_2\ldots x_n:\ x_i \in \{0,1\},\ i = 1,2,\ldots,n\},$$

and for two vertices $x = x_1x_2\ldots x_n$ and $y = y_1y_2\ldots y_n$,

$$xy \in E \Leftrightarrow \sum_{i=1}^{n} |x_i - y_i| = 1.$$

Figure 1.4 shows graphical representations of Q_1, Q_2, Q_3 and Q_4, respectively.

Example 1.1.6 The *de Bruijn digraphs*[3] $B(d,n) = (V,E)$, where

$$V = \{x_1x_2\ldots x_n:\ x_i \in \{0,1,\ldots,d-1\},\ i = 1,2,\ldots,n\},$$

and for $x, y \in V$, if $x = x_1x_2\ldots x_n$, then

$$(x,y) \in E \Leftrightarrow y = x_2x_3\ldots x_n\alpha, \quad \alpha \in \{0,1,\ldots,d-1\}.$$

Figure 1.5 shows graphical representations of the de Bruijn digraphs $B(2,n)$ for each $n = 1,2,3$, respectively.

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[1] **J. Petersen** (Julius Petersen, 1839-1910) was a Danish mathematician. His interests in mathematics were manifold. His famous paper *Die Theorie der regulären Graphen* (*Acta Mathematica*, 1891, 15(1): 193-220) was a fundamental contribution to modern graph theory as we know it today. In 1898, he presented a small counterexample to Tait's claimed theorem about hamiltonicity of 3-regular graphs, which is nowadays known as the "Petersen Graph". Although the graph is generally credited to Petersen, it had in fact first appeared 12 years earlier, in a paper by A. B. Kempe (A memoir on the theory of mathematical form. *Philosophical Transactions of the Royal Society of London*, 1886, 177: 1-70). *A special issue of Discrete Mathematics* (1992, 100(1-3): 9-82) has been dedicated to the 150th birthday of Petersen, in which a very precise biography may be found.

[2] The hypercubes have been much studied in graph theory and computer sciences, see the survey by Hayes and Mudge (J. P. Hayes and T. N. Mudge. *Hypercube supercomputers. Proceedings of the IEEE*, 1989, 77(12): 1829-1841.) and the monograph [360].

[3] The de Bruin digraph $B(2,n)$ was proposed by de Bruijn (N. G. De Bruijn. *A combinatorial problem. Koninklije Nedderlandse van Wetenshappen Proc.*, 1946, 49A: 758-764.) and Good (I. J. Good. *Normal recurring decimals. J. London Math. Soc.*, 1946, 21: 167-169), independently.

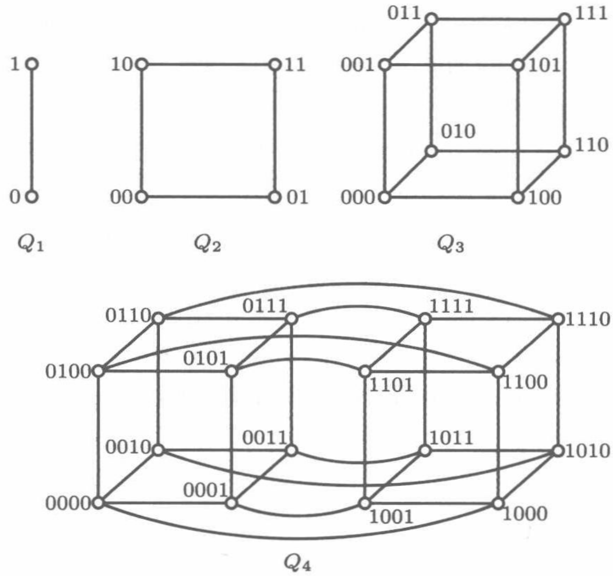

Figure 1.4 The n-cubes Q_1, Q_2, Q_3 and Q_4

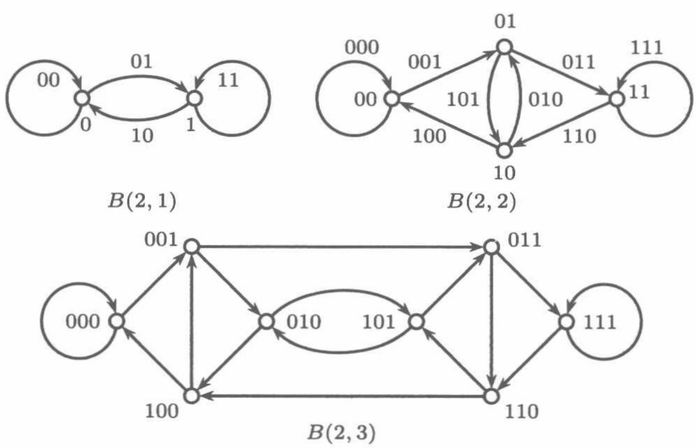

Figure 1.5 De Bruin digraphs $B(2,1)$, $B(2,2)$ and $B(2,3)$

Example 1.1.7 The *Kautz digraphs*[1] $K(d,n) = (V,E)$, where

$$V = \{x_1x_2\ldots x_n :\ x_i \in \{0,1,\ldots,d\},\ x_{i+1} \neq x_i,\ i = 1,2,\ldots,n-1\},$$

and for $x, y \in V(K(d,n))$, if $x = x_1x_2\ldots x_n$, then

$$(x,y) \in E \Leftrightarrow y = x_2x_3\ldots x_n\alpha, \quad \alpha \in \{0,1,\ldots,d\} \setminus \{x_n\}.$$

[1] The Kautz digraph was proposed by Kautz (W. H. Kautz. *Design of optimal interconnection networks for multiprocessor. Architecture and Design of Digital Computers*, Nato Advanced Summer Institute, 1969, 249-272).

Figure 1.6 shows graphical representations of the Kautz digraphs $K(2, n)$ for each $n = 1, 2, 3$, respectively.

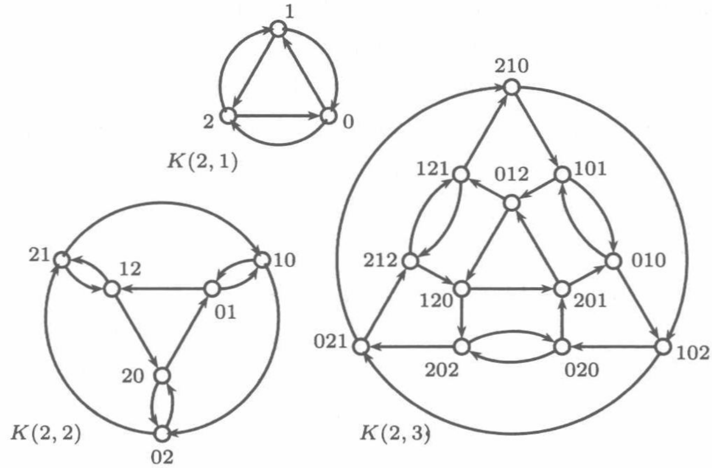

Figure 1.6 Kautz digraphs $K(2,1)$, $K(2,2)$ and $K(2,3)$

From definition of a graph, it is possible that two end-vertices of an edge are identical since it is possible that $(x, x) \in V \times V$ for any $x \in V$, such an edge is called a *loop* (see Figure 1.1, Figure 1.2 and Figure 1.5), while it is not allowable that more than one edges link a vertex to another vertex. However, in some practical applications, it is convenient and allowable that more than one edges link a vertex to another vertex, these edges are called *multi-edges*, the corresponding graph is called a *multi-graph*. A graph is called to be *loopless* if it contains no loops. A graph is called to be *simple* if it contains no loops and multi-edges edges.

An undirected graph G can be thought of as a particular digraph D, a *symmetric digraph* obtained by replacing each edge in G by two oppositely directed edges, called *symmetric edges*. Figure 1.7 shows such graphs, where (a) is an undirected graph, (b) is its symmetric digraph. Thus, to study structural properties of graphs for digraphs

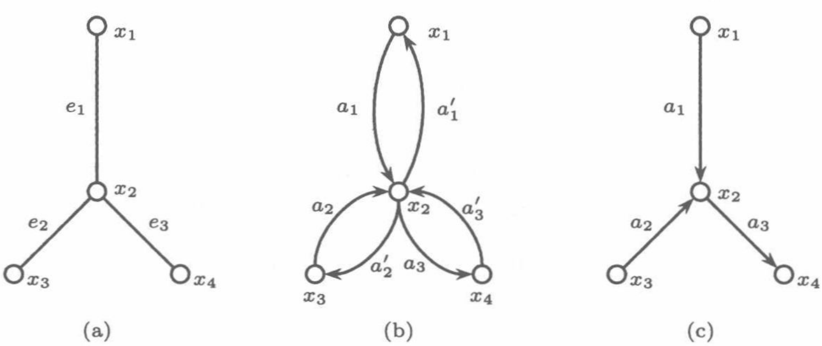

Figure 1.7 The symmetric digraph and oriented graph of an undirected graph

is more general than for undirected graphs. A digraph is said to be *asymmetric* if it contains no symmetric edges (see Figure 1.7 (c)).

There are many topics in graph theory that have no relations with direction of edges. The undirected graph (maybe multi-graph) obtained from a digraph D by removing the orientation of all edges is called the *underlying graph* of D. Conversely, the digraph obtained from an undirected graph G by specifying an orientation of each edge of G is called an *oriented graph* of G.

Figure 1.7 shows such graphs, where (a) is the underlying graph of (c) and, conversely, (c) is an oriented graph of (a).

Let (V, E) be a graph. The number of vertices, $\upsilon = |V|$, is called *order* of the graph; the number of edges, $\varepsilon = |E|$, is called *size* of the graph. A graph is said to be *edgeless*[1] if $\varepsilon = 0$. An edgeless graph of order υ is often denoted by K_υ^c. An edgeless graph is said to be *trivial* if $\upsilon = 1$, and all other graphs *non-trivial*. A graph is *finite* if υ and ε are both finite.

Throughout this book, all graphs are always considered to be finite. The letter G always denotes a graph, which is directed or undirected according to the context if it is not specially noted. Sometimes, to emphasize, we use the letter D to denote a digraph. When just one graph is under discussion, the letters υ and ε always denote order and size of the graph, respectively.

The notations $\lfloor r \rfloor$ and $\lceil r \rceil$ denote the greatest integer not exceeding the real number r and the smallest integer not less than r, respectively. The notation

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!}$$

denotes the number of k-combinations of n distinct objects ($k \leqslant n$).

We conclude this section with an example to show that using a graph can make some statements intuitive, simple and clear.

Example 1.1.8 *At a gathering of any six people, some three of them are either mutual acquaintances or complete strangers to each other*[2].

Proof. Use points A, B, C, D, E, F to denote these six people, respectively. Draw a solid line joining two points if two people have known each other, a dashed line otherwise. Use G to denote the resulting diagram. We only need to prove that G certainly contains either a solid triangle or a dashed triangle. Consider a point

[1] Some authors prefer to use the term "*empty graph*" rather than "edgeless graph" (see for example [42, 43]). This usage may be found inconvenient in problems involving graph operations, such as $G_1 \cap G_2 = \varnothing$ (see Section 1.4 of this book), called a *null graph* sometime to avoid confusion.

[2] See: C. W. Bostwick, J. Rainwater and J. D. Baum, E1321, *The American Mathematical Monthly*, 1959, 66(2): 141-142.